CUL-H02710 – 58 – 02577607

58

AMERICAN
JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

G. D. BIRKHOFF
HARVARD UNIVERSITY

H. WEYL
THE INSTITUTE FOR ADVANCED STUDY

F. D. MURNAGHAN
THE JOHNS HOPKINS UNIVERSITY

R. L. WILDER
UNIVERSITY OF MICHIGAN

A. WINTNER
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

G. BIRKHOFF
N. DUNFORD
E. P. LANE
D. MONTGOMERY
J. L. SYNGE
R. P. AGNEW
C. CHEVALLEY
G. A. HEDLUND
S. B. MYERS
N. E. STEENROD
V. G. GROVE
M. H. HEINS
D. C. LEWIS
T. RADÓ
H. S. WALL

PUBLISHED UNDER THE JOINT AUSPICES OF
THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

VOLUME LXV
1943

CALCUTTA UNIVERSITY CENTRAL LIBRARY

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

PRINTED IN THE UNITED STATES OF AMERICA
BY J. H. FURST COMPANY, BALTIMORE, MARYLAND

INDEX

PAGE

rtin, Emil and Whaples, George. The theory of simple rings, . 87

aer, Reinhold. Radical ideals, ... 537

aer, Reinhold. Rings with duals, ... 569

l, E. T. Interpolated denumerants and Lambert series, . . . 382

ergman, Stefan. The behavior of the kernel function at boundary points of the second order, ... 679

gin, D. G. Linear topological spaces, ... 637

amp, Chester C. A convergence proof involving an inseparable multiple contour integral, ... 216

valley, Claude. A new kind of relationship between matrices, . 521

oolidge, Julian L. Systems of rational curves, ... 257

ressel, F. G. and Elliott, E. R. A class of solutions for the heat equation and associated boundary value problems, . . . 408

exter, Glenn Edward. The calculus of non-contradiction, . . 171

oob, J. L. and Leibler, R. A. On the spectral analysis of a certain transformation, ... 263

ilenberg, Samuel and Harrold, O. G., Jr. Continua of finite linear measure I, ... 137

Elliott, E. R. See Dressel and Elliott, page 408.

Harrold, O. G., Jr. See Eilenberg and Harrold, page 137.

Hartman, Philip and Wintner, Aurel. Integrability in the large and dynamical stability, ... 273

Haviland, E. K. On the singularities and the qualitative behavior of the solutions of the problem of central forces, ... 477

Hughes, Howard K. See Miller and Hughes, page 423.

Kac, M. On the distribution of values of trigonometric sums with linearly independent frequencies, ... 609

Ladue, Mary Elizabeth. Conformal geometry of horn angles of higher order, ... 455

PA

LECAINE, JEANNE. The linear q-difference equation of the second order,

LEE, HWA-CHUNG. A kind of even-dimensional differential geometry and its application to exterior calculus, ... 4

LEHNER, JOSEPH. Ramanujan identities involving the partition function for the moduli 11^a, ... 492

LEIBLER, R. A. See Doob and Leibler, page 263.

LEVI, HOWARD. A characterization of polynomial rings by means of order relations, ... 221

MACLANE, S. and SCHILLING, O. F. G. Groups of algebras over an algebraic number field, ...

MARTIN, MONROE H. The rectilinear motion of a gas, ...

MAXIMOFF, ISAIAH. On functions of class 1 having the property of Darboux, ...

MILLER, EDWIN W. A note on Souslin's problem, ... 673

MILLER, GLENN T. and HUGHES, HOWARD K. Analytic continuation of functions defined by factorial series, ... 4

MITCHELL, JOSEPHINE. On double Sturm-Liouville series, . . . 61

MONTGOMERY, DEANE and ZIPPIN, LEO. A class of transformation groups in E_n, ... 601

NEWSOM, C. V. The asymptotic behavior of a class of entire functions, 4

NIVEN, IVAN. Two observations concerning algebraic equations for matrices, ... 66

PA, CHENKUO. The projective theory of surfaces in ruled space, I, . 715

PHILLIPS, R. S. On weakly compact subsets of a Banach space, . . 108

POST, EMIL L. Formal reductions of the general combinatorial decision problem, ... 197

PRENOWITZ, WALTER. Projective geometries as multigroups, . . 235

RADEMACHER, HANS. On the Bloch-Landau constant, . . . 387

RADÓ, TIBOR. On a problem of Geöcze, ... 361

ROSENTHALL, E. On some cubic diophantine equations, . . . 663

SCHIFFER, MENAHEM. Variation of the Green function and theory of the p-valued functions, ... 341

SCHILLING, O. F. G. See MacLane and Schilling, page 299.

PAGE

SCHILLING, O. F. G. Normal extensions of relatively complete fields, . 309

SIEGEL, CARL LUDWIG. Symplectic geometry, ... 1

SPENCER, D. C. A function-theoretic identity, ... 147

SU, BUCHIN. Osculating conics of the plane sections through a point of a surface, ... 439

SU, BUCHIN. Plane sections through a point of a non-holonomic surface, 702

SZEGÖ, G. On conjugate trigonometric polynomials, ... 532

TRANSUE, W. R. Representation of subharmonic functions in the neighborhood of a point, ... 335

WHAPLES, GEORGE. See Artin and Whaples, page 87.

WHITMAN, PHILIP M. Splittings of a lattice, ... 179

WIENER, NORBERT and WINTNER, AUREL. The discrete chaos, . . 279

WINTNER, AUREL. See Hartman and Wintner, page 273.

WINTNER, AUREL. See Wiener and Wintner, page 279.

ZIPPIN, LEO. See Montgomery and Zippin, page 601.

SYMPLECTIC GEOMETRY.[*]

By Carl Ludwig Siegel.

CALCUTTA

I. INTRODUCTION.

1. Our present knowledge concerning functions of several complex variables $z_1, \cdots, z_m$ is much less far-reaching and complete than the classical theory in the case $m = 1$. If we want to proceed further, it seems reasonable to investigate, in the first place, a *special* class of analytic functions of m variables found by the following considerations:

Let R be the Riemann surface of an analytic function of a single variable. On account of the main theorem of uniformization, the universal covering surface U of R can be conformally mapped onto a simple domain E, which is either the unit-circle $| z | < 1$ or the finite z-plane or the complete z-plane. The conformal mappings of E onto itself form a group Ω of linear transformations, and the fundamental group of R is faithfully represented by a subgroup Δ of Ω, discontinuous in E. By the introduction of the uniformizing parameter z, the general theory of the analytic single-valued functions on R is reduced to the theory of the automorphic functions with the group Δ.

The group Ω is transitive, i. e., there exists for any two points z_1 and z_2 of E an element of Ω transforming z_1 into z_2. Moreover, there exists for every point z_1 of E an involution in Ω with the fixed point z_1, i. e., an element of Ω identical with its inverse and transforming z_1 into itself. Consequently E is a *symmetric* space, in the notation of Élie Cartan. The domain E is bounded, if we consider only the first case, the case of the unit-circle $| z | < 1$; it is well known, that this occurs if and only if U has at least two frontier points.

A generalization of the theory of automorphic functions to the case of an arbitrary number of variables requires the following three steps: 1) To determine all bounded simple domains E in the space of m complex variables, which are symmetric spaces with respect to a group Ω of analytic mappings. 2) To investigate the invariant geometric properties of E, to find the discontinuous subgroups Δ of Ω and to construct their fundamental domains. 3) To study the field of automorphic functions in E with the group Δ.

The first step has been made by Cartan; he obtained explicitly 6 different types of irreducible domains E, such that all other bounded simple symmetric analytic spaces can be derived from them by analytic mappings and topological products.

---

[*] Received February 27, 1942.

We shall consider more closely the second step. We restrict our researches to one of the six possible types, which is the most important for applications to other branches of mathematics. In this case, the number m of complex dimensions is $\frac{1}{2}n(n+1)$, with integral $n \geqq 1$, the m variables form the elements $z_{kl} = z_{lk}$ $(1 \leqq k \leqq l \leqq n)$ of a symmetric complex matrix $\mathfrak{Z} = (z_{kl})$ with n rows and E consists of all points $\mathfrak{Z}$ for which the hermitian form $\sum_{k=1}^{n}(|u_k|^2 - \sum_{l=1}^{n}|z_{kl}u_l|^2)$ in the auxiliary variables $u_1, \cdots, u_n$ is positive definite.

The third step has already been carried out, in a former publication, for the special case of the modular group of degree n. It is possible to generalize most of those results, but we shall not do so in the present paper.

2. Notations, definitions, results. All German letters denote matrices with complex elements; small German letters denote columns. The upper indices p and q in $\mathfrak{A}^{(pq)}$ designate the number p of rows and the number q of columns of the matrix $\mathfrak{A}$; instead of $\mathfrak{A}^{(pp)}$ and $\mathfrak{a}^{(p1)}$ we write more simply $\mathfrak{A}^{(p)}$ and $\mathfrak{a}^{(p)}$. If $a_1, \cdots, a_p$ are the diagonal elements of $\mathfrak{A}^{(p)}$ and if all other elements are 0, we write $\mathfrak{A} = [a_1, \cdots, a_p]$ and call $\mathfrak{A}$ a *diagonal* matrix. The letter $\mathfrak{E}$ denotes a *unit* matrix, and 0 denotes also a *zero* matrix. If $\mathfrak{B}$ is any matrix, $\mathfrak{B}'$ is the *transposed* matrix and $\bar{\mathfrak{B}}$ the *conjugate complex* matrix. We use the abbreviations $\mathfrak{B}'\mathfrak{A}\mathfrak{B} = \mathfrak{A}[\mathfrak{B}]$, $\mathfrak{B}'\mathfrak{A}\bar{\mathfrak{B}} = \mathfrak{A}\{\mathfrak{B}\}$. The inequality $\mathfrak{A} > 0$ means that $\mathfrak{A} = \bar{\mathfrak{A}}'$ is the matrix of a positive definite hermitian form, i. e., $\mathfrak{A}\{\mathfrak{x}\} > 0$ for all $\mathfrak{x} \neq 0$; obviously $\mathfrak{A} > 0$ means in the case of a real $\mathfrak{A}$, that $\mathfrak{A} = \mathfrak{A}'$ is the matrix of a positive definite quadratic form, i. e., $\mathfrak{A}[\mathfrak{x}] > 0$ for all real $\mathfrak{x} \neq 0$. The *trace* $\sigma(\mathfrak{A})$ of a matrix $\mathfrak{A}^{(p)} = (a_{kl})$ is defined by $\sigma(\mathfrak{A}) = \sum_{k=1}^{p} a_{kk}$.

We denote by $\mathfrak{Z} = (z_{kl})$ a symmetric matrix with n rows and variable complex elements $z_{kl} = z_{lk}$ $(1 \leqq k \leqq l \leqq n)$; $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$, where $\mathfrak{X} = \frac{1}{2}(\mathfrak{Z} + \bar{\mathfrak{Z}})$ and $\mathfrak{Y} = (1/2i)(\mathfrak{Z} - \bar{\mathfrak{Z}})$ are the real and imaginary part of $\mathfrak{Z}$. The condition $\mathfrak{E} - \mathfrak{Z}\bar{\mathfrak{Z}} > 0$ defines a bounded domain E which is obviously a generalization of the unit-circle. On the other hand, the domain H defined by the inequality $\mathfrak{Y} > 0$ is a generalization of the upper half-plane. It is well-known that the transformation $w = (az+b)/(cz+d)$ with real a, b, c, d and $ad - bc = 1$ is the most general analytic mapping of the upper half-plane onto itself. In order to generalize this theorem, we have to introduce the symplectic group. The *homogeneous symplectic* group Ω_0 consists of all real matrices

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A}^{(n)} & \mathfrak{B}^{(n)} \\ \mathfrak{C}^{(n)} & \mathfrak{D}^{(n)} \end{pmatrix}$$

satisfying the condition $\mathfrak{J}[\mathfrak{M}] = \mathfrak{J}$ with

$$\mathfrak{J} = \begin{pmatrix} 0 & \mathfrak{E}^{(n)} \\ -\mathfrak{E}^{(n)} & 0 \end{pmatrix}.$$

It is easily proved that the transformation

$$\mathfrak{W} = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} \tag{1}$$

maps the domain H onto itself. These transformations form the (inhomogeneous) *symplectic* group Ω obtained by identifying $\mathfrak{M}$ and $-\mathfrak{M}$.

THEOREM 1. *Every analytic mapping of H onto itself is symplectic.*

The next four theorems generalize known properties of the Poincaré model of non-euclidean geometry. For any two points $\mathfrak{Z}, \mathfrak{Z}_1$ of H we define

$$\mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1) = (\mathfrak{Z} - \mathfrak{Z}_1)(\mathfrak{Z} - \bar{\mathfrak{Z}}_1)^{-1}(\bar{\mathfrak{Z}} - \bar{\mathfrak{Z}}_1)(\bar{\mathfrak{Z}} - \mathfrak{Z}_1)^{-1}.$$

THEOREM 2. *There exists a symplectic transformation mapping a given pair $\mathfrak{Z}, \mathfrak{Z}_1$ of H into another given pair $\mathfrak{W}, \mathfrak{W}_1$ of H, if and only if the two matrices $\mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)$ and $\mathfrak{R}(\mathfrak{W}, \mathfrak{W}_1)$ have the same characteristic roots.*

Let $d\mathfrak{Z} = (dz_{kl})$ denote the matrix of the differentials dz_{kl}. The quadratic differential form

$$ds^2 = \sigma(\mathfrak{Y}^{-1} d\mathfrak{Z}\, \mathfrak{Y}^{-1} d\bar{\mathfrak{Z}}) \tag{2}$$

is invariant under Ω and defines a Riemann metric in H.

THEOREM 3. *There exists exactly one geodesic arc connecting two arbitrary points $\mathfrak{Z}, \mathfrak{Z}_1$ of H; its length ρ is given by*

$$\rho^2 = \sigma\left(\log^2 \frac{1 + \mathfrak{R}^{\frac{1}{2}}}{1 - \mathfrak{R}^{\frac{1}{2}}}\right)$$

with $\mathfrak{R} = \mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)$ and

$$\log^2 \frac{1 + \mathfrak{R}^{\frac{1}{2}}}{1 - \mathfrak{R}^{\frac{1}{2}}} = 4\mathfrak{R}\left(\sum_{k=0}^{\infty} \frac{\mathfrak{R}^k}{2k+1}\right)^2.$$

THEOREM 4. *All geodesics are symplectic images of the curves $\mathfrak{Z} = i[p_1^s, \cdots, p_n^s]$, where $p_1, \cdots, p_n$ are arbitrary positive constants satisfying $\sum_{k=1}^{n} \log^2 p_k = 1$.*

Let $\mathfrak{X} = (x_{kl})$, $\mathfrak{Y}^{-1} = (Y_{kl})$ and dv be the euclidean volume element in the space with the $n(n+1)$ rectangular cartesian coördinates x_{kl}, Y_{kl}

$(1 \leq k \leq l \leq n)$. It is easily shown that $2^{n(n-1)/2} dv$ is the volume element for the symplectic metric (2).

THEOREM 5. *The Euler characteristic of a closed manifold F with the metric* (2) *is*

$$\chi = c_n(-\pi)^{-n(n+1)/2} \int_F dv, \tag{3}$$

where c_n denotes a positive rational number depending only upon n; in particular $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{8}$, $c_3 = \frac{45}{256}$.

The theorems 6 and 7 are concerned with the generalization of the Fuchsian groups and their fundamental domains. Let Δ be a subgroup of the symplectic group Ω. Two points $\mathfrak{Z}, \mathfrak{W}$ of H are called *equivalent* under Δ, if (1) holds for a matrix $\mathfrak{M}$ of Δ. The group Δ is *discontinuous,* if no set of equivalent points has a limit point in H. A domain F in H is a *fundamental domain* for Δ, if the images of F under Δ cover H without gaps and overlappings. A domain F is called a *star,* if there exists an inner point $\mathfrak{Z}_0$ of F such that for every point $\mathfrak{Z}$ of F the whole geodesic arc between $\mathfrak{Z}$ and $\mathfrak{Z}_0$ belongs to F.

THEOREM 6. *A fundamental domain F of a discontinuous group Δ may be chosen as a star bounded by analytic surfaces and such that every compact domain in H is covered by a finite number of images of F under Δ.*

A discontinuous group Δ is called of the *first kind,* if there exists a *normal* fundamental domain F having the following three properties: 1) Every compact domain in H is covered by a finite number of images of F; 2) only a finite number of images of F are neighbors of F; 3) the integral

$$V(\Delta) = \int_F dv$$

converges. The space H is called *compact relative to* Δ, if there exists for every infinite sequence of points $\mathfrak{Z}_k$ $(k = 1, 2, 3, \cdots)$ in H a compact sequence $\mathfrak{W}_k$ such that $\mathfrak{W}_k$ is equivalent to $\mathfrak{Z}_k$ under Δ.

THEOREM 7. *If H is compact relative to a discontinuous group Δ, then Δ is of the first kind and has a compact normal fundamental domain.*

Let us assume that a discontinuous group Δ has no fixed point in H, i. e., that no transformation of Δ except the identical one has a fixed point in H. Identifying equivalent frontier points of a fundamental domain F, we obtain a closed manifold, if H is compact relative to Δ. The Euler number of this manifold is then given by Theorem 5. If H is not compact relative to Δ, we obtain an open manifold. It is probable that Theorem 5 still holds good for this open manifold, provided Δ is of the first kind; it may easily be proved that this is true in the case $n = 1$.

The rest of the paper deals with two special classes of groups Δ defined by arithmetical properties. The simplest and most important example of a discontinuous subgroup of Ω is given by the *modular* group Γ of degree n consisting of all symplectic matrices $\mathfrak{M}^{(2n)}$ with rational integral elements.

THEOREM 8. *The modular group of degree n is a discontinuous group of the first kind.*

Let K be a totally real algebraic number-field of degree $h \geq 1$, $K(\sqrt{-r})$ a totally imaginary quadratic field over K and s a positive number in K such that all other conjugates of s are negative. Let $\mathfrak{S}^{(2n)}$ be a skew-symmetric matrix and $\mathfrak{H}^{(2n)}$ a hermitian matrix, both with elements from $K(\sqrt{-r})$ and non-singular. We assume that all conjugates of $\mathfrak{H}$ except $\mathfrak{H}$ and $\bar{\mathfrak{H}}$ are positive and that $\mathfrak{S}$ and $\mathfrak{H}$ are connected by the relation $\mathfrak{H}\bar{\mathfrak{S}}^{-1}\bar{\mathfrak{H}} = s\mathfrak{S}$. Let $\Lambda(\mathfrak{S}, \mathfrak{H})$ denote the group of matrices $\mathfrak{U}$ with integral elements of $K(\sqrt{-r})$ satisfying the two conditions $\mathfrak{S}[\mathfrak{U}] = \mathfrak{S}$ and $\mathfrak{H}\{\mathfrak{U}\} = \mathfrak{H}$. Then there exists a constant matrix $\mathfrak{C}$ such that $\mathfrak{C}^{-1}\mathfrak{U}\mathfrak{C} = \mathfrak{M}$ is symplectic, and $\mathfrak{C}^{-1}\Lambda(\mathfrak{S}, \mathfrak{H})\mathfrak{C} = \Delta(\mathfrak{S}, \mathfrak{H})$ is a subgroup of Ω. The modular group Γ is a particular case of these groups $\Delta(\mathfrak{S}, \mathfrak{H})$, namely the case $h = 1$, $\mathfrak{S} = \mathfrak{J}$, $\mathfrak{H} = i\mathfrak{J}$, $r = 1$.

THEOREM 9. *The group $\Delta(\mathfrak{S}, \mathfrak{H})$ is discontinuous and of the first kind. In the case $h > 1$, the space H is compact relative to $\Delta(\mathfrak{S}, \mathfrak{H})$.*

For every ideal κ of $K(\sqrt{-r})$, we denote by $\Lambda_\kappa(\mathfrak{S}, \mathfrak{H})$ the congruence subgroup of $\Lambda(\mathfrak{S}, \mathfrak{H})$ defined by the condition $\mathfrak{U} \equiv \mathfrak{E} \pmod{\kappa}$, and by $\Delta_\kappa(\mathfrak{S}, \mathfrak{H}) = \mathfrak{C}^{-1}\Lambda_\kappa(\mathfrak{S}, \mathfrak{H})\mathfrak{C}$ the corresponding subgroup of $\Delta(\mathfrak{S}, \mathfrak{H})$.

THEOREM 10. *Let ρ be a prime ideal of $K(\sqrt{-r})$ and κ the least power of ρ such that p is not divisible by κ^{p-1}, where p denotes the rational prime number divisible by ρ. Then $\Delta_\kappa(\mathfrak{S}, \mathfrak{H})$ has no fixed point in H.*

On account of the theorems 5 and 10, the calculation of the integral $V(\Delta)$ for $\Delta = \Delta(\mathfrak{S}, \mathfrak{H})$ is important. Applying the Gauss-Dirichlet method from analytic number theory, we obtain in the case of the modular group of degree n a curious connection with Riemann's ζ-function. Using the abbreviation $\xi(t) = \pi^{-(t/2)}\Gamma(t/2)\zeta(t)$, so that $\xi(t) = \xi(1-t)$ is the functional equation of $\zeta(t)$, we have

THEOREM 11. *The symplectic volume of the fundamental domain of the modular group is*

$$V(\Gamma) = 2\prod_{k=1}^{n} \xi(2k).$$

This formula may be written in a different way, suggested by the results of the analytic theory of quadratic forms. Consider a domain Q in the space of the real skew-symmetric matrices $\mathfrak{Q}^{(2n)} = (q_{kl})$, with the rectangular cartesian coördinates q_{kl} $(1 \leq k < l \leq 2n)$, and denote by L the corresponding part of the space of the real matrices $\mathfrak{L}^{(2n)}$ defined by the condition $\mathfrak{J}[\mathfrak{L}] = \mathfrak{Q}$, the coördinates in L being the $4n^2$ elements of $\mathfrak{L}$. Obviously L is invariant under any mapping $\mathfrak{L} \rightarrow \mathfrak{M}\mathfrak{L}$ with symplectic $\mathfrak{M}$. Let L_0 be a fundamental domain in L with respect to the homogeneous modular group, and let $v(L_0)$, $v(Q)$ be the euclidean volumes of L_0 and Q. We define

$$d_0(\Gamma) = \lim_{Q \to \mathfrak{J}} \frac{v(L_0)}{v(Q)}, \tag{4}$$

where Q runs over a sequence of domains tending to the single point $\mathfrak{J}$. On the other hand, let p be a rational prime number and E_p the number of modulo p incongruent integral solutions $\mathfrak{M}$ of the congruence $\mathfrak{J}[\mathfrak{M}] \equiv \mathfrak{J}$ (mod p). Since there are, modulo p, exactly $p^{n(2n-1)}$ integral skew-symmetric matrices $\mathfrak{Q}$ and p^{4n^2} integral matrices $\mathfrak{L}$, the expression

$$d_p(\Gamma) = p^{n(2n-1)} E_p^{-1}$$

may be considered as the p-adic analogue of $d_0(\Gamma)$. As a consequence of Theorem 11, we obtain

THEOREM 12. *Let p run over all prime numbers, then*

$$d_0(\Gamma) = \prod_{p} d_p(\Gamma).$$

It is possible to generalize this theorem for the case of an arbitrary group $\Delta(\mathfrak{G}, \mathfrak{H})$ instead of Γ.

Two subgroups Δ and Δ_1 of Ω are *conjugate*, if the relation $\Delta_1 = \mathfrak{F}^{-1}\Delta\mathfrak{F}$ holds for a symplectic matrix $\mathfrak{F}$. More generally, Δ and Δ_1 are called *commensurable*, if they contain conjugate subgroups of finite index. It is important, for the theory of automorphic functions, to know whether two given groups Δ and Δ_1 are commensurable or not. Let $\Delta = \Delta(\mathfrak{G}, \mathfrak{H})$, $\Delta_1 = \Delta(\mathfrak{G}_1, \mathfrak{H}_1)$ and let K_1, r_1, s_1 have the same meaning for $\mathfrak{G}_1, \mathfrak{H}_1$ that K, r, s have for $\mathfrak{G}, \mathfrak{H}$.

THEOREM 13. *The two groups $\Delta(\mathfrak{G}, \mathfrak{H})$ and $\Delta(\mathfrak{G}_1, \mathfrak{H}_1)$ are commensurable if, and only if, $K = K_1$ and the ternary quadratic forms $rsx^2 - ry^2 + sz^2$ and $r_1s_1x^2 - r_1y^2 + s_1z^2$ are equivalent in K.*

In the particular case $n = 2$, another class of discontinuous subgroups of Ω is given by the theory of *units* of quinary quadratic forms. Let K be again a totally real field of degree h. We consider a quadratic form $\mathfrak{T}[\mathfrak{x}]$ of 5 variables with coefficients from K and assume that $\mathfrak{T}[\mathfrak{x}]$ has the signature 2, 3, whereas all other conjugates of $\mathfrak{T}[\mathfrak{x}]$ are definite. Let $\Lambda(\mathfrak{T})$ be the group of all integral matrices $\mathfrak{U}$ in K satisfying $\mathfrak{T}[\mathfrak{U}] = \mathfrak{T}$, $|\mathfrak{U}| = 1$. On account of the spin representation of the orthogonal group, either $\Lambda(\mathfrak{T})$ itself or a subgroup of index 2 is then isomorphic to a certain subgroup $\Delta(\mathfrak{T})$ of Ω. Concerning these groups $\Delta(\mathfrak{T})$, there are analogues of the theorems 9, 10, 12, 13; in particular, analogous to Theorem 9, we have

THEOREM 14. *The group $\Delta(\mathfrak{T})$ is discontinuous and of the first kind. In the case $h > 1$, the space H is compact relative to $\Delta(\mathfrak{T})$.*

It would be interesting to seek discontinuous subgroups of Ω which are not commensurable with any of the groups $\Delta(\mathfrak{S}, \mathfrak{H})$ and $\Delta(\mathfrak{T})$. In the case $n = 1$, we may start with an arbitrary polygon satisfying certain conditions, and use the reflection method, but this simple geometric principle breaks down for $n > 1$.

---

3. Literature.

C. B. Allendoerfer, "The Euler number of a Riemann manifold," *American Journal of Mathematics* **62**, pp. 243-248 (1940).

E. Cartan, "Sur les domaines bornés homogènes de l'espace de n variables complexes," *Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität* **11**, pp. 116-162 (1936).

W. Fenchel, "On total curvatures of Riemannian manifolds: I," *The Journal of the London Mathematical Society* **15**, pp. 15-22 (1940).

R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen*, Vol. 1, B. G. Teubner, Leipzig (1897).

G. Fubini, "A remark on general Fuchsian groups," *Proceedings of the National Academy of Sciences* **26**, pp. 695-700 (1940).

———, "The distance in general Fuchsian geometries," *Proceedings of the National Academy of Sciences* **26**, pp. 700-708 (1940).

G. Giraud, "Sur une classe de groupes discontinus de transformations birationnelles quadratiques et sur les fonctions de trois variables indépendantes restant invariables par ces transformations," *Annales Scientifiques de l'Ecole Normale Supérieure* (3) **32**, pp. 237-403 (1915).

———, "Sur les groupes de transformations semblables arithmétiques de certaines formes quadratiques quinaires indéfinies et sur les fonctions de trois variables indépendantes invariantes par des groupes isomorphes aux précédents," *Annales Scientifiques de l'Ecole Normale Supérieure* (3) **33**, pp. 330-362 (1916).

P. Humbert, "Théorie de la réduction des formes quadratiques définies positives dans un corps algébrique K fini," *Commentarii Mathematici Helvetici* **12**, pp. 263-306 (1940).

H. Minkowski, "Diskontinuitätsbereich für arithmetische Äquivalenz," *Journal für die reine und angewandte Mathematik* **129**, pp. 220-274 (1905).

C. L. Siegel, "Einführung in die Theorie der Modulfunktionen n-ten Grades," *Mathematische Annalen* **116**, pp. 617-657 (1939).

————, "Einheiten quadratischer Formen," *Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität* **13**, pp. 209-239 (1940).

M. Sugawara, "Über eine allgemeine Theorie der Fuchsschen Gruppen und Theta-Reihen," *Annals of Mathematics* (2) **41**, pp. 488-494 (1940).

————, "A generalization of Poincaré-space," *Proceedings of the Imperial Academy of Japan* **16**, pp. 373-377 (1940).

E. Witt, "Eine Identität zwischen Modulformen zweiten Grades," *Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität* **14**, pp. 323-337 (1941).

II. THE SYMPLECTIC GROUP.

4. The linear substitution $z = i\frac{1+z_0}{1-z_0}$ maps the unit-circle $z_0\bar{z}_0 < 1$ onto the upper half-plane $\frac{1}{2i}(z-\bar{z}) > 0$. We shall prove that there is an immediate generalization to the case $n > 1$.

Let $\mathfrak{Z}_0$ be a point of E, i. e., $\mathfrak{Z}_0 = \mathfrak{Z}'_0$, $\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0 > 0$. If $\mathfrak{x}$ is a solution of $(\mathfrak{E}-\mathfrak{Z}_0)\mathfrak{x} = 0$, then $\bar{\mathfrak{x}} = \bar{\mathfrak{Z}}_0\bar{\mathfrak{x}}$, $\mathfrak{x}' = \mathfrak{x}'\mathfrak{Z}_0$ and consequently $(\mathfrak{E}-\mathfrak{Z}_0\bar{\mathfrak{Z}}_0)\{\mathfrak{x}\} = \mathfrak{x}'\bar{\mathfrak{x}} - \mathfrak{x}'\mathfrak{Z}_0\bar{\mathfrak{Z}}_0\bar{\mathfrak{x}} = 0$, $\mathfrak{x} = 0$. This proves $|\mathfrak{E}-\mathfrak{Z}_0| \neq 0$ and the existence of the matrix

$$i(\mathfrak{E}+\mathfrak{Z}_0)(\mathfrak{E}-\mathfrak{Z}_0)^{-1} = \mathfrak{Z}. \tag{5}$$

Obviously $\mathfrak{Z} = \mathfrak{Z}'$ and

$$\frac{1}{2i}(\mathfrak{Z}-\bar{\mathfrak{Z}}) = \tfrac{1}{2}(\mathfrak{E}-\mathfrak{Z}_0)^{-1}((\mathfrak{E}+\mathfrak{Z}_0)(\mathfrak{E}-\bar{\mathfrak{Z}}_0) + (\mathfrak{E}-\mathfrak{Z}_0)(\mathfrak{E}+\bar{\mathfrak{Z}}_0))(\mathfrak{E}-\bar{\mathfrak{Z}}_0)^{-1}$$
$$= (\mathfrak{E}-\mathfrak{Z}_0\bar{\mathfrak{Z}}_0)\{(\mathfrak{E}-\mathfrak{Z}_0)^{-1}\} > 0;$$

hence $\mathfrak{Z}$ is a point of H. On the other hand, let $\mathfrak{Z}$ be an arbitrary point of H, i. e., $\mathfrak{Z} = \mathfrak{Z}'$, $\frac{1}{2i}(\mathfrak{Z}-\bar{\mathfrak{Z}}) > 0$. If $\mathfrak{x}$ is a solution of $(\mathfrak{Z}+i\mathfrak{E})\mathfrak{x} = 0$, then $\bar{\mathfrak{Z}}\bar{\mathfrak{x}} = i\bar{\mathfrak{x}}$, $\mathfrak{x}'\mathfrak{Z} = -i\mathfrak{x}'$ and consequently $\frac{1}{2i}(\mathfrak{Z}-\bar{\mathfrak{Z}})\{\mathfrak{x}\} = \frac{1}{2i}(\mathfrak{x}'\mathfrak{Z}\bar{\mathfrak{x}} - \mathfrak{x}'\bar{\mathfrak{Z}}\bar{\mathfrak{x}}) = -\mathfrak{x}'\bar{\mathfrak{x}} \leq 0$, $\mathfrak{x} = 0$. This proves $|\mathfrak{Z}+i\mathfrak{E}| \neq 0$ and the existence of the matrix

$$(\mathfrak{Z}-i\mathfrak{E})(\mathfrak{Z}+i\mathfrak{E})^{-1} = \mathfrak{Z}_0. \tag{6}$$

Obviously $\mathfrak{Z}_0 = \mathfrak{Z}'_0$ and

$$\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0 = (\mathfrak{Z} + i\mathfrak{E})^{-1}((\mathfrak{Z} + i\mathfrak{E})(\bar{\mathfrak{Z}} - i\mathfrak{E}) - (\mathfrak{Z} - i\mathfrak{E})(\bar{\mathfrak{Z}} + i\mathfrak{E}))(\bar{\mathfrak{Z}} - i\mathfrak{E})^{-1}$$
$$= \frac{2}{i}(\mathfrak{Z} - \bar{\mathfrak{Z}})\{(\mathfrak{Z} + i\mathfrak{E})^{-1}\} > 0;$$

hence $\mathfrak{Z}_0$ is a point of E. Moreover (6) follows from (5), and vice versa.

5. The homogeneous symplectic group Ω_0 consists of all matrices

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$$

with real elements satisfying $\mathfrak{M}'\mathfrak{J}\mathfrak{M} = \mathfrak{J}$. Since $\mathfrak{J}^{-1} = -\mathfrak{J}$, we have then also $\mathfrak{M}\mathfrak{J}\mathfrak{M}' = \mathfrak{J}$; hence $\mathfrak{M}'$ is symplectic and

(7) $$\mathfrak{A}\mathfrak{B}' = \mathfrak{B}\mathfrak{A}', \qquad \mathfrak{C}\mathfrak{D}' = \mathfrak{D}\mathfrak{C}', \qquad \mathfrak{A}\mathfrak{D}' - \mathfrak{B}\mathfrak{C}' = \mathfrak{E}.$$

Let $\mathfrak{Z}$ be a point of H, i. e.,

$$\mathfrak{J}\begin{bmatrix}\mathfrak{Z}\\ \mathfrak{E}\end{bmatrix} = 0, \qquad \frac{1}{2i}\mathfrak{J}\left\{\begin{matrix}\mathfrak{Z}\\ \mathfrak{E}\end{matrix}\right\} > 0.$$

The matrices $\mathfrak{A}\mathfrak{Z} + \mathfrak{B} = \mathfrak{P}$, $\mathfrak{C}\mathfrak{Z} + \mathfrak{D} = \mathfrak{Q}$ satisfy

$$\mathfrak{M}\begin{pmatrix}\mathfrak{Z}\\ \mathfrak{E}\end{pmatrix} = \begin{pmatrix}\mathfrak{P}\\ \mathfrak{Q}\end{pmatrix}, \qquad \mathfrak{J}\begin{bmatrix}\mathfrak{P}\\ \mathfrak{Q}\end{bmatrix} = 0, \qquad \frac{1}{2i}\mathfrak{J}\left\{\begin{matrix}\mathfrak{P}\\ \mathfrak{Q}\end{matrix}\right\} > 0,$$

$$\mathfrak{P}'\mathfrak{Q} = \mathfrak{Q}'\mathfrak{P}, \qquad \frac{1}{2i}(\mathfrak{P}'\bar{\mathfrak{Q}} - \mathfrak{Q}'\bar{\mathfrak{P}}) > 0.$$

If $\mathfrak{x}$ is a solution of $\mathfrak{Q}\mathfrak{x} = 0$, then $\bar{\mathfrak{Q}}\bar{\mathfrak{x}} = 0$, $\mathfrak{x}'\mathfrak{Q}' = 0$, $\frac{1}{2i}(\mathfrak{P}'\bar{\mathfrak{Q}} - \mathfrak{Q}'\bar{\mathfrak{P}})\{\mathfrak{x}\} = 0$, whence $\mathfrak{x} = 0$, $|\mathfrak{Q}| \neq 0$. This proves the existence of

$$(\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} = \mathfrak{P}\mathfrak{Q}^{-1} = \mathfrak{W},$$

with $\mathfrak{W} = \mathfrak{W}'$, $\frac{1}{2i}(\mathfrak{W} - \bar{\mathfrak{W}})\{\mathfrak{Q}\} > 0$, $\frac{1}{2i}(\mathfrak{W} - \bar{\mathfrak{W}}) > 0$. Consequently the fractional linear transformation

$$\mathfrak{W} = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} = (\mathfrak{Z}\mathfrak{C}' + \mathfrak{D}')^{-1}(\mathfrak{Z}\mathfrak{A}' + \mathfrak{B}')$$

maps H into itself. Since $\mathfrak{Z}(-\mathfrak{C}'\mathfrak{W} + \mathfrak{A}') = \mathfrak{D}'\mathfrak{W} - \mathfrak{B}'$ and

(8) $$\mathfrak{M}^{-1} = \mathfrak{J}^{-1}\mathfrak{M}'\mathfrak{J} = \begin{pmatrix} \mathfrak{D}' & -\mathfrak{B}' \\ -\mathfrak{C}' & \mathfrak{A}' \end{pmatrix},$$

we obtain $|-\mathfrak{C}'\mathfrak{W} + \mathfrak{A}'| \neq 0$ and $\mathfrak{Z} = (\mathfrak{D}'\mathfrak{W} - \mathfrak{B}')(-\mathfrak{C}'\mathfrak{W} + \mathfrak{A}')^{-1}$; hence H is mapped onto itself.

It is easily seen that two symplectic matrices $\mathfrak{M}$ and $\mathfrak{M}_1$ define the same symplectic mapping $\mathfrak{W} = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$, if and only if $\mathfrak{M}_1 = \pm \mathfrak{M}$. Hence the inhomogeneous symplectic group Ω is the factor group of Ω_0 obtained by identifying $\mathfrak{M}$ and $-\mathfrak{M}$.

We have

$$\begin{pmatrix} i\mathfrak{C}' + \mathfrak{D}' & -i\mathfrak{A}' - \mathfrak{B}' \\ 0 & \mathfrak{E} \end{pmatrix} \mathfrak{M} \begin{pmatrix} \mathfrak{E} & i\mathfrak{E} \\ 0 & \mathfrak{E} \end{pmatrix} = \begin{pmatrix} \mathfrak{E} & 0 \\ \mathfrak{C} & i\mathfrak{C} + \mathfrak{D} \end{pmatrix}.$$

On the other hand $| i\mathfrak{C} + \mathfrak{D} | \neq 0$, since $i\mathfrak{E}$ is a point of H. This proves $| \mathfrak{M} | = 1$.

6. The fractional linear transformation (5) maps E onto H; its matrix is

$$\mathfrak{L} = \begin{pmatrix} i\mathfrak{E} & i\mathfrak{E} \\ -\mathfrak{E} & \mathfrak{E} \end{pmatrix}$$

and satisfies

$$\mathfrak{J}[\mathfrak{L}] = 2i\mathfrak{J}, \qquad \frac{1}{i}\mathfrak{J}\{\mathfrak{L}\} = 2\mathfrak{K}$$

with

$$\mathfrak{K} = \begin{pmatrix} -\mathfrak{E} & 0 \\ 0 & \mathfrak{E} \end{pmatrix}.$$

Let $\mathfrak{M}$ be an arbitrary symplectic matrix, i. e., $\mathfrak{J}[\mathfrak{M}] = \mathfrak{J}$, $\mathfrak{J}\{\mathfrak{M}\} = \mathfrak{J}$. Then

$$\mathfrak{L}^{-1}\mathfrak{M}\mathfrak{L} = \mathfrak{M}_0 = \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{B}_0 \\ \mathfrak{C}_0 & \mathfrak{D}_0 \end{pmatrix}$$

fulfills the conditions $\mathfrak{J}[\mathfrak{M}_0] = \mathfrak{J}$, $\mathfrak{K}\{\mathfrak{M}_0\} = \mathfrak{K}$, whence $\mathfrak{J}[\mathfrak{M}_0] = \mathfrak{J}$, $\mathfrak{F}\bar{\mathfrak{M}}_0 = \mathfrak{M}_0\mathfrak{F}$ with

$$\mathfrak{F} = \mathfrak{J}\mathfrak{K} = \begin{pmatrix} 0 & \mathfrak{E} \\ \mathfrak{E} & 0 \end{pmatrix},$$

or more explicitly

(9) $$\mathfrak{A}_0\mathfrak{B}'_0 = \mathfrak{B}_0\mathfrak{A}'_0, \quad \mathfrak{A}_0\bar{\mathfrak{A}}'_0 - \mathfrak{B}_0\bar{\mathfrak{B}}'_0 = \mathfrak{E}, \quad \mathfrak{C}_0 = \bar{\mathfrak{B}}_0, \quad \mathfrak{D}_0 = \bar{\mathfrak{A}}_0.$$

The corresponding transformation

(10) $$\mathfrak{W}_0 = (\mathfrak{A}_0\mathfrak{Z}_0 + \mathfrak{B}_0)(\bar{\mathfrak{B}}_0\mathfrak{Z}_0 + \bar{\mathfrak{A}}_0)^{-1}$$

maps E onto itself, and all these transformations form the group $\mathfrak{L}^{-1}\Omega\mathfrak{L} = \Omega_E$.

We shall prove that Ω_E is transitive. Let $\mathfrak{Z}_0$ be any point of E, i. e., $\mathfrak{Z}_0 = \mathfrak{Z}'_0$, $\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0 > 0$. It is sufficient to prove the existence of a transformation (10) mapping $\mathfrak{Z}_0$ into 0. We choose $\mathfrak{A}_0$ such that $\mathfrak{A}_0(\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0)\bar{\mathfrak{A}}' = \mathfrak{E}$ and define $\mathfrak{B}_0 = -\mathfrak{A}_0\mathfrak{Z}_0$; then (9) is satisfied and (10) has obviously the required property. Consequently Ω is also transitive.

A mapping of Ω_E has the fixed point 0, if and only if $\mathfrak{B}_0 = 0$; then $\mathfrak{A}_0$ is unitary, by (9). Consequently this mapping has the particular form

$$\mathfrak{W}_0 = \mathfrak{U}'\mathfrak{Z}_0\mathfrak{U},$$

where $\mathfrak{U}$ denotes an arbitrary unitary matrix, i. e., a matrix satisfying $\mathfrak{U}'\bar{\mathfrak{U}} = \mathfrak{E}$. Since (5) maps 0 into $i\mathfrak{E}$, the formula

$$\frac{\mathfrak{W} - i\mathfrak{E}}{\mathfrak{W} + i\mathfrak{E}} = \mathfrak{U}' \frac{\mathfrak{Z} - i\mathfrak{E}}{\mathfrak{Z} + i\mathfrak{E}} \mathfrak{U}$$

gives all symplectic transformations having $i\mathfrak{E}$ as a fixed point.

7. By the results of the preceding section, the proof of Theorem 1 is reduced to the proof of the following statement: *Let* $\mathfrak{Z}_0 \rightarrow \mathfrak{W}_0$ *be an analytic mapping of* E *onto itself with the fixed point* 0; *then* $\mathfrak{W}_0 = \mathfrak{U}'\mathfrak{Z}_0\mathfrak{U}$ *with unitary constant* $\mathfrak{U}$.

Let $\mathfrak{Z}_0 = (z_{kl})$ be an arbitrary point of E and denote by r_k $(k = 1, \cdots, n)$ the characteristic roots of the hermitian matrix $\mathfrak{Z}_0\bar{\mathfrak{Z}}_0$; then $r_k \geq 0$ and also $r_k < 1$, by $\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0 > 0$; we may assume $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n < 1$. The matrix $\mathfrak{Z} = t\mathfrak{Z}_0$ is again a point of E, if the complex scalar factor t satisfies the condition $r_n t\bar{t} < 1$. Let $\mathfrak{W} = \mathfrak{W}(t)$ be the image of $\mathfrak{Z}$ under the analytic mapping $\mathfrak{Z}_0 \rightarrow \mathfrak{W}_0$. The elements of the matrix $\mathfrak{W}$ are analytic functions of the single complex variable t, for given $\mathfrak{Z}_0$; they are regular in the circle $r_n t\bar{t} < 1$ and a fortiori in the unit-circle $t\bar{t} \leq 1$. Consequently

$$\mathfrak{W} = \sum_{k=1}^{\infty} t^k \mathfrak{W}_k \qquad (r_n t\bar{t} > 1), \tag{11}$$

where the coefficients $\mathfrak{W}_k$ are matrices depending only upon $\mathfrak{Z}_0$. On the other hand, $\mathfrak{W}$ may be expressed as a power series in the variables tz_{kl}, converging for sufficiently small values of $t\bar{t}$. Since this expansion is unique, the matrix $t^k\mathfrak{W}_k$ is exactly the aggregate of the terms of order k in that power series. This proves, in particular, that the power series representation $\mathfrak{W}_0 = \sum_{k=1}^{\infty} \mathfrak{W}_k$ converges everywhere in E, if we do not split up the polynomials $\mathfrak{W}_k$ into their single terms.

Since $\mathfrak{Z}_0 \rightarrow \mathfrak{W}_0$ maps E onto itself, we have $\mathfrak{E} - \mathfrak{W}\bar{\mathfrak{W}}' > 0$ for $t\bar{t} = 1$. Integrating over that circle we obtain

$$\frac{1}{2\pi i} \int_{t\bar{t}=1} (\mathfrak{E} - \mathfrak{W}\bar{\mathfrak{W}}') \frac{dt}{t} > 0,$$

whence by (11)

$$\mathfrak{E} - \sum_{k=1}^{\infty} \mathfrak{W}_k \bar{\mathfrak{W}}_k > 0 \tag{12}$$

and in particular

$$\mathfrak{E} - \mathfrak{W}_1 \bar{\mathfrak{W}}_1 > 0. \tag{13}$$

The $\frac{1}{2}n(n+1)$ elements w_{kl} $(1 \leq k \leq l \leq n)$ of $\mathfrak{W}_1 = (w_{kl})$ are linear functions of the independent variables z_{kl} $(1 \leq k \leq l \leq n)$; let D be their determinant. Since $\mathfrak{W}_1$ is the linear part of the power series for $\mathfrak{W}_0$, the functional determinant of the $\frac{1}{2}n(n+1)$ independent elements of $\mathfrak{W}_0$ with respect to the variables z_{kl} is also D, at the point $\mathfrak{Z}_0 = 0$. If we interchange $\mathfrak{W}_0$ and $\mathfrak{Z}_0$, the determinant D is replaced by D^{-1}. In order to prove Theorem 1, we may therefore assume $D\bar{D} \geq 1$.

Consider now the linear mapping $\mathfrak{Z}_0 \rightarrow \mathfrak{W}_1$, with the determinant D. By (13), the domain E is mapped onto a domain E_1 contained in E. Let $v(E)$ and $v(E_1)$ be the euclidean volumes of E and E_1, the real and imaginary parts of the z_{kl} being rectangular cartesian coördinates. Then $v(E_1) = D\bar{D}v(E) \geq v(E)$, whence $D\bar{D} = 1$, $E_1 = E$, and the boundary of E is mapped onto itself. We take $\mathfrak{Z}_0 = \mathfrak{U}'\mathfrak{P}\mathfrak{U}$ with unitary $\mathfrak{U}$ and $\mathfrak{P} = [p_1, \cdots, p_n]$; obviously $\mathfrak{Z}_0$ is a boundary point of E, if $-1 \leq p_k \leq 1$ $(k = 1, \cdots, n)$ and at least one $p_k = \pm 1$. On the other hand, the determinant $|\mathfrak{E} - \mathfrak{W}_1\bar{\mathfrak{W}}_1|$ is a polynomial in $p_1, \cdots, p_n$, of total degree $2n$. Since $|\mathfrak{E} - \mathfrak{W}_1\bar{\mathfrak{W}}_1|$ vanishes on the boundary of E, this polynomial is divisible by $\prod_{k=1}^{n}(1 - p_k^2)$, of total degree $2n$. Moreover the constant terms in both polynomials have the value 1; hence

$$|\mathfrak{E} - \mathfrak{W}_1\bar{\mathfrak{W}}_1| = |\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0| \tag{14}$$

for $\mathfrak{Z}_0 = \mathfrak{U}'\mathfrak{P}\mathfrak{U}$, where $\mathfrak{U}$ is an arbitrary unitary matrix and $\mathfrak{P}$ an arbitrary real diagonal matrix. We use now the following lemma, the proof of which will be given in Section **9**.

LEMMA 1. *Let* $\mathfrak{Z}$ *be a complex symmetric matrix and* $\mathfrak{P}$ *the diagonal matrix* $[q_1^{\frac{1}{2}}, \cdots, q_n^{\frac{1}{2}}]$, *where* $q_1, \cdots, q_n$ *denote the characteristic roots of* $\mathfrak{Z}\bar{\mathfrak{Z}}$. *There exists a unitary matrix* $\mathfrak{U}$ *such that* $\mathfrak{Z} = \mathfrak{U}'\mathfrak{P}\mathfrak{U}$.

On account of this lemma, the relationship (14) holds also identically in $\mathfrak{Z}_0 = (z_{kl})$. Since $\mathfrak{W}_1$ is linear in all z_{kl}, we obtain

$$|\lambda\mathfrak{E} - \mathfrak{W}_1\bar{\mathfrak{W}}_1| = |\lambda\mathfrak{E} - \mathfrak{Z}_0\bar{\mathfrak{Z}}_0|$$

identically in λ. This proves that $\mathfrak{Z}_0\bar{\mathfrak{Z}}_0$ and $\mathfrak{W}_1\bar{\mathfrak{W}}_1$ have the same characteristic roots. Applying again Lemma 1, we find

(15) $$\mathfrak{W}_1 = \mathfrak{U}'\mathfrak{Z}_0\mathfrak{U}$$
with unitary $\mathfrak{U}$.

By (12), the inequality $\mathfrak{E} - \mathfrak{W}_1\bar{\mathfrak{W}}_1 - \mathfrak{W}_k\bar{\mathfrak{W}}_k > 0$ holds for $k = 2, 3, \cdots$ and every $\mathfrak{Z}_0$ in E. Choose, in particular, $\mathfrak{Z}_0 = u \exp i\mathfrak{S}$ with real symmetric $\mathfrak{S}$ and $0 < u < 1$. Then, by (15),

$$(1 - u^2)\mathfrak{E} - \mathfrak{W}_k\bar{\mathfrak{W}}_k > 0 \qquad (k = 2, 3, \cdots);$$

hence $\mathfrak{W}_k$ tends to 0, if u tends to 1, and $\mathfrak{W}_k = 0$ for $\mathfrak{Z}_0 = \exp i\mathfrak{S}$ and arbitrary real symmetric $\mathfrak{S}$. But $\mathfrak{W}_k$ is analytic and consequently $\mathfrak{W}_k = 0$ also for $\mathfrak{Z}_0 = \exp i\mathfrak{S}$ with complex symmetric $\mathfrak{S}$. This proves that $\mathfrak{W}_k$ vanishes identically.

8. In order to complete the proof of Theorem 1, we have to prove that the unitary matrix $\mathfrak{U}$ in (15) can be chosen as a constant matrix. Let

(16) $$\mathfrak{W}_1 = \mathfrak{W}_1(\mathfrak{Z}_0) = \sum_{k \leq l} z_{kl}\mathfrak{A}_{kl}$$

with constant matrices $\mathfrak{A}_{kl}$ and define

(17) $$\mathfrak{W}_1^* = \mathfrak{W}_1^*(\mathfrak{Z}_0) = \sum_{k \leq l} z_{kl}\bar{\mathfrak{A}}_{kl};$$

whence $\bar{\mathfrak{W}}_1 = \mathfrak{W}_1^*(\bar{\mathfrak{Z}}_0)$. Now $\mathfrak{W}_1\bar{\mathfrak{W}}_1 = \mathfrak{E}$ for $\mathfrak{Z}_0\bar{\mathfrak{Z}}_0 = \mathfrak{E}$ and consequently

(18) $$\mathfrak{W}_1(\mathfrak{Z}_0)\mathfrak{W}_1^*(\mathfrak{Z}_0^{-1}) = \mathfrak{E}$$

for $\mathfrak{Z}_0 = \exp i\mathfrak{S}$ with arbitrary real symmetric $\mathfrak{S}$. Since $\mathfrak{W}_1$ and $\mathfrak{W}_1^*$ are analytic, we infer again that (18) is an identity in $\mathfrak{Z}_0$.

Putting $z_{kk} = z_k \neq 0$ $(k = 1, \cdots, n)$, $\mathfrak{Z}_1 = [z_1, \cdots, z_n]$, $\mathfrak{Z}_2 = \mathfrak{Z}_0 - \mathfrak{Z}_1$ and using the Taylor series in the neighborhood of $\mathfrak{Z}_2 = 0$, we find

$$\mathfrak{Z}_0^{-1} = \mathfrak{Z}_1^{-1}(\mathfrak{E} + \mathfrak{Z}_2\mathfrak{Z}_1^{-1})^{-1} = \mathfrak{Z}_1^{-1} - \mathfrak{Z}_1^{-1}\mathfrak{Z}_2\mathfrak{Z}_1^{-1} + \cdots$$
$$(\mathfrak{W}_1(\mathfrak{Z}_1) + \mathfrak{W}_1(\mathfrak{Z}_2))(\mathfrak{W}_1^*(\mathfrak{Z}_1^{-1}) - \mathfrak{W}_1^*(\mathfrak{Z}_1^{-1}\mathfrak{Z}_2\mathfrak{Z}_1^{-1}) + \cdots) = \mathfrak{E},$$

hence in particular

(19) $$\mathfrak{W}_1(\mathfrak{Z}_1)\mathfrak{W}_1^*(\mathfrak{Z}_1^{-1}) = \mathfrak{E},$$

(20) $$\mathfrak{W}_1(\mathfrak{Z}_2)\mathfrak{W}_1^*(\mathfrak{Z}_1^{-1}) = \mathfrak{W}_1(\mathfrak{Z}_1)\mathfrak{W}_1^*(\mathfrak{Z}_1^{-1}\mathfrak{Z}_2\mathfrak{Z}_1^{-1}).$$

It follows from (16), (17) and (19) that

$$\sum_{k,l=1}^{n} z_k z_l^{-1}\mathfrak{A}_k\bar{\mathfrak{A}}_l = \mathfrak{E}$$

with $\mathfrak{A}_k = \mathfrak{A}_{kk}$ $(k = 1, \cdots, n)$, whence

(21) $$\mathfrak{A}_k\bar{\mathfrak{A}}_l = 0 \qquad (k \neq l).$$

By (15), the matrix $\mathfrak{A}_k\bar{\mathfrak{A}}_k$ has the characteristic roots $1, 0, \cdots, 0$. Without loss of generality, we may replace $\mathfrak{W}_1$ by $\mathfrak{U}_1'\mathfrak{W}_1\mathfrak{U}_1$, for any constant unitary matrix $\mathfrak{U}_1$. On account of Lemma 1, we may therefore assume that $\mathfrak{A}_1 = [1, 0, \cdots, 0]$. Then, by (21), the matrices $\mathfrak{A}_2, \cdots, \mathfrak{A}_n$ have the form

$$\mathfrak{A}_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{B}_k^{(n-1)} \end{pmatrix} \qquad (k = 2, \cdots, n).$$

By induction, applying again Lemma 1 and (21), we may assume

$$\mathfrak{A}_k = [e_{k1}, e_{k2}, \cdots, e_{kn}] \qquad (k = 1, \cdots, n)$$

with $e_{kl} = 0\,(k \neq l)$ and $e_{kk} = 1$; hence

$$\mathfrak{W}_1(\mathfrak{Z}_1) = \mathfrak{Z}_1. \tag{22}$$

It follows from (20) and (22) that

$$\mathfrak{W}_1(\mathfrak{Z}_2) = \mathfrak{Z}_1\mathfrak{W}^*_1(\mathfrak{Z}_1^{-1}\mathfrak{Z}_2\mathfrak{Z}_1^{-1})\mathfrak{Z}_1,$$

whence

$$z_k z_l \mathfrak{A}_{kl} = \mathfrak{Z}_1\bar{\mathfrak{A}}_{kl}\mathfrak{Z}_1 \qquad (k \neq l).$$

Consequently

$$\mathfrak{W}_1 = (a_{kl}z_{kl})$$

with real $a_{kl} = a_{lk}$ and $a_{kk} = 1$. Since the matrix $[\pm 1, \pm 1, \cdots, \pm 1]$ is unitary, we may, moreover, assume $a_{1l} \geq 0$ $(l = 2, \cdots, n)$.

The expression $|\mathfrak{W}_1|\,|\mathfrak{Z}_0|^{-1}$ is a rational function of the z_{kl} and has, by (15), the constant absolute value 1; hence it is identically constant. On the other hand, both determinants $|\mathfrak{W}_1|$ and $|\mathfrak{Z}_0|$ contain the term $z_1z_2\cdots z_n = z$ with the same coefficient 1. This proves $|\mathfrak{W}_1| = |\mathfrak{Z}_0|$. The term $(z_1z_l)^{-1}zz_{1l}^2$ $(l = 2, \cdots, n)$ has in $|\mathfrak{W}_1|$ the coefficient $-a_{1l}^2$ and in $|\mathfrak{Z}_0|$ the coefficient -1, hence $a_{1l} = 1$. The term $(z_1z_kz_l)^{-1}zz_{1k}z_{1l}z_{kl}$ $(1 < k < l)$ has then in $|\mathfrak{W}_1|$ the coefficient $2a_{kl}$ and in $|\mathfrak{Z}_0|$ the coefficient 2, hence $a_{kl} = 1$ and $\mathfrak{W}_1 = \mathfrak{Z}_0$.

9. It remains to prove Lemma 1. There exists a unitary matrix $\mathfrak{U}_1$ such that

$$\mathfrak{Z}\bar{\mathfrak{Z}} = \mathfrak{P}^2\{\mathfrak{U}_1\}.$$

Then the matrix $\mathfrak{Z}[\mathfrak{U}_1^{-1}] = \mathfrak{F}$ is symmetric and $\mathfrak{F}\bar{\mathfrak{F}} = \mathfrak{P}^2$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the real and imaginary parts of $\mathfrak{F} = \mathfrak{F}_1 + i\mathfrak{F}_2$. Since $\mathfrak{P}$ is real, we obtain $\mathfrak{F}_1\mathfrak{F}_2 = \mathfrak{F}_2\mathfrak{F}_1$; consequently the two real symmetric matrices $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are permutable. This proves the existence of a real orthogonal matrix $\mathfrak{O}$ such that $\mathfrak{F}_1[\mathfrak{O}]$ and $\mathfrak{F}_2[\mathfrak{O}]$ are both diagonal matrices. Hence $\mathfrak{F}[\mathfrak{O}] = \mathfrak{R}$ is also a diagonal matrix $[r_1, \cdots, r_n]$ and $\mathfrak{R}\bar{\mathfrak{R}} = \mathfrak{P}^2[\mathfrak{O}]$. The numbers $r_k\bar{r}_k$ $(k = 1, \cdots, n)$ are therefore a permutation of $q_1, \cdots, q_n$, and we may obviously assume $r_k\bar{r}_k = q_k$.

The diagonal matrix $\mathfrak{U}_2 = [s_1, \cdots, s_n]$ with $s_k = r_k^{1/2} q_k^{-(1/4)}$ $(k = 1, \cdots, n)$ is unitary and $\mathfrak{P}[\mathfrak{U}_2] = \mathfrak{R}$. Defining $\mathfrak{U} = \mathfrak{U}_2\mathfrak{O}'\mathfrak{U}_1$, we have $\mathfrak{P}[\mathfrak{U}] = \mathfrak{R}[\mathfrak{O}'\mathfrak{U}_1] = \mathfrak{F}[\mathfrak{U}_1] = \mathfrak{Z}$; q. e. d.

10. Consider the symplectic mappings in the case $n = 1$, i. e.,

$$w = \frac{az + b}{cz + d} \tag{23}$$

with real a, b, c, d and $ad - bc = 1$. It is well-known that there exists a transformation (23) mapping two given points z, z_1 of the upper half-plane into two other given points w, w_1 of the upper half-plane, if and only if $R(z, z_1) = R(w, w_1)$, where $R(z, z_1)$ denotes the cross-ratio $\dfrac{z - z_1}{z - \bar{z}_1} \dfrac{\bar{z} - \bar{z}_1}{\bar{z} - z_1}$. Theorem 2 is the generalization to the case of an arbitrary n.

Let $\mathfrak{Z}, \mathfrak{Z}_1$ be two points of H and $\mathfrak{W}, \mathfrak{W}_1$ their images under the symplectic mapping $\mathfrak{W} = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ with the matrix $\mathfrak{M}$. We have

$$\begin{aligned} \mathfrak{Z}_1 - \mathfrak{Z} = (\mathfrak{Z}_1\mathfrak{E})\mathfrak{J}\begin{pmatrix}\mathfrak{Z}\\ \mathfrak{E}\end{pmatrix} &= (\mathfrak{Z}_1\mathfrak{E})\mathfrak{M}'\mathfrak{J}\mathfrak{M}\begin{pmatrix}\mathfrak{Z}\\ \mathfrak{E}\end{pmatrix} \\ &= (\mathfrak{C}\mathfrak{Z}_1 + \mathfrak{D})'(\mathfrak{W}_1 - \mathfrak{W})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}), \end{aligned} \tag{24}$$

$$\bar{\mathfrak{Z}}_1 - \mathfrak{Z} = (\mathfrak{C}\bar{\mathfrak{Z}}_1 + \mathfrak{D})'(\bar{\mathfrak{W}}_1 - \mathfrak{W})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}). \tag{25}$$

Now $(\mathfrak{Z} - \bar{\mathfrak{Z}}_1)^{-1}$ exists, since $\mathfrak{Z} - \bar{\mathfrak{Z}}_1$ is a point of H; consequently

$$\begin{aligned} &(\mathfrak{Z} - \mathfrak{Z}_1)(\mathfrak{Z} - \bar{\mathfrak{Z}}_1)^{-1} \\ &\qquad = (\mathfrak{Z}_1\mathfrak{C}' + \mathfrak{D}')(\mathfrak{W} - \mathfrak{W}_1)(\mathfrak{W} - \bar{\mathfrak{W}}_1)^{-1}(\bar{\mathfrak{Z}}_1\mathfrak{C}' + \mathfrak{D}')^{-1}. \end{aligned}$$

Introducing the cross-ratio

$$\mathfrak{R} = \mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1) = (\mathfrak{Z} - \mathfrak{Z}_1)(\mathfrak{Z} - \bar{\mathfrak{Z}}_1)^{-1}(\bar{\mathfrak{Z}} - \bar{\mathfrak{Z}}_1)(\bar{\mathfrak{Z}} - \mathfrak{Z}_1)^{-1}$$

and putting $\mathfrak{R}^* = \mathfrak{R}(\mathfrak{W}, \mathfrak{W}_1)$, $\mathfrak{Q} = (\mathfrak{C}\mathfrak{Z}_1 + \mathfrak{D})'$, we find

$$\mathfrak{R} = \mathfrak{Q}\mathfrak{R}^*\mathfrak{Q}^{-1}. \tag{26}$$

Hence the matrices $\mathfrak{R}$ and $\mathfrak{R}^*$ have the same characteristic roots.

Choose in particular $\mathfrak{Z}_1 = i\mathfrak{E}$, $\mathfrak{Z} = i\mathfrak{T}$ with $\mathfrak{T} = [t_1, \cdots, t_n]$ and $1 \leq t_1 \leq t_2 \leq \cdots \leq t_n$. Then $\mathfrak{R} = [r_1, \cdots, r_n]$ with

$$r_k = \left(\frac{t_k - 1}{t_k + 1}\right)^2 \qquad (k = 1, \cdots, n),$$

whence $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n < 1$ and

$$t_k = \frac{1 + r_k^{\frac{1}{2}}}{1 - r_k^{\frac{1}{2}}}. \tag{27}$$

In this case, the diagonal matrix $\mathfrak{T}$ is uniquely determined by the characteristic roots $r_1, \cdots, r_n$ of $\mathfrak{R}$. In order to complete the proof of Theorem 2, we have only to prove

LEMMA 2. *Let $\mathfrak{Z}, \mathfrak{Z}_1$ be two arbitrary points of H. There exists a symplectic transformation mapping $\mathfrak{Z}, \mathfrak{Z}_1$ into $i\mathfrak{T}, i\mathfrak{E}$ with $\mathfrak{T} = [t_1, \cdots, t_n]$ and $1 \leq t_1 \leq t_2 \leq \cdots \leq t_n$.*

Since Ω is transitive, we may already assume $\mathfrak{Z}_1 = i\mathfrak{E}$. If $\mathfrak{T} = [t_1, \cdots, t_n]$ with $1 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, then $(\mathfrak{T} - \mathfrak{E})(\mathfrak{T} + \mathfrak{E})^{-1} = \mathfrak{P} = [p_1, \cdots, p_n]$ with $p_k = (t_k - 1)(t_k + 1)^{-1}$ $(k = 1, \cdots, n)$, $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n < 1$, and vice versa. By (5), (6) and the results of Section **6**, we have only to prove that there exists for every point $\mathfrak{Z}$ of E a unitary matrix $\mathfrak{U}_1$ satisfying $\bar{\mathfrak{U}}'_1 \mathfrak{Z} \mathfrak{U}_1 = \mathfrak{P} = [p_1, \cdots, p_n]$ with $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n < 1$. This follows from Lemma 1: We choose $\mathfrak{U}_1 = \mathfrak{U}^{-1}$ and $p_k = q_k^{\frac{1}{2}}$; since $q_1, \cdots, q_n$ are the characteristic roots of the hermitian matrix $\mathfrak{Z}\bar{\mathfrak{Z}}$ and $\mathfrak{E} - \mathfrak{Z}\bar{\mathfrak{Z}} > 0$, we may assume $0 \leq q_1 \leq q_2 \leq \cdots \leq q_n < 1$, and $p_1, \cdots, p_n$ have the required property.

On account of the symplectic invariance of the characteristic roots $r_1, \cdots, r_n$ of $\mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)$, the diagonal elements t_k of $\mathfrak{T}$ are given by (27). This proves that those characteristic roots are always real numbers of the interval $0 \leq r < 1$.

III. THE SYMPLECTIC METRIC.

11. We consider the cross-ratio

$$\mathfrak{R} = (\mathfrak{Z} - \mathfrak{Z}_1)(\mathfrak{Z} - \bar{\mathfrak{Z}}_1)^{-1}(\bar{\mathfrak{Z}} - \bar{\mathfrak{Z}}_1)(\bar{\mathfrak{Z}} - \mathfrak{Z}_1)^{-1}$$

as a function of $\mathfrak{Z}_1$, for any given $\mathfrak{Z}$ in H. Since the two factors $\mathfrak{Z} - \mathfrak{Z}_1$ and $\bar{\mathfrak{Z}} - \bar{\mathfrak{Z}}_1$ vanish for $\mathfrak{Z}_1 = \mathfrak{Z}$, the second differential of $\mathfrak{R}$, at the point $\mathfrak{Z}_1 = \mathfrak{Z}$, has the value

$$d^2\mathfrak{R} = 2d\mathfrak{Z}(\mathfrak{Z} - \bar{\mathfrak{Z}})^{-1}d\bar{\mathfrak{Z}}(\bar{\mathfrak{Z}} - \mathfrak{Z})^{-1} = \tfrac{1}{2}d\mathfrak{Z}\mathfrak{Y}^{-1}d\bar{\mathfrak{Z}}\mathfrak{Y}^{-1},$$

where $\mathfrak{Y}$ denotes the imaginary part of $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$. On the other hand, by (26), the trace $\sigma(\mathfrak{R})$ of $\mathfrak{R} = \mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)$ is invariant under any cogredient symplectic transformation of the points $\mathfrak{Z}, \mathfrak{Z}_1$. Moreover $d^2\sigma(\mathfrak{R}) = \sigma(d^2\mathfrak{R})$, and consequently the hermitian differential form

$$ds^2 = \sigma(\mathfrak{Y}^{-1}d\mathfrak{Z}\mathfrak{Y}^{-1}d\bar{\mathfrak{Z}}) \tag{28}$$

is invariant under Ω. Introducing $\mathfrak{X} = (x_{kl})$ and $\mathfrak{Y} = (y_{kl})$, we obtain

$$ds^2 = \sigma(\mathfrak{Y}^{-1}d\mathfrak{X}\mathfrak{Y}^{-1}d\mathfrak{X} + \mathfrak{Y}^{-1}d\mathfrak{Y}\mathfrak{Y}^{-1}d\mathfrak{Y}) \tag{29}$$

and in particular, for $\mathfrak{Z} = i\mathfrak{E}$,

$$ds^2 = \sum_{k=1}^{n} (dx_{kk}^2 + dy_{kk}^2) + 2\sum_{k<l} (dx_{kl}^2 + dy_{kl}^2).$$

Since Ω is transitive in H, the quadratic differential form ds^2 is obviously positive definite everywhere in H.

Let us determine the most general quadratic differential invariant Q of the symplectic group. On account of the transitivity of Ω, we have only to find Q at the point $\mathfrak{Z} = i\mathfrak{E}$ of H or, if we use the variable $\mathfrak{Z}_0 = (\mathfrak{Z} - i\mathfrak{E})(\mathfrak{Z} + i\mathfrak{E})^{-1}$ already defined in (6), at the point $\mathfrak{Z}_0 = 0$ of E. Then Q becomes a quadratic form of the elements of $\mathfrak{S} = d\mathfrak{Z}_0$ and $\bar{\mathfrak{S}} = d\bar{\mathfrak{Z}}_0$ which is, by the result of Section **6**, invariant under all transformations $\mathfrak{S} \to \mathfrak{U}'\mathfrak{S}\mathfrak{U}$ with unitary $\mathfrak{U}$. By Lemma 1, there exists for any complex symmetric $\mathfrak{S}$ a unitary matrix $\mathfrak{U}$, such that $\mathfrak{U}'\mathfrak{S}\mathfrak{U} = \mathfrak{P} = [p_1, \cdots, p_n]$ with real p_k $(k = 1, \cdots, n)$, where the p_k^2 are the characteristic roots of $\mathfrak{S}\bar{\mathfrak{S}}$. This proves that Q is a quadratic function of $p_1, \cdots, p_n$ alone. Let $k_1, \cdots, k_n$ be a permutation of the numbers $1, \cdots, n$ and ϵ_l $(l = 1, \cdots, n)$ a fourth root of unity; then the matrix $\mathfrak{U}_0$ of the substitution $s_{k_l} \to \epsilon_l s_l$ $(l = 1, \cdots, n)$ is unitary and $\mathfrak{U}'_0\mathfrak{P}\mathfrak{U}_0 = [q_1, \cdots, q_n]$ with $q_l = \epsilon_l^2 p_{k_l} = \pm p_{k_l}$ $(l = 1, \cdots, n)$. Hence Q is a symmetric polynomial in $p_1^2, \cdots, p_n^2$,

$$Q = \lambda \sum_{k=1}^{n} p_k^2 = \lambda\sigma(\mathfrak{S}\bar{\mathfrak{S}})$$

with constant λ. Consequently any quadratic differential invariant of the symplectic group is a constant multiple of ds^2.

12. We are now interested in the properties of the geodesics for the symplectic metric (28). In order to find the shortest arc connecting two arbitrarily given points $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ of H, we have, by Lemma 2, only to investigate the special case $\mathfrak{Z}_1 = i\mathfrak{E}$, $\mathfrak{Z}_2 = i\mathfrak{T} = i[t_1, \cdots, t_n]$, $1 \leq t_1 \leq t_2 \leq \cdots \leq t_n$; moreover, we may obviously assume $\mathfrak{Z}_1 \neq \mathfrak{Z}_2$, i. e., $t_n > 1$. Let now $\mathfrak{Z} = \mathfrak{Z}(u)$ be any curve connecting these two points in H and having a piecewise continuous tangent, $\mathfrak{Z}(0) = i\mathfrak{E}$, $\mathfrak{Z}(1) = i\mathfrak{T}$. We may put

$$\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Q}[\mathfrak{D}],$$

where $\mathfrak{Q} = [q_1, \cdots, q_n]$ with $q_k > 0$ $(k = 1, \cdots, n)$ and $\mathfrak{D}$ denotes a real orthogonal matrix; moreover $\mathfrak{X}$, $\mathfrak{Q}$, $\mathfrak{D}$ have again piecewise continuous derivatives; $\mathfrak{Q} = \mathfrak{E}$, $\mathfrak{D} = \mathfrak{E}$, $\mathfrak{X} = 0$ for $u = 0$; $\mathfrak{Q} = \mathfrak{T}$, $\mathfrak{D} = \mathfrak{E}$, $\mathfrak{X} = 0$ for $u = 1$. This arc has the length

$$s = \int_0^1 \sigma^{\frac{1}{2}}(\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\bar{\mathfrak{Z}}})\,du,$$

where the dot denotes differentiation with respect to u.

By (29), we have $s \geqq s_1$, where s_1 denotes the length of the curve

$$\mathfrak{Z} = i\mathfrak{Q}[\mathfrak{D}] \tag{30}$$

also connecting $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$, and $s > s_1$, if both curves do not coincide.

We use the abbreviation $\dot{\mathfrak{D}}\mathfrak{D}' = \mathfrak{F} = (f_{kl})$. Since $\mathfrak{D}\mathfrak{D}' = \mathfrak{E}$, we have $\dot{\mathfrak{D}}\mathfrak{D}' = -\mathfrak{D}\dot{\mathfrak{D}}'$, and consequently $\mathfrak{F}$ is skew-symmetric. Differentiating the equation $\mathfrak{D}\mathfrak{Y}\mathfrak{D}' = \mathfrak{Q}$, we obtain

$$\begin{aligned}
\mathfrak{D}\dot{\mathfrak{Y}}\mathfrak{D}' &= \dot{\mathfrak{Q}} - \mathfrak{F}\mathfrak{Q} + \mathfrak{Q}\mathfrak{F}\\
\mathfrak{D}\mathfrak{Y}^{-1}\dot{\mathfrak{Y}}\mathfrak{Y}^{-1}\dot{\mathfrak{Y}}\mathfrak{D}' &= \mathfrak{Q}^{-1}(\dot{\mathfrak{Q}} - \mathfrak{F}\mathfrak{Q} + \mathfrak{Q}\mathfrak{F})\mathfrak{Q}^{-1}(\dot{\mathfrak{Q}} - \mathfrak{F}\mathfrak{Q} + \mathfrak{Q}\mathfrak{F})\\
&= \mathfrak{F}^2 + \mathfrak{Q}^{-1}\mathfrak{F}^2\mathfrak{Q} + \mathfrak{Q}^{-1}\dot{\mathfrak{Q}}\mathfrak{Q}^{-1}\dot{\mathfrak{Q}} - \mathfrak{Q}^{-1}\mathfrak{F}\mathfrak{Q}\mathfrak{F} - \mathfrak{F}\mathfrak{Q}^{-1}\mathfrak{F}\mathfrak{Q}\\
&\quad - \mathfrak{Q}^{-1}\dot{\mathfrak{Q}}\mathfrak{Q}^{-1}\mathfrak{F}\mathfrak{Q} - \mathfrak{Q}^{-1}\mathfrak{F}\dot{\mathfrak{Q}} + \mathfrak{Q}^{-1}\dot{\mathfrak{Q}}\mathfrak{F} + \mathfrak{F}\mathfrak{Q}^{-1}\dot{\mathfrak{Q}}\\
\sigma(\mathfrak{Y}^{-1}\dot{\mathfrak{Y}}\mathfrak{Y}^{-1}\dot{\mathfrak{Y}}) &= 2\sigma(\mathfrak{F}^2) - 2\sigma(\mathfrak{F}\mathfrak{Q}\mathfrak{F}\mathfrak{Q}^{-1}) + \sigma(\mathfrak{Q}^{-1}\dot{\mathfrak{Q}}\mathfrak{Q}^{-1}\dot{\mathfrak{Q}})\\
&= \sum_{k,l=1}^{n} f_{kl}{}^2 \frac{(q_k - q_l)^2}{q_k q_l} + \sum_{k=1}^{n} \left(\frac{\dot{q}_k}{q_k}\right)^2. \qquad (31)
\end{aligned}$$

On the other hand, the formula

$$\sum_{k=1}^{n} Q_k{}^2 = (c_1Q_1 + \cdots + c_nQ_m)^2 + \tfrac{1}{2}\sum_{k,l=1}^{n}(c_kQ_l - c_lQ_k)^2 \tag{32}$$

holds for $c_1{}^2 + c_2{}^2 + \cdots + c_n{}^2 = 1$ and in particular with

$$Q_k = \frac{\dot{q}_k}{q_k}, \qquad c_k = \rho^{-1}\log t_k \qquad (k = 1, \cdots, n),$$

where

$$\rho = (\sum_{k=1}^{n} \log^2 t_k)^{\frac{1}{2}} > 0. \tag{33}$$

By (31) and (32),

$$s_1 \geqq \int_0^1 (c_1Q_1 + \cdots + c_nQ_n)\,du = \sum_{k=1}^{n} c_k \log t_k = \rho,$$

with the sign of inequality, if not all the three conditions

$$(34)\qquad f_{kl}(q_k - q_l) = 0 \qquad (k, l = 1, \cdots, n),$$

$$(35)\qquad c_k Q_l - c_l Q_k = 0 \qquad (k, l = 1, \cdots, n),$$

$$(36)\qquad \sum_{k=1}^{n} c_k Q_k \geq 0$$

are fulfilled. By (35),

$$\log t_n \ \log q_k = \log t_k \ \log q_n \qquad (k = 1, \cdots, n),$$

whence

$$(37)\qquad q_k = t_k^{\gamma} \qquad (k = 1, \cdots, n)$$

with $\gamma = \gamma(u)$, $\gamma(0) = 0$, $\gamma(1) = 1$. By (36), the function $\gamma(u)$ is monotone. By (34) and (37),

$$f_{kl}(t_k - t_l) = 0 \qquad (k, l = 1, \cdots, n)$$

$$\mathfrak{D}\dot{\mathfrak{D}}'\mathfrak{T} + \mathfrak{T}\dot{\mathfrak{D}}\mathfrak{D}' = 0$$

$$(\mathfrak{D}'\mathfrak{T}\mathfrak{D})^{\bullet} = \dot{\mathfrak{D}}'\mathfrak{T}\mathfrak{D} + \mathfrak{D}'\mathfrak{T}\dot{\mathfrak{D}} = 0$$

$$\mathfrak{D}'\mathfrak{T}\mathfrak{D} = \mathfrak{T};$$

consequently $\mathfrak{T}\mathfrak{D} = \mathfrak{D}\mathfrak{T}$ and, by (37), also $\mathfrak{Q}\mathfrak{D} = \mathfrak{D}\mathfrak{Q}$. This proves, by (30), that the minimum ρ of s is attained, if and only if $\mathfrak{Z} = i[t_1{}^{\gamma}, \cdots, t_n{}^{\gamma}]$, where $\gamma(u)$ is a monotone function with $\gamma(0) = 0$, $\gamma(1) = 1$. We may replace $\gamma(u)$ by u and obtain the curve $\mathfrak{Z} = i[t_1{}^{u}, \cdots, t_n{}^{u}]$ as the unique solution. Introducing the length of arc $\tau = \rho u$, we have $\mathfrak{Z} = i[e^{c_1\tau}, \cdots, e^{c_n\tau}]$ with $c_k = \rho^{-1} \log t_k$ $(k = 1, \cdots, n)$ and $c_1{}^2 + \cdots + c_n{}^2 = 1$.

13. Let $\mathfrak{Z}$ and $\mathfrak{Z}_1$ be again two arbitrary points of H. By (33) and the results of Section **10**, the symplectic distance $\rho = \rho(\mathfrak{Z}, \mathfrak{Z}_1)$ is given by

$$(38)\qquad \rho^2 = \sum_{k=1}^{n} \log^2 \frac{1 + r_k^{\frac{1}{2}}}{1 - r_k^{\frac{1}{2}}},$$

where $r_1, \cdots, r_n$ denote the characteristic roots of the cross-ratio $\mathfrak{R} = \mathfrak{R}(\mathfrak{Z}, \mathfrak{Z}_1)$. Since

$$\log^2 \frac{1 + r^{\frac{1}{2}}}{1 - r^{\frac{1}{2}}} = 4r \left(\sum_{k=0}^{\infty} \frac{r^k}{2k + 1} \right)^2 \qquad (0 \leq r < 1)$$

and

$$\sum_{k=1}^{n} r_k{}^l = \sigma(\mathfrak{R}^l) \qquad (l = 1, 2, \cdots),$$

we may write

$$\rho^2 = \sigma \left(\log^2 \frac{1 + \mathfrak{R}^{\frac{1}{2}}}{1 - \mathfrak{R}^{\frac{1}{2}}} \right)$$

with

$$\log^2 \frac{1+\mathfrak{R}^{\frac{1}{2}}}{1-\mathfrak{R}^{\frac{1}{2}}} = 4\mathfrak{R}\left(\sum_{k=0}^{\infty} \frac{\mathfrak{R}^k}{2k+1}\right)^2 .$$

By the results of this and the preceding section, Theorem 3 and Theorem 4 are completely proved.

14. In order to calculate the differential equation of the geodesics and the tensor of curvature, we determine the first variation δs. We consider any curve $\mathfrak{Z} = \mathfrak{Z}(s)$ $(0 \leq s \leq s_0)$, where the parameter s denotes the length of arc. Then $\sigma(\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\bar{\mathfrak{Z}}}) = 1$ and

$$\delta s = \tfrac{1}{2} \int_0^{s_0} \delta\sigma(\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\bar{\mathfrak{Z}}})\, ds.$$

Using the abbreviations $\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\bar{\mathfrak{Z}}} = \mathfrak{V}$, $\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1} = \mathfrak{W}$ and denoting by R the real part, we find

$$\delta\sigma(\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\bar{\mathfrak{Z}}}) = 2R\sigma(\mathfrak{V}\delta\mathfrak{Y}^{-1} + (\mathfrak{W}\delta\bar{\mathfrak{Z}})^{\cdot} - \dot{\mathfrak{W}}\delta\bar{\mathfrak{Z}})$$

$$\delta\mathfrak{Y}^{-1} = \frac{i}{2}\,\mathfrak{Y}^{-1}(\delta\mathfrak{Z} - \delta\bar{\mathfrak{Z}})\mathfrak{Y}^{-1}$$

$$\dot{\mathfrak{W}} = \mathfrak{Y}^{-1}\ddot{\mathfrak{Z}}\mathfrak{Y}^{-1} + i\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\mathfrak{Z}}\mathfrak{Y}^{-1} - \frac{i}{2}\,\mathfrak{Y}^{-1}(\mathfrak{V} + \bar{\mathfrak{V}})\mathfrak{Y}^{-1},$$

whence

$$\delta s = -R \int_0^{s_0} \sigma(\mathfrak{Y}^{-1}(\ddot{\mathfrak{Z}} + i\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\mathfrak{Z}})\mathfrak{Y}^{-1}\delta\bar{\mathfrak{Z}})\, ds.$$

Consequently

$$\ddot{\mathfrak{Z}} = -i\dot{\mathfrak{Z}}\mathfrak{Y}^{-1}\dot{\mathfrak{Z}} \tag{39}$$

is the differential equation of the geodesic lines.

It is easy to perform the integration without using Section **12.** On account of the symplectic invariance of the geodesics and the transitivity of Ω, it is sufficient to integrate (39) for the initial point $\mathfrak{Z} = i\mathfrak{E}$ and an arbitrary direction $\dot{\mathfrak{Z}} = \mathfrak{K}$ through this point; obviously $\sigma(\mathfrak{K}\bar{\mathfrak{K}}) = 1$. By Section **6,** the mapping

$$\frac{\mathfrak{Z} - i\mathfrak{E}}{\mathfrak{Z} + i\mathfrak{E}} \rightarrow \mathfrak{U}' \frac{\mathfrak{Z} - i\mathfrak{E}}{\mathfrak{Z} + i\mathfrak{E}}\, \mathfrak{U} \tag{40}$$

is symplectic for arbitrary unitary $\mathfrak{U}$. Under this mapping, the direction $\mathfrak{K}$ through $i\mathfrak{E}$ is replaced by $\mathfrak{U}'\mathfrak{K}\mathfrak{U}$. By Lemma 1, we can determine $\mathfrak{U}$ such that $\mathfrak{U}'\mathfrak{K}\mathfrak{U} = i\mathfrak{G} = i[g_1, \cdots, g_n]$ with $0 \leq g_1 \leq \cdots \leq g_n$; since $\sigma(\mathfrak{K}\bar{\mathfrak{K}}) = 1$, we have $g_1{}^2 + \cdots + g_n{}^2 = 1$. It is now sufficient to integrate (39) for the initial conditions $\mathfrak{Z} = i\mathfrak{E}$, $\dot{\mathfrak{Z}} = i\mathfrak{G}$. Obviously the solution is $\mathfrak{Z} = i \exp s\mathfrak{G}$. Consequently the most general geodesic line is

$$(\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} = i[e^{g_1 s}, \cdots, e^{g_n s}],$$

where the left-hand side is an arbitrary symplectic transformation of $\mathfrak{Z}$ and $g_1, \cdots, g_n$ are arbitrary real numbers with $0 \leq g_1 \leq \cdots \leq g_n$, $g_1^2 + \cdots + g_n^2 = 1$.

15. We shall now establish directly, without using Section **12**, that there exists exactly one geodesic through two arbitrarily given points $\mathfrak{Z}_1$ and $\mathfrak{Z}_2 \neq \mathfrak{Z}_1$ of H. We may again assume $\mathfrak{Z}_1 = i\mathfrak{E}$, $\mathfrak{Z}_2 = i\mathfrak{T} = i[t_1, \cdots, t_n]$, $1 \leq t_1 \leq \cdots \leq t_n$,

$$\rho = \Big(\sum_{k=1}^{n} \log^2 t_k\Big)^{\frac{1}{2}} > 0.$$

Defining $g_k = \rho^{-1} \log t_k$ $(k = 1, \cdots, n)$, we obtain the geodesic line

$$\mathfrak{Z} = i \exp s\mathfrak{G}. \tag{41}$$

Since (40) is the most general symplectic mapping with the fixed point $i\mathfrak{E}$, any geodesic through this point has an equation

$$\frac{\mathfrak{Z} - i\mathfrak{E}}{\mathfrak{Z} + i\mathfrak{E}} = \mathfrak{U}' \frac{\exp s\mathfrak{H} - \mathfrak{E}}{\exp s\mathfrak{H} + \mathfrak{E}} \mathfrak{U} \tag{42}$$

with unitary $\mathfrak{U}$ and $\mathfrak{H} = [h_1, \cdots, h_n]$, $0 \leq h_1 \leq \cdots \leq h_n$, $h_1^2 + \cdots + h_n^2 = 1$. If this curve goes also through the point $\mathfrak{Z} = i\mathfrak{T}$ of (41), we obtain

$$\mathfrak{U}\left(\frac{\mathfrak{E} - \mathfrak{T}}{\mathfrak{E} + \mathfrak{T}}\right)^2 = \left(\frac{\mathfrak{E} - \exp s_0\mathfrak{H}}{\mathfrak{E} + \exp s_0\mathfrak{H}}\right)^2 \mathfrak{U}$$

for a certain $s_0 > 0$, whence $\mathfrak{T} = \exp s_0\mathfrak{H}$, $s_0\mathfrak{H} = \rho\mathfrak{G}$, $s_0 = \rho$, $\mathfrak{H} = \mathfrak{G}$. Moreover $\mathfrak{U}$ and $(\exp s_0\mathfrak{H} - \mathfrak{E})(\exp s_0\mathfrak{H} + \mathfrak{E})^{-1}$ are permutable, hence also $\mathfrak{U}$ and $(\exp s\mathfrak{H} - \mathfrak{E})(\exp s\mathfrak{H} + \mathfrak{E})^{-1}$. Putting $s = \rho$ in (42), we find $\mathfrak{U}'\mathfrak{U} = \mathfrak{E}$, and consequently the geodesics (41) and (42) coincide.

This result proves again, by a general theorem from the calculus of variations, that there exists exactly one shortest arc between two arbitrarily given points of H.

16. By the minimum property of the geodesic arc, the symplectic distance $\rho(\mathfrak{Z}_1, \mathfrak{Z}_2)$ satisfies the triangle inequality

$$\rho(\mathfrak{Z}_1, \mathfrak{Z}_3) \leq \rho(\mathfrak{Z}_1, \mathfrak{Z}_2) + \rho(\mathfrak{Z}_2, \mathfrak{Z}_3)$$

for three arbitrary points $\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3$ of H, and the sign of equality is true, if and only if $\mathfrak{Z}_2$ is a point on the uniquely determined geodesic arc between $\mathfrak{Z}_1$

and $\mathfrak{Z}_3$. Obviously $\rho(\mathfrak{Z}, \mathfrak{Z}_1)$ is a continuous function of $\mathfrak{Z}$. If G is any compact point set in H, then $\rho(\mathfrak{Z}, \mathfrak{Z}_1) \leq c$ for all $\mathfrak{Z}$ in G, where c is a positive constant depending only upon $\mathfrak{Z}_1$ and G. Let us prove that also the converse statement is true: If c is an arbitrary positive constant and $\mathfrak{Z}_1$ any given point of H, then the inequality $\rho(\mathfrak{Z}, \mathfrak{Z}_1) \leq c$ defines a compact set G of points $\mathfrak{Z}$ in H. It is sufficient to prove this in the special case $\mathfrak{Z}_1 = i\mathfrak{E}$. By the definition of the cross-ratio,

$$\mathfrak{R}(\mathfrak{Z}, i\mathfrak{E}) = \mathfrak{Z}_0\bar{\mathfrak{Z}}_0,$$

where $\mathfrak{Z}_0 = (\mathfrak{Z} - i\mathfrak{E})(\mathfrak{Z} + i\mathfrak{E})^{-1}$ is the image of $\mathfrak{Z}$ under the transformation (6) mapping H onto E. We infer from (38) that the characteristic roots r_k of the hermitian matrix $\mathfrak{Z}_0\bar{\mathfrak{Z}}_0$ satisfy an inequality

$$0 \leq r_k \leq \vartheta < 1 \qquad (k = 1, \cdots, n)$$

for all $\mathfrak{Z}$ with $\rho(\mathfrak{Z}, \mathfrak{Z}_1) \leq c$, where ϑ depends only upon c. Since an arbitrary symmetric matrix $\mathfrak{Z}_0$ is a point of E, if the characteristic roots of $\mathfrak{Z}_0\bar{\mathfrak{Z}}_0$ are < 1, all limit points of the images $\mathfrak{Z}_0$ of the points $\mathfrak{Z}$ in G belong to E again, and consequently G is compact.

17. Let

$$ds^2 = \sum_{k,l=1}^{m} g_{kl}\,dx_k\,dx_l \tag{43}$$

be a Riemann metric in an m-dimensional space and let

$$\ddot{x}_k = -\sum_{p,q=1}^{m} \{pq, k\}\dot{x}_p\dot{x}_q \qquad (k = 1, \cdots, m)$$

be the differential equations of the geodesics, so that $\{pq, k\}$ denotes the Christoffel symbol of the second kind. The Riemann tensor of curvature R is obtained in the following way: Define two covariant differentials $\delta_1 u_k$ and $\delta_2 u_k$ by

$$\delta_r u_k = -\sum_{p,q} \{pq, k\} u_p \delta_r x_q \qquad (r = 1, 2); \tag{44}$$

then

$$R = \sum_{k,l} g_{kl} v_k (\delta_1\delta_2 - \delta_2\delta_1) u_l = \sum_{k,l,p,q} R_{klpq} u_k v_l \delta_1 x_p \delta_2 x_q, \tag{45}$$

where u_k, v_l, $\delta_1 x_p$, $\delta_2 x_q$ are the components of 4 covariant vectors.

In the case of the symplectic metric (28), the differential equations of the geodesics are given by (39). Instead of (44), we may write now

$$2i\delta_r\mathfrak{U} = \mathfrak{U}\mathfrak{Y}^{-1}\delta_r\mathfrak{Z} + \delta_r\mathfrak{Z}\mathfrak{Y}^{-1}\mathfrak{U} \qquad (r = 1, 2)$$

with complex symmetric $\mathfrak{U} = (u_{kl})$, and we obtain

$$-4\delta_1\delta_2\mathfrak{U} = (\mathfrak{U}\mathfrak{Y}^{-1}\delta_1\bar{\mathfrak{Z}} + \delta_1\mathfrak{Z}\mathfrak{Y}^{-1}\mathfrak{U})\mathfrak{Y}^{-1}\delta_2\mathfrak{Z} + \delta_2\mathfrak{Z}\mathfrak{Y}^{-1}(\mathfrak{U}\mathfrak{Y}^{-1}\delta_1\mathfrak{Z} + \delta_1\bar{\mathfrak{Z}}\mathfrak{Y}^{-1}\mathfrak{U})$$

$$4(\delta_1\delta_2 - \delta_2\delta_1)\mathfrak{U} = \mathfrak{F}\mathfrak{Y}^{-1}\mathfrak{U} + \mathfrak{U}\mathfrak{Y}^{-1}\mathfrak{F}', \tag{46}$$

where

$$\mathfrak{F} = \delta_1\mathfrak{Z}\mathfrak{Y}^{-1}\delta_2\bar{\mathfrak{Z}} - \delta_2\mathfrak{Z}\mathfrak{Y}^{-1}\delta_1\bar{\mathfrak{Z}}. \tag{47}$$

Introducing

$$\mathfrak{G} = \mathfrak{U}\mathfrak{Y}^{-1}\bar{\mathfrak{V}} - \mathfrak{V}\mathfrak{Y}^{-1}\bar{\mathfrak{U}} \tag{48}$$

with complex symmetric $\mathfrak{V} = (v_{kl})$, we find, by (28), (45) and (46),

$$\begin{aligned} 4R &= \sigma(\mathfrak{Y}^{-1}\mathfrak{V}\bar{\mathfrak{F}}\mathfrak{Y}^{-1}\bar{\mathfrak{U}} + \mathfrak{Y}^{-1}\mathfrak{F}\mathfrak{Y}^{-1}\mathfrak{U}\mathfrak{Y}^{-1}\bar{\mathfrak{V}}) \\ &= \sigma(\mathfrak{Y}^{-1}\mathfrak{F}\mathfrak{Y}^{-1}\mathfrak{G}) = -\sigma(\mathfrak{Y}^{-1}\mathfrak{F}\mathfrak{Y}^{-1}\bar{\mathfrak{G}}'). \end{aligned} \tag{49}$$

In order to determine the Gaussian curvature, we have to take two arbitrary different directions $\delta_1\mathfrak{Z}$ and $\delta_2\mathfrak{Z}$ at the point $\mathfrak{Z}$ and to choose $\mathfrak{U} = \delta_1\mathfrak{Z}$, $\mathfrak{V} = \delta_2\mathfrak{Z}$. Then

$$R = -\tfrac{1}{4}\sigma(\mathfrak{Y}^{-1}\mathfrak{F}\mathfrak{Y}^{-1}\bar{\mathfrak{F}}') \leq 0,$$

where the sign of equality is true only for $\mathfrak{F} = 0$. Consequently the curvature is negative for $\delta_1\mathfrak{Z}\mathfrak{Y}^{-1}\delta_2\bar{\mathfrak{Z}} \neq \delta_2\mathfrak{Z}\mathfrak{Y}^{-1}\delta_1\bar{\mathfrak{Z}}$, and 0 otherwise.

It may easily be seen, that the contracted tensor of curvature is $\frac{n+1}{2}\,ds^2$; hence we have an Einstein metric with cosmological term.

18. Allendoerfer and Fenchel proved independently the following generalization of the Gauss-Bonnet formula concerning the curvatura integra of a closed two-dimensional surface. Let F be a closed manifold with the Riemann metric (43) and an even number m of dimensions. For every permutation $k_1, \cdots, k_m$ of the numbers $1, \cdots, m$ we define $\epsilon_{k_1\ldots k_m} = 1$, if the permutation is even, and $= -1$, if the permutation is odd. Let g be the determinant $|g_{kl}|$ and

$$K = (2^{m/2} g m!)^{-1} \Sigma R_{k_1k_2l_1l_2} R_{k_3k_4l_3l_4} \cdots R_{k_{m-1}k_m l_{m-1}l_m} \epsilon_{k_1k_2\ldots k_{m-1}k_m} \epsilon_{l_1l_2\ldots l_{m-1}l_m}, \tag{50}$$

where the summation is extended over all permutations $k_1, \cdots, k_m$ and $l_1, \cdots, l_m$ of $1, \cdots, m$; moreover let $d\omega$ be the volume element in the given metric. Then the Euler characteristic of F has the value

$$\chi = \pi^{-(m+1)/2}\,\Gamma\left(\frac{m+1}{2}\right)\int_F K\,d\omega. \tag{51}$$

For practical purposes, the sum on the right-hand side of (50) may be calculated in the following manner: Determine the single terms of the polynomial

$$R^{m/2} = (\Sigma R_{klpq} u_k v_l u^*_p v^*_q)^{m/2}$$

and replace every product $u_{k_1} v_{k_2} \cdots u_{k_{m-1}} v_{k_m} u^*_{l_1} v^*_{l_2} \cdots u^*_{l_{m-1}} v^*_{l_m}$ by $\epsilon_{k_1 \ldots k_m} \epsilon_{l_1 \ldots l_m}$.

In the case of the symplectic metric, the transitive group Ω of isometric mappings exists, and consequently the invariant K is constant in the whole space H. In order to find this constant value, we may assume $\mathfrak{Z} = i\mathfrak{E}$. Writing $\mathfrak{U}^*, \mathfrak{V}^*$ instead of $\delta_1 \mathfrak{Z}, \delta_2 \mathfrak{Z}$, we obtain, by (47), (48) and (49),

$$R = \tfrac{1}{4} \sigma(\mathfrak{Q}\mathfrak{Q}^*)$$

with

$$\mathfrak{Q} = \mathfrak{U}\bar{\mathfrak{V}} - \mathfrak{V}\bar{\mathfrak{U}}, \qquad \mathfrak{Q}^* = \mathfrak{U}^*\bar{\mathfrak{V}}^* - \mathfrak{V}^*\bar{\mathfrak{U}}^*,$$

where $\mathfrak{U}, \mathfrak{V}, \mathfrak{U}^*, \mathfrak{V}^*$ are indeterminate symmetric complex matrices. Hence

$$R^\nu = 2^{-2\nu} \sum q_{k_1 l_1} \cdots q_{k_\nu l_\nu} q^*_{k_1 l_1} \cdots q^*_{k_\nu l_\nu} \qquad (\nu = 1, 2, \cdots) \tag{52}$$

with

$$q_{kl} = \sum_{r=1}^{n} (u_{kr} \bar{v}_{rl} - v_{kr} \bar{u}_{rl}), \qquad q^*_{kl} = \sum_{r=1}^{n} (u^*_{kr} \bar{v}^*_{rl} - v^*_{kr} \bar{u}^*_{rl}),$$

where $k_1, l_1, \cdots, k_\nu, l_\nu$ run independently from 1 to n. We choose

$$\nu = \frac{m}{2} = \frac{n(n+1)}{2}.$$

Let us denote the ν elements u_{kl} $(1 \leq k \leq l \leq n)$ of $\mathfrak{U} = (u_{kl})$ in lexicographic order by $u_1, \cdots u_\nu$, their conjugates by $u_{\nu+1}, \cdots, u_m$, and introduce the corresponding notation for the elements of $\mathfrak{V}, \mathfrak{U}^*, \mathfrak{V}^*$. Replacing every term $u_{g_1} v_{g_2} \cdots u_{g_{m-1}} v_{g_m}$ in $q_{k_1 l_1} \cdots q_{k_\nu l_\nu}$ by $\epsilon_{g_1 \ldots g_m}$, we get the expression $2^\nu \eta_{k_1 l_1 \ldots k_\nu l_\nu}$ with

$$\eta_{k_1 l_1 \ldots k_\nu l_\nu} = \Sigma \epsilon_{g_1 \ldots g_m}, \tag{53}$$

where $g_1, \cdots, g_m$ run over all permutations of $1, \cdots, m$ such that the m conditions

$$u_{k_h r_h} = u_{g_{2h-1}}, \qquad \bar{v}_{l_h r_h} = v_{g_{2h}} \qquad (h = 1, \cdots, \nu)$$

have a solution $r_1, \cdots, r_\nu$. By (52), the sum of the right-hand side of (50) has, for $\mathfrak{Z} = i\mathfrak{E}$, the positive integral value

$$a_n = \Sigma \eta^2_{k_1 l_1 \ldots k_\nu l_\nu}. \tag{54}$$

Moreover g is the determinant of the quadratic form $\sigma(\mathfrak{Y}^{-1} d\mathfrak{Z} \mathfrak{Y}^{-1} d\bar{\mathfrak{Z}})$ of the variables dz_{kl} and $d\bar{z}_{kl}$ $(1 \leq k \leq l \leq n)$; hence

$$g = (-1)^\nu 2^{-2n},$$

for $\mathfrak{Z} = i\mathfrak{E}$, and consequently

$$K = (-1)^{\nu} 2^{2n-\nu} \frac{a_n}{(2\nu)\,!} \,. \tag{55}$$

On the other hand,

$$d\omega = 2^{\nu-n} \,|\, \mathfrak{Y} \,|^{n-1} \prod_{k \leq l} (dx_{kl} dy_{kl}) = 2^{\nu-n} \prod_{k \leq l} (dx_{kl} dY_{kl}) \tag{56}$$

with $(Y_{kl}) = \mathfrak{Y}^{-1}$. Since

$$\pi^{-\nu-\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2}) \, \frac{2^{2n-\nu}}{(2\nu)\,!} \, 2^{\nu-n} = (2^{n^2} \pi^{\nu} \nu\,!)^{-1},$$

we obtain, by (51), (55) and (56),

$$\chi = c_n (-\pi)^{-n(n+1)/2} \int_F dv,$$

where the positive rational number

$$c_n = \frac{a_n}{2^{n^2} [(n^2 + n)/2]\,!}$$

is defined in (53), (54) and

$$dv = \prod_{k \leq l} (dx_{kl} dY_{kl}) \tag{57}$$

denotes the euclidean volume element in the space of $\mathfrak{X}, \mathfrak{Y}^{-1}$.

We find by direct calculation $a_1 = 1$, $a_2 = 6 \cdot 3\,!$, $a_3 = 90 \cdot 6\,!$ and therefore $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{8}$, $c_3 = \frac{45}{256}$, but a simple explicit formula for a_n and c_n in the case of an arbitrary n is not known.

The proof of Theorem 5 is now accomplished.

IV. DISCONTINUOUS GROUPS.

19. A group of mappings of H onto itself is called discontinuous (in H), if for every $\mathfrak{Z}$ of H the set of images of $\mathfrak{Z}$ has no limit point in H. Since H can be covered by a countable number of compact domains, the number of elements of a discontinuous group is either finite or countably infinite. We shall assume, moreover, that the mappings are analytic. Then, by Theorem 1, they form a subgroup Δ of the symplectic group.

On the other hand, let us consider the definition of a *discrete* group. A group of matrices $\mathfrak{M}$ with real (or complex) elements is called discrete, if every infinite sequence of different $\mathfrak{M}$ diverges. It is obvious that a discontinuous group of symplectic matrices is discrete. Let us now prove the converse of this statement. If Δ is a non-discontinuous group of symplectic mappings

$$\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}, \tag{58}$$

we can find a point $\mathfrak{Z}$ of H and an infinite sequence of different matrices

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$$

in Δ, such that the corresponding sequence (58) tends to a limit $\mathfrak{Z}^*_1$ in H. Denoting by $\mathfrak{Y}, \mathfrak{Y}^*, \mathfrak{Y}^*_1$ the imaginary parts of $\mathfrak{Z}, \mathfrak{Z}^*, \mathfrak{Z}^*_1$, we have, by (25),

$$\mathfrak{Y}^* = \mathfrak{Y}\{(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}\}. \tag{59}$$

Now $\mathfrak{Y}^*$ is bounded, $\mathfrak{Y}$ is fixed and $\mathfrak{Y} > 0$. By (59), the sequence of matrices $(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ is bounded; moreover, the square of the absolute value of the determinant $|\mathfrak{C}\mathfrak{Z} + \mathfrak{D}|$ tends to the limit $|\mathfrak{Y}| \cdot |\mathfrak{Y}^*_1|^{-1}$; hence also $\mathfrak{C}\mathfrak{Z} + \mathfrak{D}$ is bounded. Since $\mathfrak{C}$ is the imaginary part of $(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})\mathfrak{Y}^{-1}$ and $\mathfrak{D} = (\mathfrak{C}\mathfrak{Z} + \mathfrak{D}) - \mathfrak{C}\mathfrak{Z}$, the matrices $\mathfrak{C}, \mathfrak{D}$ are bounded. It follows from $\mathfrak{A}\mathfrak{Z} + \mathfrak{B} = \mathfrak{Z}^*(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})$ that also $\mathfrak{A}$ and $\mathfrak{B}$ are bounded. Consequently there exists a converging subsequence of matrices $\mathfrak{M}$, and Δ is non-discrete.

20. For any point set P in H, we define the diameter $\delta(P)$ as the least upper bound of the distance $\rho(\mathfrak{Z}, \mathfrak{Z}^*)$, where $\mathfrak{Z}$ and $\mathfrak{Z}^*$ run independently over all points of P. The diameter is finite, if P is compact. The distance $\rho(P, P^*)$ of two point sets P, P^* in H is defined as the greatest lower bound of the distance $\rho(\mathfrak{Z}, \mathfrak{Z}^*)$, $\mathfrak{Z}$ running over P and $\mathfrak{Z}^*$ over P^*. By the triangle inequality,

$$\delta(P + P^*) \leq \rho(P, P^*) + \delta(P) + \delta(P^*). \tag{60}$$

Let $D_1, D_2, \cdots$ be the elements of the discontinuous group Δ and let D_1 be the identity. We denote by $D_k(P) = P_k$ the image of the set P under D_k $(k = 1, 2, \cdots)$. We assume now that P and another point set Q in H are compact. We shall prove that the distance $\rho(Q, P_k)$ tends to infinity with k. Let us first consider the case of two points $Q = \mathfrak{Z}$, $P = \mathfrak{Z}_1$. By Section **16**, the condition $\rho(\mathfrak{Z}, \mathfrak{Z}^*) \leq c$ defines, for arbitrarily given $c > 0$ and variable $\mathfrak{Z}^*$, a compact point set in H; hence the inequality $\rho(\mathfrak{Z}, \mathfrak{Z}_k) \leq c$ holds only for a finite number of indices k so that $\rho(\mathfrak{Z}, \mathfrak{Z}_k) \to \infty$ for $k \to \infty$. Consider now the general case for P and Q. By (60),

$$\rho(Q, P_k) \geq \delta(Q + P_k) - \delta(Q) - \delta(P_k).$$

Choosing a point $\mathfrak{Z}$ in Q and a point $\mathfrak{Z}_1$ in P, we have $\delta(Q + P_k) \geq \rho(\mathfrak{Z}, \mathfrak{Z}_k) \to \infty$ and $\delta(P_k) = \delta(P)$, whence $\rho(Q, P_k) \to \infty$.

A point $\mathfrak{Z}$ of H is called a fixed point of Δ, if $D_k(\mathfrak{Z}) = \mathfrak{Z}$ for at least one index $k > 1$. Let P be an arbitrary compact domain in H, e. g., the domain $\rho(\mathfrak{Z}, i\mathfrak{E}) \leq 1$. Since $\rho(P, P_k) \to \infty$ for $k \to \infty$, there exists only a finite number of values k, such that the equation $D_k(\mathfrak{Z}) = \mathfrak{Z}$ has a solution $\mathfrak{Z}$ in P. But this equation, for any $k > 1$, defines an algebraic manifold in H, and consequently we may construct a point of P which is not contained in any of those manifolds; in other words, we may certainly construct a point $\mathfrak{Z}_1$ of H which is not a fixed point. The images $\mathfrak{Z}_k = D_k(\mathfrak{Z}_1)$ of $\mathfrak{Z}_1$ are then all different one from another.

21. We denote by F the set of all points $\mathfrak{Z}$ satisfying all the inequalities

$$\rho(\mathfrak{Z}, \mathfrak{Z}_1) \leq \rho(\mathfrak{Z}, \mathfrak{Z}_k) \qquad (k = 2, 3, \cdots). \tag{61}$$

It follows from this definition that F is closed, with respect to H; but F is not necessarily compact. Let $G = H - F$ be the complement of G in H and let B be the frontier of F and $F_0 = F - B$ the set of inner points of F.

Obviously, the set G consists of all points $\mathfrak{Z}$ satisfying the inequality $\rho(\mathfrak{Z}, \mathfrak{Z}_1) > \rho(\mathfrak{Z}, \mathfrak{Z}_k)$ for at least one value of k; hence all the points $\mathfrak{Z}_2, \mathfrak{Z}_3, \cdots$ belong to G.

Let us now consider a point $\mathfrak{Z}$ which fulfills all conditions

$$\rho(\mathfrak{Z}, \mathfrak{Z}_1) < \rho(\mathfrak{Z}, \mathfrak{Z}_k) \qquad (k = 2, 3, \cdots); \tag{62}$$

the point $\mathfrak{Z} = \mathfrak{Z}_1$ is an example. The differences $\rho(\mathfrak{Z}, \mathfrak{Z}_k) - \rho(\mathfrak{Z}, \mathfrak{Z}_1)$ $(k = 2, 3, \cdots)$ are all positive and tend to infinity with k and, consequently, they have a positive minimum μ. The points $\mathfrak{Z}^*$ of the geodesic sphere $\rho(\mathfrak{Z}, \mathfrak{Z}^*) < \frac{1}{2}\mu$ form a neighborhood of $\mathfrak{Z}$. It follows from

$$\rho(\mathfrak{Z}^*, \mathfrak{Z}_k) - \rho(\mathfrak{Z}^*, \mathfrak{Z}_1) > \rho(\mathfrak{Z}, \mathfrak{Z}_k) - \tfrac{1}{2}\mu - \rho(\mathfrak{Z}, \mathfrak{Z}_1) - \tfrac{1}{2}\mu \geq 0 \qquad (k = 2, 3, \cdots),$$

that all these points belong to F. Consequently $\mathfrak{Z}$ is a point of F_0.

Consider next the case where all conditions (61) are fulfilled, with the sign of equality for at least one index $k = l > 1$. Then $\mathfrak{Z} \neq \mathfrak{Z}_l$. Let $\mathfrak{Z}^*$ be an arbitrary point on the geodesic arc joining $\mathfrak{Z}$ and $\mathfrak{Z}_l$, different from $\mathfrak{Z}_l$. Since $\rho(\mathfrak{Z}, \mathfrak{Z}_1) = \rho(\mathfrak{Z}, \mathfrak{Z}_l)$ and $\mathfrak{Z}_1 \neq \mathfrak{Z}_l$, the point $\mathfrak{Z}^*$ does not lie at the same time on the geodesic arc between $\mathfrak{Z}$ and $\mathfrak{Z}_1$ so that

$$\rho(\mathfrak{Z}, \mathfrak{Z}_1) < \rho(\mathfrak{Z}, \mathfrak{Z}^*) + \rho(\mathfrak{Z}^*, \mathfrak{Z}_1).$$

On the other hand,

$$\rho(\mathfrak{Z}, \mathfrak{Z}_l) = \rho(\mathfrak{Z}, \mathfrak{Z}^*) + \rho(\mathfrak{Z}^*, \mathfrak{Z}_l),$$

and the inequality $\rho(\mathfrak{Z}^*, \mathfrak{Z}_1) > \rho(\mathfrak{Z}^*, \mathfrak{Z}_l)$ is proved. Consequently, the whole geodesic arc between $\mathfrak{Z}$ and $\mathfrak{Z}_1$ belongs to G, except the point $\mathfrak{Z}$ itself, which is a point of F. This proves that $\mathfrak{Z}$ is a point of B. Therefore F_0 consists of all $\mathfrak{Z}$ satisfying (62) and B consists of all $\mathfrak{Z}$ satisfying (61) with at least one sign of equality.

Let again $\mathfrak{Z}$ be a point of B, and choose a point $\mathfrak{Z}^* \neq \mathfrak{Z}$ on the geodesic arc joining $\mathfrak{Z}$ and $\mathfrak{Z}_1$; then

$$\rho(\mathfrak{Z}, \mathfrak{Z}_1) = \rho(\mathfrak{Z}, \mathfrak{Z}^*) + \rho(\mathfrak{Z}^*, \mathfrak{Z}_1).$$

Moreover

$$\rho(\mathfrak{Z}, \mathfrak{Z}_1) \leqq \rho(\mathfrak{Z}, \mathfrak{Z}_k) \leqq \rho(\mathfrak{Z}, \mathfrak{Z}^*) + \rho(\mathfrak{Z}^*, \mathfrak{Z}_k) \qquad (k = 2, 3, \cdots),$$

where the sign of equality cannot be true in both places. Hence $\rho(\mathfrak{Z}^*, \mathfrak{Z}_1) < \rho(\mathfrak{Z}^*, \mathfrak{Z}_k)$, i. e., $\mathfrak{Z}^*$ is a point of F_0. This proves that any geodesic ray through $\mathfrak{Z}_1$ either lies completely in F or intersects B in exactly one point. The domain F is a star formed by geodesic arcs through $\mathfrak{Z}_1$.

The boundary B of F consists of parts of the analytic surfaces $\rho(\mathfrak{Z}, \mathfrak{Z}_1) = \rho(\mathfrak{Z}, \mathfrak{Z}_k)$ for certain values of $k > 1$. It is not generally true that the number of these values is finite. However, if we consider a compact domain P in H, the distance $\rho(P, \mathfrak{Z}_k)$ tends to infinity with k; consequently only a finite number of these bounding surfaces enter into P.

Let $\mathfrak{Z}$ be an arbitrary point of H. Since $\rho(\mathfrak{Z}, \mathfrak{Z}_k) \to \infty$, there exists a positive integer r, such that

$$(63) \qquad \rho(\mathfrak{Z}, \mathfrak{Z}_r) \leqq \rho(\mathfrak{Z}, \mathfrak{Z}_k) \qquad (k = 1, 2, \cdots).$$

Then the point $\mathfrak{Z}^* = D_r^{-1}(\mathfrak{Z})$ satisfies the conditions (61). Consequently $\mathfrak{Z}$ is equivalent to a point of F. On the other hand, a point $\mathfrak{Z}$ cannot satisfy both conditions (62) and (63), for any $r > 1$ and all k. It follows that no point of F_0 is equivalent to any point of F, except to itself under the identical mapping D_1.

Our results contain the proof of Theorem 6.

22. If the fundamental domain F is compact, then the boundary B consists of a finite number of surfaces $\rho(\mathfrak{Z}, \mathfrak{Z}_1) = \rho(\mathfrak{Z}, \mathfrak{Z}_k)$. Now F depends upon the initial point $\mathfrak{Z}_1$; we write more explicitly $F = F(\mathfrak{Z}_1)$. We shall prove that $F(\mathfrak{Z}^*_1)$ is compact, for an arbitrary initial point $\mathfrak{Z}^*_1$, if $F(\mathfrak{Z}_1)$ is compact.

The space H is called compact relative to Δ, if there exists for every infinite sequence of points $\mathfrak{Z}^{(k)}$, $(k = 1, 2, \cdots)$, in H at least one compact sequence of images $\mathfrak{Z}_{l_k}^{(k)}$ under Δ. Obviously this condition is satisfied, if and only if there exists a compact domain G in H, such that every point $\mathfrak{Z}$

of H has at least one image $\mathfrak{Z}_k$ in G. By the minimum property of F, this domain is then also compact, and vice versa. Hence our assertion is proved.

In the classical case $n = 1$, the fundamental polygon F is compact, if and only if all vertices are elliptic; it is well-known that the uniformization of any field of algebraic functions of genus $p > 1$ leads to a discontinuous group with the required property. From the algebraic point of view, the most important discontinuous groups Δ, in the case $n = 1$, are, more generally, those having a fundamental polygon with a finite number of elliptic or parabolic vertices. They constitute the Fuchsian groups of the first kind, and the corresponding automorphic functions form algebraic function fields of a single variable.

For arbitrary n, we say that a discontinuous group Δ is of the first kind, if there exists a fundamental domain F with the following three properties: 1) Every compact domain in H is covered by a finite number of images of F; 2) only a finite number of images of F are neighbors of F; 3) the integral

$$V(\Delta) = \int_F dv \tag{64}$$

converges. In the special case $n = 1$, it is easily seen that this definition is tantamount to the ordinary definition of Fuchsian groups of the first kind.

It is now clear that Δ is certainly of the first kind, if H is compact relative to Δ; hence Theorem 7 is proved.

Let

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$$

be any symplectic matrix. Under the automorphism $\mathfrak{Z} \rightarrow (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ of H, a subgroup Δ of Ω is replaced by $\mathfrak{M}\Delta\mathfrak{M}^{-1}$, i. e., by a conjugate subgroup, and a fundamental domain of Δ is mapped onto a fundamental domain of $\mathfrak{M}\Delta\mathfrak{M}^{-1}$. Obviously all conjugate subgroups $\mathfrak{M}\Delta\mathfrak{M}^{-1}$ will be of the first kind, if Δ itself is of the first kind.

23. Let us now assume that Δ has no fixed point in H. Then the images $\mathfrak{Z}_k (k = 1, 2, \cdots)$ of an arbitrary point $\mathfrak{Z}_1$ of H are all different from each other, and the minimum of the distances $\rho(\mathfrak{Z}_k, \mathfrak{Z}_1)$ $(k = 2, 3, \cdots)$ is a positive number $\delta = \delta(\mathfrak{Z}_1)$. The images of the geodesic sphere $\rho(\mathfrak{Z}, \mathfrak{Z}_1) < \frac{1}{2}\delta$ with the center $\mathfrak{Z}_1$ do not overlap.

Identifying equivalent points of H, we obtain a set H_Δ. We may obviously introduce the symplectic metric into H_Δ, defining a neighborhood of $\mathfrak{Z}_1$ by $\rho(\mathfrak{Z}, \mathfrak{Z}_1) < \frac{1}{2}\delta$. Starting from a fundamental domain F of Δ, we obtain a

CALCUTTA CENTRAL LIBRARY UNIVERSITY

model of H_Δ, if we identify equivalent points on the boundary of F. Assume now that H is compact relative to Δ. Then F may be considered as a closed manifold with the symplectic metric. By Theorem 5, this manifold has the Euler number

$$\chi = c_n(-\pi)^{-n(n+1)/2}V(\Delta), \tag{65}$$

where $V(\Delta)$ denotes the symplectic volume (6) of F.

Probably the formula (65) is true for all groups Δ of the first kind, provided Δ has no fixed point in H. This is easily proved in the case $n = 1$. The general proof of our suggestion would require a careful investigation of the geodesics of infinite length in the fundamental domain.

V. HERMITIAN FORMS.

24. In the case $n = 1$, we know three different methods of constructing discontinuous groups of the first kind, namely an analytic, a geometric and an arithmetic method. The analytic method starts with a Riemann surface of finite genus and applies the theory of uniformization. The geometric method uses the principle of reflection for a circular polygon with a finite number of elliptic and parabolic vertices, the angles at the elliptic vertices being aliquot parts of π. The arithmetic method depends upon the theory of units of indefinite binary hermitian forms, in an imaginary quadratic ring over a totally real algebraic number field of finite degree. It is not known, to what extent the analytic and the geometric method may be generalized; however, we shall show in the following sections, that there is a generalization of the arithmetic method to the case of an arbitrary n.

LEMMA 3. *Let $\mathfrak{H}^{(2n)}$ be a hermitian and $\mathfrak{S}^{(2n)}$ a non-singular skew-symmetric matrix with complex elements. If*

$$\mathfrak{H}\bar{\mathfrak{S}}^{-1}\bar{\mathfrak{H}} = \mathfrak{S}, \tag{66}$$

then there exists a matrix $\mathfrak{C}$ such that $\mathfrak{H}\{\mathfrak{C}\} = i\mathfrak{J}$ and $\mathfrak{S}[\mathfrak{C}] = \mathfrak{J}$.

Putting $\mathfrak{S}^{-1}\mathfrak{H} = \mathfrak{F}$, we have $\mathfrak{F}\bar{\mathfrak{F}} = \mathfrak{E}$, by (66), whence

$$(\mathfrak{F} + i\lambda\mathfrak{E})(\bar{\mathfrak{F}} + i\bar{\lambda}\mathfrak{E}) + (\mathfrak{F} - i\lambda\mathfrak{E})(\bar{\mathfrak{F}} - i\bar{\lambda}\mathfrak{E}) = 0$$

for any scalar λ of absolute value 1. Choosing this λ such that $|\mathfrak{F} + i\lambda\mathfrak{E}| \neq 0$, we obtain

$$\frac{\mathfrak{F} - i\lambda\mathfrak{E}}{\mathfrak{F} + i\lambda\mathfrak{E}} = i\mathfrak{T}$$

with real $\mathfrak{T}$; consequently $\mathfrak{E} - i\mathfrak{T} = 2i\lambda(\mathfrak{F} + i\lambda\mathfrak{E})^{-1}$, $|\mathfrak{E} - i\mathfrak{T}| \neq 0$,

$$\mathfrak{F} = i\lambda \frac{\mathfrak{E} + i\mathfrak{T}}{\mathfrak{E} - i\mathfrak{T}}.$$

Let $\mathfrak{C}_1 = \lambda^{\frac{1}{2}}(\mathfrak{E} + i\mathfrak{T})$, $\mathfrak{H}_1 = \mathfrak{H}\{\mathfrak{C}_1\}$, $\mathfrak{G}_1 = \mathfrak{G}[\mathfrak{C}_1]$; then

$$\mathfrak{G}_1^{-1}\mathfrak{H}_1 = \mathfrak{C}_1^{-1}\mathfrak{F}\bar{\mathfrak{C}}_1 = i\mathfrak{E};\ \bar{\mathfrak{H}}_1 = \mathfrak{H}'_1 = i\mathfrak{G}'_1 = -i\mathfrak{G}_1 = -\mathfrak{H}_1;$$

hence $\mathfrak{H}_1$ is pure imaginary and $\mathfrak{G}_1 = -i\mathfrak{H}_1$ real. There exists a real matrix $\mathfrak{C}_2$ satisfying $\mathfrak{G}_1[\mathfrak{C}_2] = \mathfrak{J}$, and $\mathfrak{C} = \mathfrak{C}_1\mathfrak{C}_2$ has the required property.

25. Let K be a totally real algebraic number field of finite degree h and let $K^{(1)}, \cdots, K^{(h)}$ be its conjugate fields, $K = K^{(1)}$. If r is any positive number of K, then the field $K_0 = K(\sqrt{-r})$, of degree $2h$, is imaginary. We consider a hermitian matrix $\mathfrak{H} = \mathfrak{H}^{(2n)}$ and a non-singular skew-symmetric matrix $\mathfrak{G}$, both with elements in K_0, and we assume that the relationship

$$\mathfrak{H}\bar{\mathfrak{G}}^{-1}\bar{\mathfrak{H}} = s\mathfrak{G} \tag{67}$$

holds with a positive scalar factor s. Obviously s is then a number of K.

The matrices $\mathfrak{U}$ with integral elements in K_0, satisfying the two conditions

$$\mathfrak{G}[\mathfrak{U}] = \mathfrak{G}, \qquad \mathfrak{H}\{\mathfrak{U}\} = \mathfrak{H},$$

constitute a multiplicative group $\Lambda = \Lambda(\mathfrak{G}, \mathfrak{H})$. Applying Lemma 3 with $s^{-\frac{1}{2}}\mathfrak{H}$ instead of $\mathfrak{H}$, we obtain a complex matrix $\mathfrak{C}$, such that $\mathfrak{G}[\mathfrak{C}] = \mathfrak{J}$ and $\mathfrak{H}\{\mathfrak{C}\} = is^{\frac{1}{2}}\mathfrak{J}$. Consequently the elements $\mathfrak{C}^{-1}\mathfrak{U}\mathfrak{C} = \mathfrak{M}$ of the group $\mathfrak{C}^{-1}\Lambda\mathfrak{C} = \Delta_0 = \Delta_0(\mathfrak{G}, \mathfrak{H})$ satisfy $\mathfrak{J}[\mathfrak{M}] = \mathfrak{J}$ and $\mathfrak{J}\{\mathfrak{M}\} = \mathfrak{J}$, whence $\mathfrak{J}[\mathfrak{M}] = \mathfrak{J}$ and $\bar{\mathfrak{M}} = \mathfrak{M}$. This proves that Δ_0 is a subgroup of the homogeneous symplectic group Ω_0. Identifying $\mathfrak{U}$ and $-\mathfrak{U}$, i. e., $\mathfrak{M}$ and $-\mathfrak{M}$, we obtain a subgroup $\Delta = \Delta(\mathfrak{G}, \mathfrak{H})$ of Ω.

The matrix $\mathfrak{C}$ is not uniquely determined. If also $\mathfrak{H}\{\mathfrak{C}^*\} = is^{\frac{1}{2}}\mathfrak{J}$ and $\mathfrak{G}[\mathfrak{C}^*] = \mathfrak{J}$, then $\mathfrak{C}^{-1}\mathfrak{C}^* = \mathfrak{B}$ is symplectic, and vice versa. Using $\mathfrak{C}^*$ instead of $\mathfrak{C}$, we have to replace Δ by $\mathfrak{B}^{-1}\Delta\mathfrak{B}$; hence the class of conjugate subgroups $\mathfrak{B}^{-1}\Delta\mathfrak{B}$ in Ω is uniquely determined by $\mathfrak{G}$ and $\mathfrak{H}$.

Obviously $\Lambda(\mathfrak{G}, a\mathfrak{H}) = \Lambda(\mathfrak{G}, \mathfrak{H})$ for any number $a \neq 0$ of K. Therefore we may assume r and s to be integers. Henceforth we shall, moreover, assume that r is totally positive and that all conjugates of $\mathfrak{H}$ except $\mathfrak{H}$ and $\bar{\mathfrak{H}}$ are positive. Let r_l and the pair $\mathfrak{H}_l, \bar{\mathfrak{H}}_l$ be the image of r and the pair $\mathfrak{H}, \bar{\mathfrak{H}}$ under the isomorphism $K \rightarrow K^{(l)}$ $(l = 1, \cdots, h)$. For any element $\mathfrak{U}$ of Λ, we have $\mathfrak{H}_l\{\mathfrak{U}_l\} = \mathfrak{H}_l$, where $\mathfrak{U}_l, \bar{\mathfrak{U}}_l$ $(l = 1, \cdots, h)$ denote the $2h$ conjugates of $\mathfrak{U} = \mathfrak{U}_1$. Since $\mathfrak{H}_l$ is positive for $l > 1$, the matrices $\mathfrak{U}_2, \cdots, \mathfrak{U}_h$ are bounded. If also $\mathfrak{U}$ itself is bounded, then all conjugates of $\mathfrak{U}$ are bounded.

On the other hand, there exists only a finite number of integers in K_0 with bounded conjugates. Consequently Λ and Δ_0 are discrete groups and, by Section **19**, the group $\Delta(\mathfrak{G}, \mathfrak{H})$ is discontinuous.

It follows from (67) that $\mathfrak{H} = -s\bar{\mathfrak{H}}^{-1}\{\mathfrak{G}\}$. Since s is positive, $\mathfrak{H} = \mathfrak{H}_1$ is necessarily the matrix of an indefinite hermitian form. Since $\mathfrak{H}_l$ is positive for $l > 1$, the conjugates $s_2, \cdots, s_h$ of the positive number $s = s_1$ are all negative.

The most important example of a group $\Delta(\mathfrak{G}, \mathfrak{H})$ is provided by $\mathfrak{G} = \mathfrak{J}$, $\mathfrak{H} = i\mathfrak{J}$, $r = 1$, $h = 1$. Then we may choose $\mathfrak{C} = \mathfrak{E}$, and $\Delta_0(\mathfrak{G}, \mathfrak{H})$ consists of all symplectic matrices $\mathfrak{M}$ with rational integral elements. We call this group the homogeneous modular group of degree n and denote it by Γ_0. Identifying the elements $\mathfrak{M}$ and $-\mathfrak{M}$ of Γ_0, we obtain the (inhomogeneous) modular group Γ.

26. Two subgroups Δ and Δ^* of Ω are called commensurable, if there exist a subgroup Δ_1 of finite index in Δ and a subgroup Δ^*_1 of finite index in Δ^*, such that Δ_1 and Δ^*_1 are conjugate subgroups of Ω. If Δ is a discontinuous group of the first kind, then the same holds for Δ^*, and we obtain $jV(\Delta) = j^*V(\Delta^*)$, where j and j^* denote the indices of the subgroups Δ_1 and Δ^*_1; consequently the quotient $V(\Delta)/V(\Delta^*)$ is a rational number.

It is easily seen that the property of commensurability is symmetric and transitive; therefore we may speak of a class of commensurable groups. We have now the problem of deciding whether two groups $\Delta(\mathfrak{G}, \mathfrak{H})$ and $\Delta(\mathfrak{G}^*, \mathfrak{H}^*)$ are commensurable or not. The complete answer is given by Theorem 13. In this section we solve only a particular case of the problem: We assume that $\mathfrak{G}^*$ and $\mathfrak{H}^*$ are also matrices of the field K_0 and that they fulfill the condition

$$\mathfrak{H}^*\bar{\mathfrak{G}}^{*-1}\bar{\mathfrak{H}}^* = s\mathfrak{G}^*$$

with the same factor s as in (67).

LEMMA 4. *Let $c_1, \cdots, c_{2n}$ be $2n$ numbers of K_0, not all 0. There exists a matrix $\mathfrak{C}_1^{(2n)} = (c_{kl})$ in K_0, such that $c_{k1} = c_k$ $(k = 1, \cdots, 2n)$ and $\mathfrak{J}[\mathfrak{C}_1] = \mathfrak{J}$. If, moreover, $c_1 \neq 0$, we may choose $c_{1l} = 0$ $(l = 2, \cdots, 2n)$.*

Put $\mathfrak{c}'_1 = (c_1 \cdots c_n)$ and $\mathfrak{c}'_2 = (c_{n+1} \cdots c_{2n})$. If $\mathfrak{c}_1 = 0$, we choose in K_0 a non-singular matrix $\mathfrak{Q}^{(n)}$ with the first row $\mathfrak{c}'_2$ and define

$$\mathfrak{C}_1 = \begin{pmatrix} 0 & -\mathfrak{Q}^{-1} \\ \mathfrak{Q}' & 0 \end{pmatrix}.$$

If $\mathfrak{c}_1 \neq 0$, we choose in K_0 a non-singular matrix $\mathfrak{P}^{(n)}$ with the first row $\mathfrak{c}'_1$ and a symmetric matrix $\mathfrak{S}$ with the first column $\mathfrak{P}\mathfrak{c}_2$; then

$$\mathfrak{C}_1 = \begin{pmatrix} \mathfrak{P}' & 0 \\ \mathfrak{P}^{-1}\mathfrak{S} & \mathfrak{P}^{-1} \end{pmatrix}$$

has the required property. In the case $c_1 \neq 0$, we may obviously choose $(c_1 0 \cdots 0)$ as the first row of $\mathfrak{P}'$.

LEMMA 5. *There exists a matrix $\mathfrak{C}_0$ in K_0 such that $\mathfrak{G}[\mathfrak{C}_0] = \mathfrak{G}^*$ and $\mathfrak{H}\{\mathfrak{C}_0\} = \mathfrak{H}^*$.*

The equation $\mathfrak{G}[\mathfrak{X}] = \mathfrak{J}$ has a solution $\mathfrak{X}$ in K_0; hence we may assume, without loss of generality, that $\mathfrak{G} = \mathfrak{G}^* = \mathfrak{J}$. Moreover it is sufficient to prove the lemma for the special case

$$\mathfrak{H}^* = \begin{pmatrix} \mathfrak{E} & 0 \\ 0 & -s\mathfrak{E} \end{pmatrix}.$$

Since none of the conjugates of $-\mathfrak{H}$ is positive, it follows from Hasse's generalization of A. Meyer's theorem, that the diophantine equation $\mathfrak{H}\{\mathfrak{x}\} = 1$ has a solution $\mathfrak{x}$ in K_0. Applying Lemma 4 with $(c_1 \cdots c_{2n}) = \mathfrak{x}'$, we construct a matrix $\mathfrak{C}_1$ in K_0 which has the first column $\mathfrak{x}$ and satisfies $\mathfrak{J}[\mathfrak{C}_1] = \mathfrak{J}$. Then $\mathfrak{H}\{\mathfrak{C}_1\}$ has the first diagonal element 1. On the other hand, the condition (67) means now

$$\mathfrak{H}\mathfrak{J}\bar{\mathfrak{H}} = -s\mathfrak{J}. \tag{68}$$

For the proof of Lemma 5, we may therefore replace $\mathfrak{H}\{\mathfrak{C}_1\}$ again by $\mathfrak{H}$.

By Lemma 4, there exists a matrix $\mathfrak{C}_2$ in K_0 with the following three properties: $\mathfrak{C}_2$ and $\mathfrak{H}$ have the same first column; $\mathfrak{C}_2$ has the first row $(1\,0 \cdots 0)$; $\mathfrak{J}[\mathfrak{C}_2] = \mathfrak{J}$. Put $\mathfrak{C}_2^{-1} = \mathfrak{C}'_3$; then also $\mathfrak{J}[\mathfrak{C}_3] = \mathfrak{J}$, and the two matrices $\mathfrak{C}'_3\mathfrak{H}$ and $\bar{\mathfrak{C}}_3$ have both the first column $(1\,0 \cdots 0)'$. Writing again $\mathfrak{H}$ instead of $\mathfrak{H}\{\mathfrak{C}_3\}$, we obtain the decomposition

$$\mathfrak{H} = \begin{pmatrix} \mathfrak{H}_1 & \mathfrak{H}_{12} \\ \bar{\mathfrak{H}}'_{12} & \mathfrak{H}_2 \end{pmatrix},$$

$$\mathfrak{H}_1^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{F}_1 \end{pmatrix}, \qquad \mathfrak{H}_{12} = \begin{pmatrix} 0 & 0 \\ \mathfrak{a} & \mathfrak{F}_{12} \end{pmatrix}, \qquad \mathfrak{H}_2 = \begin{pmatrix} t & \mathfrak{c}' \\ \mathfrak{b} & \mathfrak{F}_2 \end{pmatrix}.$$

By (68),

$$\mathfrak{H}_1\mathfrak{H}'_{12} = \mathfrak{H}_{12}\mathfrak{H}'_1, \qquad \bar{\mathfrak{H}}'_{12}\mathfrak{H}'_2 = \mathfrak{H}_2\bar{\mathfrak{H}}_{12},$$
$$\mathfrak{H}_1\mathfrak{H}'_2 - \mathfrak{H}_{12}\bar{\mathfrak{H}}_{12} = -s\mathfrak{E}, \qquad \mathfrak{H}'_1\mathfrak{H}_2 - \bar{\mathfrak{H}}_{12}\mathfrak{H}_{12} = -s\mathfrak{E},$$

whence $\mathfrak{a} = 0$, $t = -s$, $\mathfrak{b} = 0$, $\mathfrak{c} = 0$. This contains, in particular, the proof of the lemma for $n = 1$. If $n > 1$, the hermitian matrix

$$\mathfrak{F} = \begin{pmatrix} \mathfrak{F}_1 & \mathfrak{F}_{12} \\ \bar{\mathfrak{J}}'_{12} & \mathfrak{J}_2 \end{pmatrix}$$

fulfills the same conditions as $\mathfrak{H}$, with $n-1$ instead of n; and we may apply induction with respect to n. Hence the lemma is proved.

Let $\mathfrak{C}_0$ be the matrix of Lemma 5 and choose a positive rational integer q, such that the two matrices $q\mathfrak{C}_0$ and $q\mathfrak{C}_0^{-1}$ have integral elements. Since the integers of K_0 belong to a finite number of classes of residues modulo q^2, the elements $\mathfrak{U}$ of $\Lambda(\mathfrak{S}, \mathfrak{H})$ satisfying $\mathfrak{U} \equiv \mathfrak{E} \pmod{q^2}$ form a subgroup Λ_1 of finite index. Consider now the subgroup Λ_2 consisting of all $\mathfrak{U}$ with integral $\mathfrak{C}_0^{-1}\mathfrak{U}\mathfrak{C}_0 = \mathfrak{U}^*$. Obviously Λ_1 is contained in Λ_2; consequently Λ_2 is a fortiori of finite index in $\Lambda(\mathfrak{S}, \mathfrak{H})$. On the other hand, $\mathfrak{U}^*$ is an element of $\Lambda(\mathfrak{S}^*, \mathfrak{H}^*)$ with the characteristic property that $\mathfrak{C}_0\mathfrak{U}^*\mathfrak{C}_0^{-1} = \mathfrak{U}$ is integral; hence the group $\Lambda^*_2 = \mathfrak{C}_0^{-1}\Lambda_2\mathfrak{C}_0$ is of finite index in $\Lambda(\mathfrak{S}^*, \mathfrak{H}^*)$. If $\mathfrak{C}$ is the matrix of Section **25**, we have $\Delta_0(\mathfrak{S}, \mathfrak{H}) = \mathfrak{C}^{-1}\Lambda(\mathfrak{S}, \mathfrak{H})\mathfrak{C}$, and we may define $\Delta_0(\mathfrak{S}^*, \mathfrak{H}^*) = \mathfrak{C}_4^{-1}\Lambda(\mathfrak{S}^*, \mathfrak{H}^*)\mathfrak{C}_4$ with $\mathfrak{C}_4 = \mathfrak{C}_0^{-1}\mathfrak{C}$. Then $\mathfrak{C}^{-1}\Lambda_2\mathfrak{C} = \mathfrak{C}_4^{-1}\Lambda^*_2\mathfrak{C}_4$ is a common subgroup of $\Delta_0(\mathfrak{S}, \mathfrak{H})$ and $\Delta_0(\mathfrak{S}^*, \mathfrak{H}^*)$, of finite indices. This proves that $\Delta(\mathfrak{S}, \mathfrak{H})$ and $\Delta(\mathfrak{S}^*, \mathfrak{H}^*)$ are commensurable.

27. The two conditions $\mathfrak{S}[\mathfrak{U}] = \mathfrak{S}$ and $\mathfrak{H}\{\mathfrak{U}\} = \mathfrak{H}$ for the elements $\mathfrak{U}$ of $\Lambda(\mathfrak{S}, \mathfrak{H})$ may be written $\mathfrak{F}\bar{\mathfrak{U}} = \mathfrak{U}\mathfrak{F}$ and $\mathfrak{U}\mathfrak{S}^{-1}\mathfrak{U}'\mathfrak{S} = \mathfrak{E}$, with $\mathfrak{F} = \mathfrak{S}^{-1}\mathfrak{H}$. Let us consider the set R of all matrices $\mathfrak{B}$ in K_0 which satisfy the condition

$$\mathfrak{F}\bar{\mathfrak{B}} = \mathfrak{B}\mathfrak{F}. \tag{69}$$

Obviously R is a ring. By (67), the matrix

$$\bar{\mathfrak{B}} = \mathfrak{S}^{-1}\mathfrak{B}'\mathfrak{S} \tag{70}$$

is again a solution of (69), and consequently (70) defines an anti-automorphism of R. The elements $\mathfrak{U}$ of $\Lambda(\mathfrak{S}, \mathfrak{H})$ are the integral elements $\mathfrak{B} = \mathfrak{U}$ of R with the property

$$\mathfrak{U}\bar{\mathfrak{U}} = \mathfrak{E}. \tag{71}$$

By Lemma 5,

$$\mathfrak{S}[\mathfrak{C}_0] = \mathfrak{J}, \quad \mathfrak{H}\{\mathfrak{C}_0\} = \begin{pmatrix} \mathfrak{E} & 0 \\ 0 & -s\mathfrak{E} \end{pmatrix} = \mathfrak{H}^*, \quad \mathfrak{C}_0^{-1}\mathfrak{J}\mathfrak{C}_0 = \begin{pmatrix} 0 & s\mathfrak{E} \\ \mathfrak{E} & 0 \end{pmatrix} = \mathfrak{F}^*,$$

where $\mathfrak{C}_0$ is a matrix in K_0. The elements $\mathfrak{B}^* = \mathfrak{C}_0^{-1}\mathfrak{B}\mathfrak{C}_0$ of the ring $R^* = \mathfrak{C}_0^{-1}R\mathfrak{C}_0$ satisfy $\mathfrak{F}^*\bar{\mathfrak{B}}^* = \mathfrak{B}^*\mathfrak{F}^*$, whence

$$\mathfrak{B}^* = \begin{pmatrix} \mathfrak{P} & s\bar{\mathfrak{Q}} \\ \mathfrak{Q} & \bar{\mathfrak{P}} \end{pmatrix}$$

with arbitrary matrices $\mathfrak{P}^{(n)}, \mathfrak{Q}^{(n)}$ in K_0. Defining

$$2^{1/2}s^{1/4}\mathfrak{L} = \begin{pmatrix} is^{1/2}\mathfrak{E} & s^{1/2}\mathfrak{E} \\ -\mathfrak{E} & -i\mathfrak{E} \end{pmatrix}, \qquad \mathfrak{C}_0\mathfrak{L} = \mathfrak{C},$$

we obtain

$$\mathfrak{G}[\mathfrak{C}] = \mathfrak{J}, \qquad \mathfrak{H}\{\mathfrak{C}\} = is^{\frac{1}{2}}\mathfrak{J}, \tag{72}$$

$$\mathfrak{C}^{-1}\mathfrak{B}\mathfrak{C} = \mathfrak{L}^{-1}\mathfrak{B}^*\mathfrak{L} = \begin{pmatrix} \mathfrak{A}_0 + \sqrt{rs}\,\mathfrak{A}_1 & \sqrt{r}\,\mathfrak{A}_2 + \sqrt{s}\,\mathfrak{A}_3 \\ -\sqrt{r}\,\mathfrak{A}_2 + \sqrt{s}\,\mathfrak{A}_3 & \mathfrak{A}_0 - \sqrt{rs}\,\mathfrak{A}_1 \end{pmatrix} = \mathfrak{M} \tag{73}$$

with

$$\mathfrak{A}_0 = \tfrac{1}{2}(\mathfrak{P} + \bar{\mathfrak{P}}), \quad \mathfrak{A}_1 = \frac{1}{2\sqrt{-r}}(\mathfrak{Q} - \bar{\mathfrak{Q}}),$$

$$\mathfrak{A}_2 = \frac{1}{2\sqrt{-r}}(\mathfrak{P} - \bar{\mathfrak{P}}), \quad \mathfrak{A}_3 = -\tfrac{1}{2}(\mathfrak{Q} + \bar{\mathfrak{Q}});$$

consequently $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ are arbitrary matrices in K.

Consider now the generalized quaternion algebra A over K consisting of the elements $\alpha = a_0\epsilon_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3$ with arbitrary a_0, a_1, a_2, a_3 in K, where ϵ_0 is the unit and $\epsilon_1{}^2 = rs\epsilon_0$, $\epsilon_2{}^2 = -r\epsilon_0$, $\epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1 = s\epsilon_3$. We denote by $\hat{\alpha} = a_0\epsilon_0 - a_1\epsilon_1 - a_2\epsilon_2 - a_3\epsilon_3$ the conjugate quaternion. There exists the well-known representation of A, of degree 2, defined by

$$\epsilon_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon_1 = \sqrt{rs}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{74}$$

$$\epsilon_2 = \sqrt{r}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_3 = \sqrt{s}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then obviously

$$\mathfrak{M} = \sum_{k=0}^{3} \mathfrak{A}_k \times \epsilon_k, \tag{75}$$

where $\mathfrak{A}_k \times \epsilon_k$ denotes the Kronecker product of the matrices $\mathfrak{A}_k$ and ϵ_k; and consequently $\mathfrak{C}^{-1}R\mathfrak{C}$ is the ring of all matrices $M = (\alpha_{kl})$ of n rows and columns with arbitrary elements α_{kl} of A.

By (70) and (73), the condition (71) may be expressed in the form

$$\mathfrak{M}\bar{\mathfrak{M}} = \mathfrak{E} \tag{76}$$

with

$$\bar{\mathfrak{M}} = \mathfrak{C}^{-1}\bar{\mathfrak{B}}\mathfrak{C} = \mathfrak{J}^{-1}\mathfrak{M}'\mathfrak{J} = \mathfrak{A}'_0 \times \epsilon_0 - \sum_{k=1}^{3} \mathfrak{A}'_k \times \epsilon_k;$$

hence $\bar{\mathfrak{M}}$, written as a quaternion matrix, is the transpose $(\hat{\alpha}_{lk})$ of the conjugate $(\hat{\alpha}_{kl})$ of M. By (72) and Section **25**, the group $\Delta_0(\mathfrak{G}, \mathfrak{H})$ consists of all matrices $\mathfrak{M}$ of the form (75), such that (76) is satisfied and $\mathfrak{C}\mathfrak{M}\mathfrak{C}^{-1}$ is integral. On the other hand, the solutions of (76) with integral $\mathfrak{A}_k$ $(k = 0, \cdots, 3)$ in K constitute also a subgroup $\Delta_0(r, s)$ of the homogeneous symplectic group Ω_0. It follows from the argument at the end of Section **26**, that the two groups $\Delta_0(\mathfrak{G}, \mathfrak{H})$ and $\Delta_0(r, s)$ are commensurable. The problem of the commensurability of two groups $\Delta(\mathfrak{G},\mathfrak{H})$ and $\Delta(\mathfrak{G}_1,\mathfrak{H}_1)$ is therefore reduced to the corresponding problem for $\Delta_0(r,s)$ and $\Delta_0(r_1,s_1)$. The solution will be given in Chapter IX.

VI. THE FUNDAMENTAL DOMAIN OF THE MODULAR GROUP.

28. We shall construct a fundamental F for any group $\Delta(\mathfrak{G},\mathfrak{H})$ and, in particular, for the modular group Γ. The application of the general method of Chapter IV would lead to a rather complicated shape of the frontier of F, and it would then be difficult to prove that $\Delta(\mathfrak{G},\mathfrak{H})$ is a group of the first kind. Therefore we shall use another procedure applying the special arithmetic properties of these groups.

Lemma 6. *The equation*

$$\mathfrak{S}=\begin{pmatrix}\mathfrak{Y}^{-1} & 0\\ 0 & \mathfrak{Y}\end{pmatrix}\begin{bmatrix}\mathfrak{E} & -\mathfrak{X}\\ 0 & \mathfrak{E}\end{bmatrix} \tag{77}$$

defines a mapping of the space H of the matrices $\mathfrak{Z}=\mathfrak{X}+i\mathfrak{Y}$ onto the space S of the symplectic positive symmetric matrices $\mathfrak{S}$. Any symplectic transformation $\mathfrak{Z}^*=(\mathfrak{A}\mathfrak{Z}+\mathfrak{B})(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-1}$ with the matrix $\mathfrak{M}$ induces in S the transformation $\mathfrak{S}^*=\mathfrak{S}[\mathfrak{M}^{-1}]$.

Let

$$\mathfrak{S}=\begin{pmatrix}\mathfrak{S}_1 & \mathfrak{S}_{12}\\ \mathfrak{S}'_{12} & \mathfrak{S}_2\end{pmatrix}$$

be an arbitrary point of S. Since $\mathfrak{S}>0$, the inequality $\mathfrak{S}_1>0$ holds, whence $\mathfrak{S}_1^{-1}=\mathfrak{Y}>0$. Moreover $\mathfrak{S}_1\mathfrak{S}'_{12}=\mathfrak{S}_{12}\mathfrak{S}_1$ and $\mathfrak{S}_1\mathfrak{S}_2-\mathfrak{S}_{12}{}^2=\mathfrak{E}$, whence $-\mathfrak{S}_1^{-1}\mathfrak{S}_{12}=\mathfrak{X}=\mathfrak{X}'$, $\mathfrak{S}_{12}=-\mathfrak{Y}^{-1}\mathfrak{X}$ and $\mathfrak{S}_2=\mathfrak{Y}+\mathfrak{Y}^{-1}[\mathfrak{X}]$. This proves the first assertion of the lemma.

The relationship (77) can be written

$$\mathfrak{S}[\mathfrak{w}]=\mathfrak{Y}^{-1}[\mathfrak{u}-\mathfrak{X}\mathfrak{v}]+\mathfrak{Y}[\mathfrak{v}]=\mathfrak{Y}^{-1}\{\mathfrak{u}-\mathfrak{Z}\mathfrak{v}\} \tag{78}$$

with an arbitrary real column

$$\mathfrak{w}^{(2n)}=\begin{pmatrix}\mathfrak{u}^{(n)}\\ \mathfrak{v}^{(n)}\end{pmatrix}.$$

For the symplectic transformation $\mathfrak{Z}^*=\mathfrak{X}^*+i\mathfrak{Y}^*=(\mathfrak{A}\mathfrak{Z}+\mathfrak{B})(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-1}$ we obtain, by (59),

$$\mathfrak{Y}^{*-1}=\mathfrak{Y}^{-1}\{\mathfrak{Z}\mathfrak{C}'+\mathfrak{D}'\}$$

$$\mathfrak{Y}^{*-1}\{\mathfrak{u}-\mathfrak{Z}^*\mathfrak{v}\}=\mathfrak{Y}^{-1}\{(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})'\mathfrak{u}-(\mathfrak{A}\mathfrak{Z}+\mathfrak{B})'\mathfrak{v}\}=\mathfrak{Y}^{-1}\{\mathfrak{u}^*-\mathfrak{Z}\mathfrak{v}^*\},$$

where $\mathfrak{u}^* = \mathfrak{D}'\mathfrak{u} - \mathfrak{B}'\mathfrak{v}$ and $\mathfrak{v}^* = -\mathfrak{C}'\mathfrak{u} + \mathfrak{A}'\mathfrak{v}$. Now the second part of the lemma follows from (8) and (78).

29. Let P be the space of all positive symmetric matrices $\mathfrak{T}^{(m)}$ with real elements. Any $\mathfrak{T}$ in P may be uniquely expressed as $\mathfrak{T} = \mathfrak{P}[\mathfrak{Q}]$, where $\mathfrak{P} = [p_1, \cdots, p_m]$ is a diagonal matrix with $p_k > 0$ $(k = 1, \cdots, m)$ and $\mathfrak{Q} = (q_{kl})$ is a triangular matrix with $q_{kl} = 0$ for $k > l$ and $q_{kk} = 1$. If t is any positive number, the inequalities

$$(79) \qquad 0 < p_k \leq t p_{k+1}, \qquad -t \leq q_{kl} \leq t \qquad (1 \leq k < l \leq m)$$

define a compact domain $Q(t)$ in P, and any given compact set in P is contained in $Q(t)$ for sufficiently large values of t.

Let U denote the group of all different transformations $\mathfrak{T} \to \mathfrak{T}[\mathfrak{U}]$, where $\mathfrak{U}$ runs over the unimodular matrices, i. e., the matrices with rational integral elements and determinant ± 1. On account of Minkowski's theory of reduction, there exists in P a fundamental domain R with respect to U, defined by a finite number of inequalities

$$(80) \qquad L_r(\mathfrak{T}) \geq 0 \qquad (r = 1, 2, \cdots, g),$$

where $L_r(\mathfrak{T})$ denotes a certain homogeneous linear function of the elements of $\mathfrak{T}$ with rational coefficients. A point $\mathfrak{T}$ lies on the frontier of R if, and only if, the conditions (80) are fulfilled with at least one sign of equality. The images of R under U cover the whole space P without gaps and overlappings. Only a finite number of images enter into any compact part of P, and only a finite number of images are neighbors of R.

Most of the results of the theory of reduction are simple consequences of the following two known lemmata.

LEMMA 7. *There exists a positive number τ_1 depending only upon m, such that R is contained in $Q(\tau_1)$.*

LEMMA 8. *Let $\mathfrak{T}_1, \mathfrak{T}_2$ be two points of $Q(t)$ and let $\mathfrak{T}_2 = \mathfrak{T}_1[\mathfrak{F}]$, where $\mathfrak{F}$ is a matrix with rational integral elements f_{kl}. Then*

$$-\tau \leq f_{kl} \leq \tau \qquad (k, l = 1, \cdots, m),$$

where τ is a positive number depending only upon t, m and the determinant $|\mathfrak{F}|$.

Let $t_1, \cdots, t_m$ be the diagonal elements of a point $\mathfrak{T} = \mathfrak{P}[\mathfrak{Q}]$ of R. Then

$$t_l = \sum_{k=1}^{m} p_k q_{kl}^2 = p_l \Big(1 + \sum_{k=1}^{l-1} \frac{p_k}{p_l} q_{kl}^2\Big) \qquad (l = 1, \cdots, m)$$

and $p_1 p_2 \cdots p_m = |\mathfrak{T}|$. By (79) and Lemma 7, the inequality

$$\prod_{k=1}^{m} t_k < \tau_2 \, |\mathfrak{T}| \tag{81}$$

follows, where τ_2 depends only upon m.

30. The general method for constructing a fundamental domain with respect to any discontinuous group Δ uses the minimum of the distance $\rho(\mathfrak{Z}^*, \mathfrak{Z}_1)$, where $\mathfrak{Z}_1$ is given and $\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ runs over all the images of $\mathfrak{Z}$ under Δ.

Let us now choose in particular $\mathfrak{Z}_1 = i\lambda\mathfrak{E}$, with a positive scalar factor λ; we shall investigate the asymptotic behavior of the distance $\rho(\mathfrak{Z}, \mathfrak{Z}_1)$ for $\lambda \to \infty$. By (38), we have

$$\rho^2(\mathfrak{Z}, \mathfrak{Z}_1) = \sum_{k=1}^{n} \log^2 \frac{1 + r_k^{\frac{1}{2}}}{1 - r_k^{\frac{1}{2}}},$$

where $r_1, \cdots, r_n$ denote the characteristic roots of the cross-ratio

$$\mathfrak{R} = (\mathfrak{Z} - \mathfrak{Z}_1)(\mathfrak{Z} - \bar{\mathfrak{Z}}_1)^{-1}(\bar{\mathfrak{Z}} - \bar{\mathfrak{Z}}_1)(\bar{\mathfrak{Z}} - \mathfrak{Z}_1)^{-1} = \mathfrak{E} + 2i\lambda^{-1}(\mathfrak{Z} - \bar{\mathfrak{Z}}) + \cdots.$$

If $s_1, \cdots, s_n$ are the characteristic roots of the imaginary part $\mathfrak{Y}$ of $\mathfrak{Z}$, we obtain

$$r_k = 1 - 4 s_k \lambda^{-1} + \cdots \qquad (k = 1, \cdots, n),$$

whence

$$\rho^2(\mathfrak{Z}, i\lambda\mathfrak{E}) = \sum_{k=1}^{n} \log^2 (s_k^{-1}\lambda) + \omega(\lambda),$$

where $\omega(\lambda)$ is a power series in λ^{-1} without constant term. Consequently

$$\lim_{\lambda \to \infty} (\rho(\mathfrak{Z}, i\lambda\mathfrak{E}) - n^{\frac{1}{2}} \log \lambda) = n^{-\frac{1}{2}} \log |\mathfrak{Y}|^{-1}.$$

This suggests a consideration of the minimum of $|\mathfrak{Y}^*|^{-1}$.

Denoting by the sign abs $\mathfrak{F}$ the absolute value of the determinant of a matrix $\mathfrak{F}$, we have, by (59),

$$|\mathfrak{Y}^*|^{-1} = |\mathfrak{Y}|^{-1} \operatorname{abs} (\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^2.$$

In order to obtain the minimum of $|\mathfrak{Y}^*|^{-1}$, we have therefore to determine the minimum of abs $(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})$. The existence of this minimum is by no means trivial; we shall prove it now in the case of the modular group Γ.

Let

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}, \qquad \mathfrak{M}_0 = \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{B}_0 \\ \mathfrak{C}_0 & \mathfrak{D}_0 \end{pmatrix}$$

be two elements of the homogeneous modular group Γ_0 and let

$$\mathfrak{M}_0\mathfrak{M}^{-1} = \mathfrak{M}_1 = \begin{pmatrix} \mathfrak{A}_1 & \mathfrak{B}_1 \\ \mathfrak{C}_1 & \mathfrak{D}_1 \end{pmatrix}.$$

We assume that the equation

$$\text{abs}\,(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}) = \text{abs}\,(\mathfrak{C}_0\mathfrak{Z} + \mathfrak{D}_0) \tag{82}$$

holds identically for all $\mathfrak{Z}$ in H. Introducing $(\mathfrak{D}'\mathfrak{Z} - \mathfrak{B}')(-\mathfrak{C}'\mathfrak{Z} + \mathfrak{A}')^{-1}$ instead of $\mathfrak{Z}$, we obtain the necessary and sufficient condition

$$\text{abs}\,(\mathfrak{C}_1\mathfrak{Z} + \mathfrak{D}_1) = 1. \tag{83}$$

Since $|\mathfrak{C}_1\mathfrak{Z} + \mathfrak{D}_1|$ is an analytic function of the elements z_{kl} of $\mathfrak{Z}$, we infer that $|\mathfrak{C}_1\mathfrak{Z} + \mathfrak{D}_1| = c$, identically, for all complex symmetric matrices $\mathfrak{Z}$, with a constant c of absolute value 1. Putting $\mathfrak{Z} = 0$, we find $|\mathfrak{D}_1| = c$. On the other hand, the elements of $\mathfrak{D}_1$ are rational integers; consequently $\mathfrak{D}_1$ is a unimodular matrix $\mathfrak{U}$ and $c = \pm 1$. Calculating the linear terms in the identity $|\mathfrak{D}_1^{-1}\mathfrak{C}_1\mathfrak{Z} + \mathfrak{E}| = 1$, we obtain $\sigma(\mathfrak{D}_1^{-1}\mathfrak{C}_1\mathfrak{Z}) = 0$. But the matrix $\mathfrak{D}_1^{-1}\mathfrak{C}_1$ is symmetric and therefore $\mathfrak{C}_1 = 0$.

Let now $\mathfrak{M}_1$ be any modular matrix with $\mathfrak{C}_1 = 0$. The general form is

$$\mathfrak{M}_1 = \begin{pmatrix} \mathfrak{U}'^{-1} & \mathfrak{B}\mathfrak{U} \\ 0 & \mathfrak{U} \end{pmatrix} \tag{84}$$

with unimodular $\mathfrak{U}$ and integral symmetric $\mathfrak{B}$. Obviously (79) is satisfied, and $\mathfrak{M}_0 = \mathfrak{M}_1\mathfrak{M}$, with an arbitrary modular matrix $\mathfrak{M}$, gives the general solution of (82). Then

$$\mathfrak{C}_0 = \mathfrak{U}\mathfrak{C}, \qquad \mathfrak{D}_0 = \mathfrak{U}\mathfrak{D}. \tag{85}$$

It is also easily seen that $\mathfrak{M}_1 = \mathfrak{M}_0\mathfrak{M}^{-1}$ has always the form (84), if the second matrix rows $(\mathfrak{C}\mathfrak{D})$ and $(\mathfrak{C}_0\mathfrak{D}_0)$ of two modular matrices $\mathfrak{M}$ and $\mathfrak{M}_0$ are connected by (85), with unimodular $\mathfrak{U}$. The two pairs $\mathfrak{C}, \mathfrak{D}$ and $\mathfrak{C}_0, \mathfrak{D}_0$ are called *associate*.

Denoting by $\mathfrak{Y}^*$ and $\mathfrak{Y}^*{}_0$ the imaginary parts of

$$\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} \quad \text{and} \quad \mathfrak{Z}^*{}_0 = (\mathfrak{A}_0\mathfrak{Z} + \mathfrak{B}_0)(\mathfrak{C}_0\mathfrak{Z} + \mathfrak{D}_0)^{-1},$$

we obtain, by (59),

$$\mathfrak{Y}^{-1}\{\mathfrak{Z}\mathfrak{C}'_0 + \mathfrak{D}'_0\} = \mathfrak{Y}^{*-1}[\mathfrak{U}'] = \mathfrak{Y}^*{}_0^{-1}. \tag{86}$$

Let $\mathfrak{Z}$ be a given point of H. For any modular matrix $\mathfrak{M}$, we choose a unimodular matrix $\mathfrak{U}$, such that $\mathfrak{Y}^{*-1}[\mathfrak{U}'] = \mathfrak{Y}^*{}_0^{-1}$ lies in the Minkowski domain R of Section **29.** We shall now prove that abs $(\mathfrak{C}_0\mathfrak{Z} + \mathfrak{D}_0)$ tends to infinity, if $(\mathfrak{C}_0\mathfrak{D}_0)$ runs over all second matrix rows of modular matrices with the required property of $\mathfrak{Y}^*{}_0$.

Let $y_1, \cdots, y_n$ denote the diagonal elements of $\mathfrak{Y}^*{}_0^{-1}$ and $\mathfrak{c}_l, \mathfrak{d}_l$ the l-th columns of $\mathfrak{C}'_0, \mathfrak{D}'_0$ $(l = 1, \cdots, n)$. By (86),

$$\begin{aligned} \mathfrak{Y}^*{}_0^{-1} &= \mathfrak{Y}^{-1}[\mathfrak{X}\mathfrak{C}'_0 + \mathfrak{D}'_0] + \mathfrak{Y}[\mathfrak{C}'_0]; \\ y_l &= \mathfrak{Y}^{-1}[\mathfrak{X}\mathfrak{c}_l + \mathfrak{d}_l] + \mathfrak{Y}[\mathfrak{c}_l] \end{aligned} \qquad (l = 1, \cdots, n). \tag{87}$$

Consider all solutions $\mathfrak{C}_0, \mathfrak{D}_0$ of the inequality abs $(\mathfrak{C}_0\mathfrak{Z} + \mathfrak{D}_0)^2 < a$, where a is an arbitrary positive constant. By (81) and (86),

$$\prod_{l=1}^{n} y_l < \tau_2 a \,|\, \mathfrak{Y} \,|^{-1};$$

by (87),

$$\mathfrak{Y}[\mathfrak{c}_l] \leq y_l, \qquad \mathfrak{Y}^{-1}[\mathfrak{X}\mathfrak{c}_l + \mathfrak{d}_l] \leq y_l \qquad (l = 1, \cdots, n), \tag{88}$$

and $\mathfrak{Y}^{-1}[\mathfrak{d}_l] = y_l$ in the case $\mathfrak{c}_l = 0$. Since $|\, \mathfrak{C}_0\mathfrak{Z} + \mathfrak{D}_0 \,| \neq 0$, the columns $\mathfrak{c}_l, \mathfrak{d}_l$ are not both 0; hence $y_l > y$, where y is a positive number depending only upon $\mathfrak{Y}$, and

$$y_l < \tau_2 a y^{1-n} \,|\, \mathfrak{Y} \,|^{-1}.$$

By (88), we obtain only a finite number of pairs $\mathfrak{C}_0, \mathfrak{D}_0$.

On account of (82), the existence of a modular transformation $\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ with the minimum value of abs $(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})$ is established for any $\mathfrak{Z}$ in H. We determine again $\mathfrak{U}$ by the condition that $\mathfrak{Y}^{*-1}[\mathfrak{U}']$ is a point of R and define $\mathfrak{Z}^*{}_0 = (\mathfrak{A}_0\mathfrak{Z} + \mathfrak{B}_0)(\mathfrak{C}_0\mathfrak{Z} + \mathfrak{D}_0)^{-1} = \mathfrak{Z}^*[\mathfrak{U}^{-1}] + \mathfrak{B}$, by (84), where $\mathfrak{B}$ is an arbitrary symmetric matrix with rational integral elements. We may choose $\mathfrak{B}$ such that all elements of the real part of $\mathfrak{Z}^*{}_0$ lie in the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

31. Let F be the set of all points $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$ of H satisfying the following three conditions:

$$\text{abs}\,(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}) \geq 1 \tag{89}$$

for all modular transformations $\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$;

$$L_r(\mathfrak{Y}^{-1}) \geq 0 \qquad (r = 1, \cdots, g); \tag{90}$$

$$x_{kl} \geq -\tfrac{1}{2}, \qquad -x_{kl} \geq -\tfrac{1}{2} \qquad (1 \leq k \leq l \leq n) \tag{91}$$

for the elements x_{kl} of $\mathfrak{X}$. In (89), we shall omit the trivial case $\mathfrak{C} = 0$, since the corresponding condition abs $(\mathfrak{D}) \geq 1$ holds identically for all $\mathfrak{Z}$. By the result of the preceding section, the images of F under Γ cover the whole space H.

We write $\mathfrak{Y}^{-1} = \mathfrak{P}[\mathfrak{Q}]$ with $\mathfrak{P} = [p_1, \cdots, p_n]$, $\mathfrak{Q} = (q_{kl})$, $q_{kl} = 0$ $(1 \leq l < k \leq n)$, $q_{kk} = 1$ $(k = 1, \cdots, n)$ and define $\mathfrak{W}^{(n)} = (w_{kl})$ with $w_{kl} = 0$ $(k + l \neq n + 1)$, $w_{kl} = 1$ $(k + l = n + 1)$,

$$\mathfrak{P}_1 = \begin{pmatrix} \mathfrak{P} & 0 \\ 0 & \mathfrak{P}^{-1}[\mathfrak{W}] \end{pmatrix}, \qquad \mathfrak{Q}_1 = \begin{pmatrix} \mathfrak{Q} & -\mathfrak{Q}\mathfrak{X} \\ 0 & \mathfrak{W}\mathfrak{Q}'^{-1}\mathfrak{W} \end{pmatrix}, \qquad \mathfrak{W}_1 = \begin{pmatrix} \mathfrak{E} & 0 \\ 0 & \mathfrak{W} \end{pmatrix}.$$

By (77),

$$\mathfrak{S}[\mathfrak{W}_1] = \mathfrak{P}_1[\mathfrak{Q}_1]. \tag{92}$$

By Lemma 7 and (91), the absolute values of the elements of the triangular matrix $\mathfrak{Q}_1$ are less than a number depending only upon n. Denoting the diagonal elements of $\mathfrak{P}_1$ by $d_1, \cdots, d_{2n}$, we have

$$d_k d_{k+1}^{-1} = p_k p_{k+1}^{-1} \ (1 \leq k < n), \ = p_n^2 \ (k = n), \ = p_{2n-k} p_{2n-k+1}^{-1} \ (n < k < 2n).$$

By Lemma 7,

$$0 < d_k \leq \tau_1 d_{k+1} \qquad (k \neq n), \tag{93}$$

where τ_1 depends only upon $m = 2n$.

We apply (89) for the particular modular transformation with

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{E}^{(n-1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{D} = \begin{pmatrix} \mathfrak{E}^{(n-1)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Denoting by $x_n + iy_n$ the last diagonal element of $\mathfrak{Z}$, we obtain the inequality $x_n^2 + y_n^2 \geq 1$. By (91), we have moreover $x_n^2 \leq \frac{1}{4}$, whence $y_n^2 \geq \frac{3}{4}$. But $\mathfrak{Y} = \mathfrak{P}^{-1}[\mathfrak{Q}'^{-1}]$, and consequently $y_n = p_n^{-1}$,

$$0 < d_n \leq (4/3)\, d_{n+1}. \tag{94}$$

By (92), (93) and (94), $\mathfrak{S}[\mathfrak{W}_1]$ is contained in a domain $Q(\tau_3)$ of the space P of positive symmetric matrices with $2n$ rows, where τ_3 depends only upon n.

Consider now any modular transformation $\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ different from identity, i. e.,

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} \neq \pm \mathfrak{E},$$

and assume that $\mathfrak{Z}$ and $\mathfrak{Z}^*$ both are points of F. By Lemma 6, $\mathfrak{S} = \mathfrak{S}^*[\mathfrak{M}]$,

$$\mathfrak{S}[\mathfrak{W}_1] = \mathfrak{S}^*[\mathfrak{W}_1][\mathfrak{W}_1\mathfrak{M}\mathfrak{W}_1].$$

Applying Lemma 8 with

$$\mathfrak{T}_1 = \mathfrak{S}^*[\mathfrak{W}_1], \quad \mathfrak{T}_2 = \mathfrak{S}[\mathfrak{W}_1], \quad \mathfrak{F} = \mathfrak{W}_1\mathfrak{M}\mathfrak{W}_1, \quad |\mathfrak{F}| = 1,$$

we conclude that $\mathfrak{M}$ belongs to a finite set of modular matrices $\mathfrak{M}_1, \cdots, \mathfrak{M}_h$, independent of $\mathfrak{Z}$ and $\mathfrak{Z}^*$.

On account of the minimum property of $|\mathfrak{Y}|^{-1}$, we have $|\mathfrak{Y}'| = |\mathfrak{Y}^*|$, whence abs $(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}) = 1$. If $\mathfrak{C} \neq 0$, then the sign of equality is true in one of the conditions (89). If $\mathfrak{C} = 0$, then $\mathfrak{M}$ has the form (84),

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{U}'^{-1} & \mathfrak{S}\mathfrak{U} \\ 0 & \mathfrak{U} \end{pmatrix}$$

and $\mathfrak{Y}^{*-1} = \mathfrak{Y}^{-1}[\mathfrak{U}']$. In the case $\mathfrak{U} \neq \pm \mathfrak{E}$, the sign of equality is true in one of the conditions (90). In the case $\mathfrak{U} = \pm \mathfrak{E}$, we have $\mathfrak{Z}^* = \mathfrak{Z} + \mathfrak{S}$, $\mathfrak{X}^* = \mathfrak{X} + \mathfrak{S}$ with integral symmetric $\mathfrak{S} \neq 0$, and then the sign of equality holds in one of the conditions (91).

We have proved the following statement: If two points $\mathfrak{Z}$ and $\mathfrak{Z}^*$ of F are equivalent under a modular transformation with the matrix $\mathfrak{M} \neq \pm \mathfrak{E}$, then $\mathfrak{M}$ is one of the matrices

$$\mathfrak{M}_s = \begin{pmatrix} \mathfrak{A}_s & \mathfrak{B}_s \\ \mathfrak{C}_s & \mathfrak{D}_s \end{pmatrix} \qquad (s = 1, \cdots, h)$$

and the conditions

$$\text{abs}\,(\mathfrak{C}_s\mathfrak{Z} + \mathfrak{D}_s) \geqq 1 \qquad (s = 1, \cdots, h), \tag{95}$$

$$L_r(\mathfrak{Y}^{-1}) \geqq 0 \qquad (r = 1, \cdots, g), \tag{96}$$

$$x_{kl} \geqq -\tfrac{1}{2}, \qquad -x_{kl} \geqq -\tfrac{1}{2} \qquad (1 \leqq k \leqq l \leqq n) \tag{97}$$

are fulfilled with at least one sign of equality.

32. Since (95) is contained in (89), every point $\mathfrak{Z}$ of F satisfies (95), (96) and (97). We prove, now, that the converse is true, namely, that all the inequalities (89) follow from (95), (96) and (97). We shall demonstrate at the same time, that then the stronger inequalities abs $(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}) > 1$ hold, if $\mathfrak{C}, \mathfrak{D}$ is not associate with one of the pairs $\mathfrak{C}_s, \mathfrak{D}_s$ $(s = 1, \cdots, h)$.

LEMMA 9. *Let $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$ be a point of H and $\mathfrak{Z}_\lambda = \mathfrak{X} + i\lambda\mathfrak{Y}$, with an arbitrary scalar factor λ. If $(\mathfrak{C}\mathfrak{D})$ is the second matrix row of any symplectic matrix and $\mathfrak{C} \neq 0$, then the inequality* abs $(\mathfrak{C}\mathfrak{Z}_\lambda + \mathfrak{D}) >$ abs $(\mathfrak{C}\mathfrak{Z}_\mu + \mathfrak{D})$ *holds for $\lambda > \mu > 0$.*

The determinant $|\mathfrak{C}\mathfrak{Z}^\lambda + \mathfrak{D}| = \phi(\lambda)$ is a polynomial in λ. For any λ with positive real part, $\mathfrak{Z}_\lambda$ is a point of H and consequently $\phi(\lambda) \neq 0$. Moreover $\overline{\phi(\lambda)} = \phi(-\bar{\lambda})$; hence all zeros of $\phi(\lambda)$ are pure imaginary. This proves that for real λ the expression abs $(\mathfrak{C}\mathfrak{Z}_\lambda + \mathfrak{D})^2$ is a polynomial in λ^2 with real non-negative coefficients. It remains only to prove that this polynomial is not identically constant.

Assume now that

$$\text{abs}\,(\mathfrak{C}\mathfrak{Z}^\lambda + \mathfrak{D}) = \text{abs}\,(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D}) = \text{abs}\,(\mathfrak{C}\mathfrak{X} + \mathfrak{D}),$$

identically in λ. Then abs $(\mathfrak{C}\mathfrak{X}+\mathfrak{D}) \neq 0$ and

$$\text{abs}\,(\mathfrak{Y}^{-1}+i\lambda(\mathfrak{C}\mathfrak{X}+\mathfrak{D})^{-1}\mathfrak{C}) = |\,\mathfrak{Y}\,|^{-1}. \tag{98}$$

Since $(\mathfrak{C}\mathfrak{X}+\mathfrak{D})^{-1}\mathfrak{C}$ is a real symmetric matrix and $\mathfrak{Y}^{-1} > 0$, there exists a real matrix $\mathfrak{K}^{(n)}$ and a diagonal matrix $\mathfrak{R} = [r_1, \cdots, r_n]$, such that

$$\mathfrak{Y}^{-1}[\mathfrak{K}] = \mathfrak{E}, \qquad ((\mathfrak{C}\mathfrak{X}+\mathfrak{D})^{-1}\mathfrak{C})[\mathfrak{K}] = \mathfrak{R}. \tag{99}$$

By (98) and (99),

$$\prod_{k=1}^{n}(1+r_k{}^2\lambda^2) = 1$$

for all real values of λ, and consequently $\mathfrak{R} = 0$, $\mathfrak{C} = 0$, which is a contradiction. This completes the proof of the lemma.

We denote again by $(\mathfrak{C}\mathfrak{D})$ the second matrix rows of the modular matrices. Let $\mathfrak{Z}$ be a point of H satisfying all conditions (95), (96) and (97), and assume that the inequality abs $(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}) \leq 1$ holds for at least one pair $\mathfrak{C}, \mathfrak{D}$, where $\mathfrak{C} \neq 0$ and $\mathfrak{C}, \mathfrak{D}$ is not associate with one of the pairs $\mathfrak{C}_s, \mathfrak{D}_s$ $(s = 1, \cdots, h)$. By the result of Section **30**, only a finite number of non-associate pairs $\mathfrak{C}, \mathfrak{D}$ fulfill that inequality. By Lemma 9, there exists a number $\lambda \geq 1$, such that abs $(\mathfrak{C}\mathfrak{Z}_\lambda+\mathfrak{D}) \geq 1$ for all $\mathfrak{C}, \mathfrak{D}$ and abs $(\mathfrak{C}\mathfrak{Z}_\lambda+\mathfrak{D}) = 1$ for $\mathfrak{C} = \mathfrak{C}_0$, $\mathfrak{D} = \mathfrak{D}_0$, where $\mathfrak{C}_0 \neq 0$ and $\mathfrak{C}_0, \mathfrak{D}_0$ is not associate with one of the pairs $\mathfrak{C}_s, \mathfrak{D}_s$. Since $\mathfrak{Z}_\lambda$ has the real part $\mathfrak{X}$ and $L_r(\lambda^{-1}\mathfrak{Y}^{-1}) = \lambda^{-1}L_r(\mathfrak{Y}^{-1})$, all conditions (89), (90) and (91) are satisfied for $\mathfrak{Z}_\lambda$ instead of $\mathfrak{Z}$; hence $\mathfrak{Z}_\lambda$ is a point of F. On the other hand, the expression abs $(\mathfrak{C}\mathfrak{Z}_\lambda+\mathfrak{D})$ attains its minimum 1 for $\mathfrak{C} = \mathfrak{C}_0$, $\mathfrak{D} = \mathfrak{D}_0$, and consequently there exists a modular transformation $\mathfrak{Z}^*_\lambda = (\mathfrak{A}\mathfrak{Z}_\lambda+\mathfrak{B})(\mathfrak{C}\mathfrak{Z}_\lambda+\mathfrak{D})^{-1}$, such that $\mathfrak{C}, \mathfrak{D}$ is associate with $\mathfrak{C}_0, \mathfrak{D}_0$ and $\mathfrak{Z}^*_\lambda$ is a point of F. By Section **31**, $\mathfrak{C} = \mathfrak{C}_s$, $\mathfrak{D} = \mathfrak{D}_s$, and this is a contradiction. Consequently abs $(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}) \geq 1$ for all $\mathfrak{C}, \mathfrak{D}$ and abs $(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}) > 1$, if $\mathfrak{C} \neq 0$ and $\mathfrak{C}, \mathfrak{D}$ is not associate with one of the pairs $\mathfrak{C}_s, \mathfrak{D}_s$.

33. By the result of the preceding section, F may be defined by the inequalities (95), (96) and (97), in finite number. Obviously F is closed relative to H. It follows from Lemma 9 and the linearity of the conditions (96) and (97), that $\mathfrak{Z}$ is a frontier point of F, if, and only if, (95), (96) and (97) are fulfilled with at least one sign of equality.

Let $F_{\mathfrak{M}}$ be the image of F under the modular transformation $\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z}+\mathfrak{B})(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-1}$ with the matrix $\mathfrak{M} \neq \pm\,\mathfrak{E}$. If F and $F_{\mathfrak{M}}$ have a point $\mathfrak{Z}^*$ in common, then, by Section **31**, $\mathfrak{M}$ is one of the matrices $\mathfrak{M}_1, \cdots, \mathfrak{M}_h$, and $\mathfrak{Z}^*$ is a frontier point of F. Consequently the images $F_{\mathfrak{M}}$ cover H without overlappings, and F has only a finite number of neighbors.

Consider any compact domain G in H, and let G_1 be the corresponding domain in the space S of the matrices $\mathfrak{S}$ defined in Lemma 6. There exists a number t depending only upon G, such that G_1 is contained in the domain $Q(t)$ of Section **29**. We may choose $t \geqq \tau_3$, where τ_3 was defined in Section **31**. Let $\mathfrak{Z}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ be a common point of $F_{\mathfrak{M}}$ and G. Then $\mathfrak{Z}$ is a point of F and the relationship $\mathfrak{S}^* = \mathfrak{S}[\mathfrak{M}^{-1}]$ holds for the corresponding points $\mathfrak{S}^*$ and $\mathfrak{S}$ of S, by Lemma 6. It follows from the result of Section **31**, that the point $\mathfrak{S}[\mathfrak{W}_1] = \mathfrak{S}^*[\mathfrak{M}\mathfrak{W}_1]$ lies in $Q(t)$. But also $\mathfrak{S}^*$ itself is a point of $Q(t)$, and consequently, by Lemma 8, the matrix $\mathfrak{M}$ belongs to a finite set. This proves that only a finite number of images $F_{\mathfrak{M}}$ enter into the compact domain G.

For the particular value $\mathfrak{Z} = i\mathfrak{E}$, we have $| \mathfrak{C}\mathfrak{Z} + \mathfrak{D} | = c + id$, with rational integers c, d not both 0. Consequently (89) is satisfied. Also (91) holds, since $\mathfrak{X} = 0$. Moreover $\mathfrak{Y}^{-1} = \mathfrak{E}$ is a point of the Minkowski domain R. Consequently $\mathfrak{Z} = i\mathfrak{E}$ is a point of F. By Lemma 9, the whole curve $\mathfrak{Z} = i\lambda\mathfrak{E}$ $(\lambda \geqq 1)$ belongs to F. Since λ may be arbitrarily large, the fundamental domain F is not compact. Let G be any compact domain in H and consider the finite set of modular matrices $\mathfrak{M}$, such that $F_{\mathfrak{M}}$ enters into G. The set of images of G under the inverse mappings with the matrices $\mathfrak{M}^{-1}$ constitutes again a compact domain G_0. For sufficiently large values of λ, the point $i\lambda\mathfrak{E}$ of F does not lie in G_0; hence no image of this point lies in G. This proves that the space H is not compact relative to the modular group Γ.

By the results of Section **31**, the matrices $\mathfrak{X}$ and $\mathfrak{Y}^{-1}$ are bounded for all $\mathfrak{Z}$ in F. On account of (57), the integral $V(\Gamma)$ converges.

Theorem 8 is now completely proved.

VII. THE FUNDAMENTAL DOMAIN OF THE GROUP $\Delta(\mathfrak{G}, \mathfrak{H})$.

34. Let K_0 be an algebraic number field, of finite degree g over the field of rational numbers. Let g_1 of the conjugate fields be real and $2g_2$ imaginary, $g = g_1 + 2g_2$. We denote the real conjugate fields by $K_0^{(\alpha)}$ $(\alpha = 1, \cdots, g_1)$ and the pairs of conjugate complex conjugate fields by $K_0^{(\alpha)}$ and $K_0^{(\alpha+g_2)}$ $(\alpha = g_1 + 1, \cdots, g_1 + g_2)$. We consider g_1 positive symmetric matrices $\mathfrak{T}_\alpha$ $(\alpha = 1, \cdots, g_1)$ with real elements and g_2 positive hermitian matrices $\mathfrak{T}_\alpha$ $(\alpha = g_1 + 1, \cdots, g_1 + g_2)$ with complex elements, all of m rows. We denote the systems of $g_1 + g_2$ matrices $\mathfrak{T}_\alpha$ $(\alpha = 1, \cdots, g_1 + g_2)$ more shortly by $\mathfrak{T}$; they form the points of a space P of $\frac{1}{2}g_1 m(m+1) + g_2 m^2$ dimensions.

We have a unique decomposition $\mathfrak{T}_\alpha = \mathfrak{P}_\alpha\{\mathfrak{Q}_\alpha\}$ with a diagonal matrix

$\mathfrak{P}_\alpha = [p_1^{(\alpha)}, \cdots, p_m^{(\alpha)}]$, $p_k^{(\alpha)} > 0$ $(k = 1, \cdots, m)$, and a triangular matrix $\mathfrak{Q}_\alpha = (q_{kl}^{(\alpha)})$, $q_{kl}^{(\alpha)} = 0$ $(k > l)$, $q_{kk}^{(\alpha)} = 1$, where $\mathfrak{Q}_\alpha$ is real for $\alpha = 1, \cdots, g_1$. If t is any positive number > 1, the inequalities

$$0 < p_k^{(\alpha)} \leq t p_{k+1}^{(\alpha)} \quad (k < m), \qquad p_k^{(\alpha)} \leq t p_k^{(\beta)} \quad (k \leq m)$$
$$\operatorname{abs} q_{kl}^{(\alpha)} \leq t \qquad (k < l)$$

with $\alpha, \beta = 1, \cdots, g_1 + g_2$ define a compact domain $Q(t)$ in P.

A matrix $\mathfrak{U}$ with integral elements in K_0 is called unimodular if the determinant $|\mathfrak{U}|$ is an algebraic unit. The unimodular matrices $\mathfrak{U}^{(m)}$ constitute the unimodular group $\mathfrak{U}$ in K_0, of degree m. The center C of U consists of the matrices $\mathfrak{U} = u\mathfrak{E}$, where u is any root of unity in K_0. We denote by U_0 the factor group U/C. Let $\mathfrak{U}_\alpha$ be the conjugate of $\mathfrak{U}$ in $K_0^{(\alpha)}$. The transformation $\mathfrak{T}_\alpha \rightarrow \mathfrak{T}_\alpha\{\mathfrak{U}_\alpha\}$ $(\alpha = 1, \cdots, g_1 + g_2)$, or more shortly $\mathfrak{T} \rightarrow \mathfrak{T}\{\mathfrak{U}\}$, maps the space P onto itself. This mapping is the identical one, if and only if $\mathfrak{U}$ is an element of C; consequently the transformations $\mathfrak{T} \rightarrow \mathfrak{T}\{\mathfrak{U}\}$ give a faithful representation of U_0.

Minkowski's theory of reduction of positive quadratic forms is concerned with the case $g = 1$, the field of rational numbers. The generalization to the case of an arbitrary field K_0 is due to P. Humbert. He obtained the following results:

There exists in P a fundamental domain R with respect to U_0, which is the union of a finite number of convex pyramids. The faces of these pyramids have equations of the form

$$\sum_{\alpha=1}^{g_1+g_2} (\mathfrak{a}'_\alpha \mathfrak{T}_\alpha \bar{\mathfrak{b}}_\alpha + \bar{\mathfrak{a}}'_\alpha \bar{\mathfrak{T}}_\alpha \mathfrak{b}_\alpha) = 0, \tag{100}$$

where $\mathfrak{a}_\alpha$ and $\mathfrak{b}_\alpha$ are the conjugates of two columns $\mathfrak{a} \neq 0$ and $\mathfrak{b} \neq 0$ in K_0; moreover $\mathfrak{a} \neq \lambda\mathfrak{b}$ for every pure imaginary scalar factor λ. Any compact domain in P is covered by a finite number of images $R_{\mathfrak{U}}$ of R, and R has only a finite number of neighbors $R_{\mathfrak{U}}$.

LEMMA 10. *There exists a finite set L of matrices $\mathfrak{L}$ with integral elements in K_0 and a positive number τ_4 depending only upon K_0 and m, such that for every $\mathfrak{T}$ in R the point $\mathfrak{T}\{\mathfrak{L}\}$ belongs to $Q(\tau_4)$, with at least one $\mathfrak{L}$ of the set L.*

LEMMA 11. *Let $\mathfrak{T}$ and $\mathfrak{T}\{\mathfrak{F}\}$ be two points of $Q(t)$, where $\mathfrak{F}$ is an integral matrix in K_0, and let ν be the norm of $|\mathfrak{F}|$. Then $\mathfrak{F}$ belongs to a finite set of matrices depending only upon K_0, m, t and ν.*

These statements are generalizations of the lemmata 7 and 8.

35. We assume now that $K_0 = K(\sqrt{-r})$, where K is a totally real algebraic number field of degree h and r a totally positive number of K; then $g_1 = 0$, $g_2 = h$. Let $K^{(a)}$ $(\alpha = 1, \cdots, h)$ be the conjugates of $K = K^{(1)}$, r_a the conjugate of r in $K^{(a)}$ and $K_0{}^{(a)} = K^{(a)}(\sqrt{-r_a})$. In the notation of Chapter V, $\mathfrak{H}$ is a hermitian matrix and $\mathfrak{G}$ is a skew-symmetric matrix, both in K_0, satisfying the condition (67). Let $\mathfrak{H}_a, \mathfrak{G}_a$ be the conjugates of $\mathfrak{H}, \mathfrak{G}$ in $K_0{}^{(a)}$ and s_a the conjugate of s in $K^{(a)}$. We assumed moreover $s > 0$, $\mathfrak{H}_a > 0$ for $\alpha = 2, \cdots, h$; then $s_a < 0$ $(\alpha = 2, \cdots, h)$. The group $\Lambda = \Lambda(\mathfrak{G}, \mathfrak{H})$ consists of all unimodular matrices $\mathfrak{W}$ in K_0 which satisfy $\mathfrak{G}[\mathfrak{W}] = \mathfrak{G}$ and $\mathfrak{H}\{\mathfrak{W}\} = \mathfrak{H}$. We have $\Delta_0 = \mathfrak{C}^{-1}\Lambda\mathfrak{C}$, where $\mathfrak{C}$ is a complex matrix with $\mathfrak{G}[\mathfrak{C}] = \mathfrak{J}$ and $\mathfrak{H}\{\mathfrak{C}\} = is^{\frac{1}{2}}\mathfrak{J}$. Identifying $\mathfrak{W}$ and $-\mathfrak{W}$, we obtain the factor group $\Delta = \Delta(\mathfrak{G}, \mathfrak{H})$ of Δ_0. Obviously this group is not changed, if we replace $\mathfrak{G}, \mathfrak{H}$ by $a\mathfrak{G}, b\mathfrak{H}$ with arbitrary positive rational numbers a, b; consequently we may assume that $\mathfrak{G}$ and $\mathfrak{H}$ have integral elements.

The matrix $\mathfrak{S}$ of (77) is the general solution of $\mathfrak{S}' = \bar{\mathfrak{S}} > 0$, $\mathfrak{J}[\mathfrak{S}] = \mathfrak{J}$, $\mathfrak{J}\{\mathfrak{S}\} = \mathfrak{J}$. Consequently

$$\mathfrak{T}_1 = \mathfrak{S}\{\mathfrak{C}^{-1}\} \tag{101}$$

is the general solution of

$$\mathfrak{T}'_1 = \bar{\mathfrak{T}}_1 > 0, \quad \mathfrak{T}_1\bar{\mathfrak{G}}^{-1}\bar{\mathfrak{T}}_1 = -\mathfrak{G}, \quad \mathfrak{T}_1\mathfrak{H}^{-1}\mathfrak{T}_1 = s^{-1}\mathfrak{H}. \tag{102}$$

We define

$$\mathfrak{T}_a = (-s_a)^{-\frac{1}{2}}\mathfrak{H}_a \qquad (\alpha = 2, \cdots, h); \tag{103}$$

then, by (67) and (102),

$$\mathfrak{T}'_a = \bar{\mathfrak{T}}_a > 0, \quad \mathfrak{T}_a = \bar{\mathfrak{T}}_a{}^{-1}\{\mathfrak{G}_a\}, \quad \mathfrak{T}_a = |\, s_a \,|^{-1}\mathfrak{T}_a{}^{-1}\{\mathfrak{H}'_a\} \qquad (\alpha = 1, \cdots, h), \tag{104}$$

where $|\, s_a \,|$ denotes the absolute value of s_a. The matrices $\mathfrak{T}_2, \cdots, \mathfrak{T}_h$ are fixed, whereas $\mathfrak{T}_1$ depends upon the variable point $\mathfrak{Z}$ of H, by (77). The space H is mapped onto a surface T, of $n(n+1)$ dimensions, in the space R. If $\mathfrak{W}$ is any element of the group Λ, then the transformation $\mathfrak{T}_a \rightarrow \mathfrak{T}_a\{\mathfrak{W}_a{}^{-1}\}$ $(\alpha = 1, \cdots, h)$ maps T onto itself, and this mapping is the identical one, if and only if $\mathfrak{W} = \pm\mathfrak{E}$; on the other hand, by Lemma 6, the corresponding mapping in H is a symplectic transformation of $\mathfrak{Z}$, with the matrix $\mathfrak{M} = \mathfrak{C}^{-1}\mathfrak{W}\mathfrak{C}$ of the group Δ.

For any point $\mathfrak{T}$ of T, there exists a unimodular matrix $\mathfrak{U} = \mathfrak{U}_{\mathfrak{Z}}$ in K_0, such that $\mathfrak{T}\{\mathfrak{U}\}$ is a point of the domain R. By Lemma 10, we may choose a matrix $\mathfrak{L}$ of the finite set L, such that $\hat{\mathfrak{T}} = \mathfrak{T}\{\mathfrak{U}\mathfrak{L}\}$ belongs to $Q(\tau_4)$. Putting $\hat{\mathfrak{G}} = \mathfrak{G}[\mathfrak{U}\mathfrak{L}]$, $\hat{\mathfrak{H}} = \mathfrak{H}\{\mathfrak{U}\mathfrak{L}\}$ we obtain, by (104),

$$(105) \qquad \hat{\mathfrak{T}} = \bar{\hat{\mathfrak{T}}}^{-1}\{\hat{\mathfrak{G}}\}, \qquad \hat{\mathfrak{T}} = |\, s \,|^{-1}\hat{\mathfrak{T}}^{-1}\{\hat{\mathfrak{H}}'\};$$

we omit here and in the following formulae the index α which runs always from 1 to h.

Let $\mathfrak{V}$ denote the matrix of the linear transformation $x_k \rightarrow x_{2n-k+1}$ $(k = 1, \cdots, 2n)$. If $\hat{\mathfrak{T}} = \mathfrak{P}\{\mathfrak{Q}\}$ is the decomposition defined in Section **34**, then $\bar{\hat{\mathfrak{T}}}^{-1}\{\mathfrak{V}\} = \mathfrak{P}^{-1}\{\mathfrak{V}\}\{\mathfrak{V}\bar{\mathfrak{Q}}'^{-1}\mathfrak{V}\}$, and consequently the points $|\, s \,|^{-1}\hat{\mathfrak{T}}^{-1}\{\mathfrak{V}\}$ and $\hat{\mathfrak{T}}^{-1}\{\mathfrak{V}\}$ belong to a domain $Q(\tau_5)$, where τ_5 depends only upon K_0, n and s. Moreover

$$|\, \mathfrak{V}\hat{\mathfrak{H}}' \,| = (-1)^n \,|\, \mathfrak{H} \,|\, \operatorname{abs}(\mathfrak{U}\mathfrak{L})^2, \qquad |\, \mathfrak{V}\hat{\mathfrak{G}} \,| = (-1)^n \,|\, \mathfrak{G} \,|\; |\, \mathfrak{U}\mathfrak{L} \,|^2;$$

hence the norms of $|\, \mathfrak{V}\hat{\mathfrak{H}}' \,|$ and $|\, \mathfrak{V}\hat{\mathfrak{G}} \,|$ belong to a finite set. It follows now from Lemma 11 and (105), that also the matrices $\hat{\mathfrak{G}}$ and $\hat{\mathfrak{H}}$ belong to a finite set, independent of $\mathfrak{Z}$, and the same holds good for $\mathfrak{G}[\mathfrak{U}] = \hat{\mathfrak{G}}[\mathfrak{L}^{-1}]$ and $\mathfrak{H}\{\mathfrak{U}\} = \hat{\mathfrak{H}}\{\mathfrak{L}^{-1}\}$.

Choose now a complete system of points $\mathfrak{Z}_0$ in H, such that the pairs $\mathfrak{G}[\mathfrak{U}_0]$, $\mathfrak{H}\{\mathfrak{U}_0\}$ with $\mathfrak{U}_0 = \mathfrak{U}_{\mathfrak{Z}_0}$ are all different, and let V be the finite set of the unimodular matrices $\mathfrak{U}_0$. We denote by $G(\mathfrak{U}_0)$ the closed set of all points $\mathfrak{Z}$ of H, such that the corresponding point $\mathfrak{T}$ of T lies in $R_{\mathfrak{U}_0^{-1}}$, and by G the union of these $G(\mathfrak{U}_0)$, as $\mathfrak{U}_0$ runs over the elements of the set V.

Let $\mathfrak{Z}$ be again an arbitrary point of H and $\mathfrak{U} = \mathfrak{U}_{\mathfrak{Z}}$. Then there exists a uniquely determined $\mathfrak{U}_0$ in V, such that $\mathfrak{G}[\mathfrak{U}_0] = \mathfrak{G}[\mathfrak{U}]$ and $\mathfrak{H}\{\mathfrak{U}_0\} = \mathfrak{H}\{\mathfrak{U}\}$; thus $\mathfrak{U}_0\mathfrak{U}^{-1} = \mathfrak{W}$ is an element of the group Λ. Since $\mathfrak{T}\{\mathfrak{U}\}$ lies in R, the point $\mathfrak{T}\{\mathfrak{W}^{-1}\} = \mathfrak{T}\{\mathfrak{U}\mathfrak{U}_0^{-1}\}$ is contained in $R_{\mathfrak{U}_0^{-1}}$; hence $\mathfrak{Z}$ is mapped by the element $\mathfrak{M} = \mathfrak{C}^{-1}\mathfrak{W}\mathfrak{C}$ of Δ into a point of $G(\mathfrak{U}_0)$. This proves that any $\mathfrak{Z}$ in H is under Δ equivalent to at least one point of G; we call this point a *reduced image* of $\mathfrak{Z}$.

36. Let us assume that there exists in H a compact domain B, such that $\mathfrak{T}\{\mathfrak{U}_{\mathfrak{Z}}\}$ is a boundary point of R, relative to P, for all $\mathfrak{Z}$ in B. By Section **34**, the unimodular matrix $\mathfrak{U}_{\mathfrak{Z}}$ belongs then to a finite set, and we infer from (77), (100), (101), (103), that the expression $\mathfrak{p}'\mathfrak{S}\bar{\mathfrak{q}} + \bar{\mathfrak{p}}'\mathfrak{S}\mathfrak{q}$ has a constant value in H, where $\mathfrak{p} = \mathfrak{C}^{-1}\mathfrak{a}$ and $\mathfrak{q} = \mathfrak{C}^{-1}\mathfrak{b}$ with two columns $\mathfrak{a} \neq 0$ and $\mathfrak{b} \neq 0$ in K_0; moreover $\mathfrak{a} \neq \lambda\mathfrak{b}$ for every pure imaginary scalar factor λ. Replacing $\mathfrak{Y}$ by $\mathfrak{E} + \mathfrak{Y}$, we have the Taylor series

$$\mathfrak{S} = \mathfrak{E} + \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots, \qquad \mathfrak{S}_1 = \begin{pmatrix} -\mathfrak{Y} & -\mathfrak{X} \\ -\mathfrak{X} & \mathfrak{Y} \end{pmatrix}, \qquad \mathfrak{S}_2 = \begin{pmatrix} \mathfrak{Y}^2 & \mathfrak{Y}\mathfrak{X} \\ \mathfrak{X}\mathfrak{Y} & \mathfrak{X}^2 \end{pmatrix},$$

in the neighborhood of $\mathfrak{Y} = 0$. It follows that the real part of $\mathfrak{p}'\mathfrak{S}_2\bar{\mathfrak{q}}$ vanishes identically in the real symmetric matrices $\mathfrak{X}$ and $\mathfrak{Y}$. Since $\mathfrak{a} = \mathfrak{C}\mathfrak{p}$, $\mathfrak{b} = \mathfrak{C}\mathfrak{q}$

and $\mathfrak{a} \neq \lambda\mathfrak{b}$, for all pure imaginary values of λ, we find easily $\mathfrak{a} = \omega\mathfrak{C}\mathfrak{r}$, $\mathfrak{b} = i\omega\mathfrak{C}\mathfrak{s}$ with real $\mathfrak{r}, \mathfrak{s}$ and complex scalar $\omega \neq 0$.

On the other hand $\mathfrak{S}[\mathfrak{C}] = \mathfrak{J}$, $\mathfrak{H}\{\mathfrak{C}\} = \sqrt{-s}\,\mathfrak{J}$, whence $\bar{\mathfrak{C}} = \sqrt{-s}\,\mathfrak{H}^{-1}\mathfrak{S}\mathfrak{C}$. We obtain, therefore,

$$\bar{\omega}\sqrt{-s}\,\mathfrak{H}^{-1}\mathfrak{S}\mathfrak{a} = \omega\bar{\mathfrak{a}}.$$

But $\mathfrak{S}, \mathfrak{H}$ and $\mathfrak{a} \neq 0$ are in $K_0 = K(\sqrt{-r})$, hence

$$\bar{\omega}\sqrt{-s} = \omega(\xi + \eta\sqrt{-r})\,; \qquad s = \xi^2 + r\eta^2, \tag{106}$$

where ξ, η are numbers in K. Since $r_a > 0$, $s_a < 0$ $(a = 2, \cdots, h)$, the relationship (106) is only possible for $h = 1$; then K is the field of rational numbers.

It will be proved in Chapter IX, that $\Delta(\mathfrak{S}, \mathfrak{H})$ is commensurable with the modular group Γ, if and only if the diophantine equation (106) has a rational solution ξ, η. In this case, however, the construction and the properties of a fundamental domain for Δ follow in a simple way from the results of Chapter VI. Therefore we exclude this case for the rest of the present chapter.

For every point $\mathfrak{Z}_0$ of H, there exists now a unimodular matrix $\mathfrak{U}$ and a sequence of points $\mathfrak{Z}$ tending to $\mathfrak{Z}_0$, such that the corresponding points $\mathfrak{T}$ of T are inner points of $R_{\mathfrak{U}^{-1}}$, relative to P.

37. We denote by $F(\mathfrak{U}_0)$ the closure of the set of inner points of $G(\mathfrak{U}_0)$, relative to H. If $\mathfrak{Z}_0$ is a point of $G(\mathfrak{U}_0)$ which does not belong to $F(\mathfrak{U}_0)$, then we use the result of the last section to construct a sequence of points $\mathfrak{Z}$ tending to $\mathfrak{Z}_0$, such that they have as reduced images inner points of $G(\mathfrak{U})$, where $\mathfrak{U}$ is a certain matrix of the set V. Consequently $\mathfrak{Z}_0$ has a reduced image in $F(\mathfrak{U})$, and any point of H is equivalent under Δ to at least one point in one of the domains $F(\mathfrak{U}_0)$.

Let $\mathfrak{Z}^*$ be the image of $\mathfrak{Z}$ under the transformation $\mathfrak{M} = \mathfrak{C}^{-1}\mathfrak{W}\mathfrak{C}$ of Δ and let $\mathfrak{T}^*, \mathfrak{T}$ be the corresponding points in T. We assume now the existence of an inner point $\mathfrak{Z}$ of $F(\mathfrak{U}_0)$, such that $\mathfrak{Z}^*$ is a point of $F(\mathfrak{U}^*_0)$, where $\mathfrak{U}^*_0$ is also one of the matrices of the set V. By Section **36**, this holds then even under the further condition that $\mathfrak{T}\{\mathfrak{U}_0\}$ be an inner point of R, relative to P. But $\mathfrak{T}^*\{\mathfrak{U}^*_0\} = \mathfrak{T}\{\mathfrak{U}_0\}\{\mathfrak{U}_0^{-1}\mathfrak{W}^{-1}\mathfrak{U}^*_0\}$ is again a point of R, and consequently $\mathfrak{U}^*_0 = \pm\,\mathfrak{W}\mathfrak{U}_0$, $\mathfrak{S}[\mathfrak{U}^*_0] = \mathfrak{S}[\mathfrak{U}_0]$, $\mathfrak{H}\{\mathfrak{U}^*_0\} = \mathfrak{H}\{\mathfrak{U}_0\}$. Now it follows from the definition of V, that $\mathfrak{U}^*_0 = \mathfrak{U}_0$, $\mathfrak{W} = \pm\,\mathfrak{E}$. This proves that $F(\mathfrak{U}_0)$ and $F(\mathfrak{U}^*_0)$ do not overlap, if $\mathfrak{U}_0 \neq \mathfrak{U}^*_0$, and that the sum of the domains $F(\mathfrak{U}_0)$ is a fundamental domain F of Δ.

Obviously every $F(\mathfrak{U}_0)$ is bounded by a finite number of algebraic surfaces, and F has the same property. It follows immediately from Section **34**, that any compact domain in H is covered by a finite number of images $F_{\mathfrak{M}}$ of F under Δ and that F has only a finite number of neighbors $F_{\mathfrak{M}}$.

38. Let $\mathfrak{Z}$ be a point of $F(\mathfrak{U}_0)$ and $\mathfrak{S}$ the corresponding point of S. By (101) and (105),

$$\mathfrak{P} = s^{-1}\mathfrak{P}^{-1}\{\hat{\mathfrak{H}}'\{\bar{\mathfrak{Q}}^{-1}\}\}, \tag{107}$$

where

$$\left\{\begin{array}{lll} \hat{\mathfrak{H}} = \mathfrak{H}\{\mathfrak{U}_0\mathfrak{L}\}, & \mathfrak{P}\{\mathfrak{Q}\} = \mathfrak{T}_1\{\mathfrak{U}_0\mathfrak{L}\} = \mathfrak{S}\{\mathfrak{C}^{-1}\mathfrak{U}_0\mathfrak{L}\} & \\ \mathfrak{P} = [p_1, \cdots, p_{2n}], & 0 < p_k \leq \tau_4 p_{k+1} & (k = 1, \cdots, 2n-1) \\ \mathfrak{Q} = (q_{kl}), \quad q_{kl} = 0 \ (k > l), & q_{kk} = 1, \quad \operatorname{abs} q_{kl} \leq \tau_4 & (k < l). \end{array}\right. \tag{108}$$

Let d be the first diagonal element of $\hat{\mathfrak{H}}$. Then

$$p_1{}^2 \geq s^{-1}d^2 \tag{109}$$

and

$$\prod_{k=1}^{2n} p_k = (-s)^{-n} \mid \hat{\mathfrak{H}} \mid, \tag{110}$$

by (107).

We assume now $h > 1$, i. e., K is not the field of rational numbers. Then the conjugates of $\hat{\mathfrak{H}}$ in $K^{(\alpha)}(\sqrt{-r_\alpha})$ $(\alpha = 2, \cdots, h)$ are positive, and consequently $d \neq 0$. On the other hand, $\mathfrak{U}_0$ and $\mathfrak{L}$ belong to a finite set. It follows from (108), (109), (110), that $\mathfrak{S}$ is bounded for all $\mathfrak{Z}$ in the fundamental domain F. By (77), the matrices $\mathfrak{Y}^{-1}$, $\mathfrak{Y} + \mathfrak{Y}^{-1}[\mathfrak{X}]$, $\mathfrak{Y}^{-1}\mathfrak{X}$ are bounded, hence also $\mathfrak{Y}$, $\mid \mathfrak{Y} \mid^{-1}$, $\mathfrak{X}$. This proves that F is compact.

In the remaining case $h = 1$, F is not necessarily compact, and the proof of the convergence of the volume integral $V(\Delta)$ requires more detailed estimates. This proof may be given by the same method which leads to the analogous result in the theory of units of indefinite quadratic forms; we omit it here.

It is also not difficult to prove that the space H is compact relative to $\Delta(\mathfrak{G}, \mathfrak{H})$, in the case $h = 1$, if and only if $n = 1$ and Δ is not commensurable with the elliptic modular group, and then also F is compact.

39. The congruence subgroup $\Lambda_\kappa(\mathfrak{G}, \mathfrak{H})$ of $\Lambda(\mathfrak{G}, \mathfrak{H})$ consists of all elements of Λ satisfying $\mathfrak{W} \equiv \mathfrak{E} \pmod{\kappa}$, where κ is a given ideal in K_0. Let ρ be a prime ideal of K_0 and p the rational prime number which is divisible by ρ; let κ be a power of ρ, such that p is not divisible by κ^{p-1}. If the transformation with the matrix $\mathfrak{C}^{-1}\mathfrak{W}\mathfrak{C} \neq \pm \mathfrak{E}$ has a fixed point in H, then $\mathfrak{W}$ is of finite order, since Δ is discontinuous in H.

We shall prove that $\Lambda_\kappa(\mathfrak{G}, \mathfrak{H})$ contains no element of finite order except $\mathfrak{E}$. Otherwise we may assume

$$\mathfrak{W}^q = \mathfrak{E}, \quad \mathfrak{W} = \mathfrak{E} + \mathfrak{R}, \quad \mathfrak{R} \equiv 0 \pmod{\rho^a}, \quad \mathfrak{R} \not\equiv 0 \pmod{\rho^{a+1}},$$

where q is a rational prime number and ρ^a divisible by κ. Then

$$\sum_{l=1}^{q} \binom{q}{l} \mathfrak{R}^l = 0; \quad \mathfrak{R} \equiv -q^{-1}\mathfrak{R}^q \pmod{\rho^{2a}}.$$

Since

$$q^{-1}\mathfrak{R}^q \equiv 0 \pmod{\rho^{a+1}},$$

we arrive at a contradiction.

The proofs of Theorems 9 and 10 are finished.

VIII. THE VOLUME OF THE FUNDAMENTAL DOMAIN OF THE MODULAR GROUP.

40. In the interval $0 \leq x \leq 1$, we consider an arbitrary monotone function $f(x)$, such that

$$f(1) = 0, \qquad \int_0^1 f(x) x^{n-1} dx = 1; \tag{111}$$

an example is $f(x) = n(n+1)(1-x)$. For $x \geq 1$, we put $f(x) = 0$. Let $\mathfrak{Z}$ be a point of H and $\mathfrak{S}$ the positive symmetric matrix defined in (77). For any $\epsilon > 0$, we define

$$\phi(\epsilon, \mathfrak{Z}) = \epsilon^{2n} \sum_{\mathfrak{w} \neq 0} f(\mathfrak{S}[\epsilon\mathfrak{w}]), \tag{112}$$

where $\mathfrak{w}$ runs over all lattice points $\neq 0$; this is a finite sum.

Lemma 12. *If ϵ tends to 0 through positive values and $\mathfrak{Z}$ is fixed, then*

$$\lim \phi(\epsilon, \mathfrak{Z}) = \frac{\pi^n}{(n-1)!}.$$

On account of the definition of the integral, we have

$$\lim_{\epsilon \to 0} \phi(\epsilon, \mathfrak{Z}) = \int_{\mathfrak{S}[\mathfrak{q}] \leq 1} f(\mathfrak{S}[\mathfrak{q}]) d\mathfrak{q},$$

where $d\mathfrak{q}$ denotes the euclidean volume element in the space of the real vectors $\mathfrak{q}$ of $2n$ dimensions. Since the volume of the ellipsoid $\mathfrak{S}[\mathfrak{q}] \leq x$ is

$$J(x) = \frac{\pi^n}{n!} x^n \qquad (x \geq 0),$$

we obtain

$$\int\limits_{\mathfrak{S}[\mathfrak{q}]\leq 1} f(\mathfrak{S}[\mathfrak{q}])\,d\mathfrak{q} = \int_0^1 f(x)\,\frac{dJ(x)}{dx}\,dx = \frac{\pi^n}{(n-1)\,!}\int_0^1 f(x)x^{n-1}dx = \frac{\pi^n}{(n-1)\,!}\,.$$

41. Let F be the fundamental domain of the modular group, in the space H of $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$, and denote by dv the euclidean volume element in the space of $\mathfrak{X}$ and $\mathfrak{Y}^{-1}$, of $n(n+1)$ dimensions.

LEMMA 13. *There exists an integrable positive function* $g(\mathfrak{Z})$ *of the elements of* $\mathfrak{Z}$, *independent of* ϵ, *such that*

(113) $$\phi(\epsilon, \mathfrak{Z}) \leq g(\mathfrak{Z}) \qquad (0 < \epsilon \leq 1)$$

and the integral

$$\gamma = \int_F g(\mathfrak{Z})\,dv$$

converges.

We denote by $h(\rho, \mathfrak{Z})$ the number of lattice points $\mathfrak{w}$ satisfying $\mathfrak{S}[\mathfrak{w}] \leq \rho$, where ρ is an arbitrary positive number. By (78),

$$\mathfrak{S}[\mathfrak{w}] = \mathfrak{Y}^{-1}[\mathfrak{u} - \mathfrak{X}\mathfrak{v}] + \mathfrak{Y}[\mathfrak{v}], \qquad \mathfrak{w} = \begin{pmatrix}\mathfrak{u}\\ \mathfrak{v}\end{pmatrix},$$

and consequently $h(\rho, \mathfrak{Z})$ is not larger than the number of integral solutions $\mathfrak{u}, \mathfrak{v}$ of

(114) $$\mathfrak{Y}^{-1}[\mathfrak{u} - \mathfrak{X}\mathfrak{v}] \leq \rho, \qquad \mathfrak{Y}[\mathfrak{v}] \leq \rho.$$

Put $\mathfrak{Y}^{-1} = \mathfrak{P}[\mathfrak{Q}]$ with $\mathfrak{P} = [p_1, \cdots, p_n]$ and $\mathfrak{Q} = (q_{kl})$, $q_{kl} = 0$ $(k > l)$, $q_{kk} = 1$, and let u_k, v_k, r_k $(k = 1, \cdots, n)$ denote the elements of the columns $\mathfrak{u}, \mathfrak{v}, \mathfrak{r} = \mathfrak{X}\mathfrak{v}$. The first condition (114) involves

$$p_k\{(u_k - r_k) + \sum_{l=k+1}^{n} q_{kl}(u_l - r_l)\}^2 \leq \rho \qquad (k = 1, \cdots, n);$$

this proves that the number of integral $\mathfrak{u}$ is

$$\leq \prod_{k=1}^{n}(1 + 2p_k^{-\frac{1}{2}}\rho^{\frac{1}{2}}),$$

for any given $\mathfrak{v}$. On the other hand, the second condition (114) involves

$$p_k^{-1}(v_k + \sum_{l=1}^{k-1} q^*_{kl}v_l)^2 \leq \rho,$$

where $(q^*_{kl}) = \mathfrak{Q}^{-1}$; this proves that the number of integral $\mathfrak{v}$ is

$$\leq \prod_{k=1}^{n}(1 + 2p_k^{\frac{1}{2}}\rho^{\frac{1}{2}}).$$

It follows that

$$h(\rho, \mathfrak{Z}) \leq \prod_{k=1}^{n} (1 + 2\rho^{\frac{1}{2}}(p_k^{\frac{1}{2}} + p_k^{-\frac{1}{2}}) + 4\rho).$$

Let $\gamma_1, \cdots, \gamma_4$ denote positive numbers which depend only upon n. By (93) and (94), we have $p_k < \gamma_1$ $(k = 1, \cdots, n)$, for all $\mathfrak{Z}$ in F; hence

$$h(\rho, \mathfrak{Z}) < \gamma_2(1 + \rho)^n \prod_{k=1}^{n} p_k^{-\frac{1}{2}}. \tag{115}$$

Now $0 \leq f(x) \leq f(0)$ $(0 \leq x \leq 1)$ and $f(x) = 0$ $(x \geq 1)$; consequently we infer from the definition (112) that

$$\phi(\epsilon, \mathfrak{Z}) \leq \epsilon^{2n} f(0) h(\epsilon^{-2}, \mathfrak{Z}). \tag{116}$$

By (115) and (116),

$$\phi(\epsilon, \mathfrak{Z}) < f(0)\gamma_2(1 + \epsilon^2)^n \prod_{k=1}^{n} p_k^{-\frac{1}{2}} < f(0)\gamma_3 \prod_{k=1}^{n} p_k^{-\frac{1}{2}} \qquad (0 < \epsilon \leq 1).$$

We define

$$g(\mathfrak{Z}) = f(0)\gamma_3 \prod_{k=1}^{n} p_k^{-\frac{1}{2}};$$

then (113) is fulfilled and it remains only to prove the convergence of the integral

$$\gamma_4 = \int_F \prod_{k=1}^{n} p_k^{-\frac{1}{2}} dv.$$

Instead of the elements Y_{kl} $(k \leq l)$ of $(Y_{kl}) = \mathfrak{Y}^{-1} = \mathfrak{P}[\mathfrak{Q}]$, we introduce the new variables $p_k^{\frac{1}{2}}$ $(k = 1, \cdots, n)$ and q_{kl} $(k < l)$. The functional determinant has the value

$$2^n \prod_{k=1}^{n} p_k^{n-k+\frac{1}{2}},$$

whence

$$\gamma_4 = 2^n \int_F \prod_{k=1}^{n} p_k^{n-k} dv_1$$

with

$$dv_1 = \prod_{k \leq l} dx_{kl} \prod_{k<l} dq_{kl} \prod_{k} dp_k^{\frac{1}{2}}.$$

Since $\mathfrak{X}$, $\mathfrak{P}$, $\mathfrak{Q}$ are bounded in F, the convergence is obvious.

Applying a well-known theorem of integral calculus, we obtain from Lemmata 12 and 13 the important

LEMMA 14. *The integral*

$$\psi(\epsilon) = \int_F \phi(\epsilon, \mathfrak{Z}) dv = \epsilon^{2n} \sum_{\mathfrak{w} \neq 0} \int_F f(\mathfrak{S}[\epsilon\mathfrak{w}]) dv$$

converges and

$$\lim_{\epsilon\to 0}\psi(\epsilon)=\frac{\pi^n}{(n-1)!}\int_F dv=\frac{\pi^n}{(n-1)!}V_n$$

where $V_n=V(\Gamma_n)$ is the symplectic volume of the fundamental domain for the modular group $\Gamma=\Gamma_n$ of degree n.

42. We denote by W the set of all columns $\mathfrak{w}^{(2n)}$ with coprime integral elements and by $\mathfrak{e}$ the first column of the unit matrix $\mathfrak{E}^{(2n)}$; obviously $\mathfrak{e}$ belongs to W.

LEMMA 15. *There exists a modular matrix $\mathfrak{M}$ with the first column $\mathfrak{w}$, if and only if $\mathfrak{w}$ belongs to W.*

The necessity of the condition is obvious, since $\mathfrak{M}$ has integral elements and the determinant 1. Therefore we have only to prove the existence of a modular matrix $\mathfrak{M}$ satisfying $\mathfrak{M}^{-1}\mathfrak{w}=\mathfrak{e}$, where $\mathfrak{w}$ is a given column of the set W.

Consider the column

$$\mathfrak{M}^{-1}\mathfrak{w}=\mathfrak{w}_1=\begin{pmatrix}\mathfrak{u}^{(n)}\\ \mathfrak{v}^{(n)}\end{pmatrix},$$

where $\mathfrak{M}$ is an arbitrary modular matrix, and let u, v denote the greatest common divisors of the n elements of $\mathfrak{u}, \mathfrak{v}$; we define $u=0$ or $v=0$ if $\mathfrak{u}=0$ or $\mathfrak{v}=0$. We choose now the matrix $\mathfrak{M}$, such that the sum $u+v=w$ is as small as possible. If $u<v$, we replace $\mathfrak{M}, \mathfrak{u}, \mathfrak{v}$ by $\mathfrak{M}\mathfrak{J}$, $-\mathfrak{v}$, $\mathfrak{u}$ and obtain the case $u>v$, with the same value of w; hence we may assume $u\geqq v$.

If $v>0$, we determine a unimodular matrix $\mathfrak{U}$ with the first row $v^{-1}\mathfrak{v}'$ and an integral column $\mathfrak{t}$, such that all elements of the column $\mathfrak{U}\mathfrak{u}-v\mathfrak{t}=\mathfrak{u}_1$ have an absolute value $\leqq(v/2)$; then the greatest common divisor u_1 of these elements satisfies $u_1\leqq(v/2)<u$. Let $\mathfrak{T}$ be an integral symmetric matrix having $\mathfrak{t}$ as its first column and

$$\mathfrak{M}_1=\begin{pmatrix}\mathfrak{U}^{-1} & \mathfrak{U}^{-1}\mathfrak{T}\\ 0 & \mathfrak{U}'\end{pmatrix}.$$

Since $\mathfrak{U}'^{-1}\mathfrak{v}=v\mathfrak{e}_1$ and $\mathfrak{T}\mathfrak{U}'^{-1}\mathfrak{v}=v\mathfrak{t}$, where $\mathfrak{e}_1$ denotes the first column of the unit matrix $\mathfrak{E}^{(n)}$, we find

$$(\mathfrak{M}\mathfrak{M}_1)^{-1}\mathfrak{w}=\mathfrak{M}_1^{-1}\mathfrak{w}_1=\begin{pmatrix}\mathfrak{u}_1\\ v\mathfrak{e}_1\end{pmatrix}.$$

But $\mathfrak{M}_1$ and $\mathfrak{M}\mathfrak{M}_1$ are again modular matrices and $u_1+v<w$, in contradiction to the minimum property of w.

Consequently $v = 0$ and $u = 1$, the elements of $\mathfrak{w}_1$ being coprime. Let $\mathfrak{U}_1$ be a unimodular matrix with the first row $\mathfrak{u}'$ and

$$\mathfrak{M}_2 = \begin{pmatrix} \mathfrak{U}'_1 & 0 \\ 0 & \mathfrak{U}_1^{-1} \end{pmatrix}.$$

Then $(\mathfrak{M}\mathfrak{M}_2)^{-1}\mathfrak{w} = \mathfrak{e}$ and $\mathfrak{M}\mathfrak{M}_2$ is unimodular, q. e. d.

We denote by Δ_1 the subgroup of Γ consisting of all modular matrices $\mathfrak{M}$ with the first column $\mathfrak{e}$. Obviously two arbitrary modular matrices are then and only then in the same left coset of Δ_1 relative to the homogeneous modular group Γ_0, if they have the same first column. Applying Lemma 15, we find the decomposition

$$\Gamma_0 = \sum_{\mathfrak{w} \subset W} \mathfrak{M}_{\mathfrak{w}} \Delta_1 = \sum_{\mathfrak{w} \subset W} \Delta_1 \mathfrak{M}_{\mathfrak{w}}^{-1},$$

where $\mathfrak{w}$ runs over all elements of the set W and $\mathfrak{M}_{\mathfrak{w}}$ denotes a modular matrix with the first column $\mathfrak{w}$. We choose $\mathfrak{M}_{-\mathfrak{w}} = -\mathfrak{M}_{\mathfrak{w}}$, such that $\mathfrak{M}_{\mathfrak{w}}$ and $\mathfrak{M}_{-\mathfrak{w}}$ give the same element of Γ. Let $F_{\mathfrak{w}}$ be the image of F under the transformation $\mathfrak{M}_{\mathfrak{w}}^{-1}$; then the domains $F_{\mathfrak{w}}$ cover exactly twice a fundamental domain F_0 of the group Δ_1. On the other hand

$$\int_F f(\mathfrak{S}[\epsilon\mathfrak{w}])\,dv = \int_{F_{\mathfrak{w}}} f(\mathfrak{S}[\epsilon\mathfrak{e}]\,dv,$$

by Lemma 6. Using the abbreviation $p = \mathfrak{S}[\mathfrak{e}]$ for the first diagonal element of $\mathfrak{S}$, we obtain

$$\sum_{\mathfrak{w} \subset W} \int_F f(\mathfrak{S}[\epsilon\mathfrak{w}])\,dv = 2 \int_{F_0} f(\epsilon^2 p)\,dv.$$

Now we replace ϵ by ϵl and sum over all positive integers l; then $l\mathfrak{w}$ runs exactly over all lattice points $\neq 0$, and we have

$$\psi(\epsilon) = 2\epsilon^{2n} \sum_{l=1}^{\infty} \int_{F_0} f(\epsilon^2 l^2 p)\,dv, \tag{117}$$

where $\psi(\epsilon)$ is the function defined in Lemma 14.

43. Any element of Δ_1 has the form

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix} 1 & * \\ 0 & \mathfrak{A}_1 \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} * & * \\ * & \mathfrak{B}_1 \end{pmatrix}, \tag{118}$$
$$\mathfrak{C} = \begin{pmatrix} 0 & * \\ 0 & \mathfrak{C}_1 \end{pmatrix}, \quad \mathfrak{D} = \begin{pmatrix} * & * \\ * & \mathfrak{D}_1 \end{pmatrix}.$$

It follows from (7) that also

$$\mathfrak{A}_1\mathfrak{B}'_1=\mathfrak{B}_1\mathfrak{A}'_1,\quad \mathfrak{C}_1\mathfrak{D}'_1=\mathfrak{D}_1\mathfrak{C}'_1,\quad \mathfrak{A}_1\mathfrak{D}'_1-\mathfrak{B}_1\mathfrak{C}'_1=\mathfrak{E};$$

hence

$$\mathfrak{M}_1=\begin{pmatrix}\mathfrak{A}_1 & \mathfrak{B}_1\\ \mathfrak{C}_1 & \mathfrak{D}_1\end{pmatrix}$$

belongs to the homogeneous modular group of degree $n-1$. We define

$$\mathfrak{A}_2=\begin{pmatrix}1 & 0\\ 0 & \mathfrak{A}_1\end{pmatrix},\quad \mathfrak{B}_2=\begin{pmatrix}0 & 0\\ 0 & \mathfrak{B}_1\end{pmatrix},\quad \mathfrak{C}_2=\begin{pmatrix}0 & 0\\ 0 & \mathfrak{C}_1\end{pmatrix},$$

$$\mathfrak{D}_2=\begin{pmatrix}1 & 0\\ 0 & \mathfrak{D}_1\end{pmatrix},\quad \mathfrak{M}_2=\begin{pmatrix}\mathfrak{A}_2 & \mathfrak{B}_2\\ \mathfrak{C}_2 & \mathfrak{D}_2\end{pmatrix},$$

$$\mathfrak{M}_0=\mathfrak{M}\mathfrak{M}_2^{-1}=\begin{pmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{pmatrix}\begin{pmatrix}\mathfrak{D}'_2 & -\mathfrak{B}'_2\\ -\mathfrak{C}'_2 & \mathfrak{A}'_2\end{pmatrix}=\begin{pmatrix}\mathfrak{A}_0 & \mathfrak{B}_0\\ \mathfrak{C}_0 & \mathfrak{D}_0\end{pmatrix},$$

where

$$\mathfrak{A}_0=\begin{pmatrix}1 & *\\ 0 & \mathfrak{E}\end{pmatrix},\quad \mathfrak{B}_0=\begin{pmatrix}* & *\\ * & 0\end{pmatrix},\quad \mathfrak{C}_0=\begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix},\quad \mathfrak{D}_0=\begin{pmatrix}* & *\\ * & \mathfrak{E}\end{pmatrix}.$$

Since $\mathfrak{M}_0$ is symplectic, we have $\mathfrak{C}_0\mathfrak{D}'_0=\mathfrak{D}_0\mathfrak{C}'_0$, whence $\mathfrak{C}_0=0$; moreover $\mathfrak{A}_0\mathfrak{D}'_0=\mathfrak{E}$ and $\mathfrak{A}_0\mathfrak{B}'_0=\mathfrak{B}_0\mathfrak{A}'_0$ so that $\mathfrak{A}_0=\mathfrak{U}'$, $\mathfrak{D}_0=\mathfrak{U}^{-1}$, $\mathfrak{B}_0=\mathfrak{T}\mathfrak{U}^{-1}$ with

(119) $$\mathfrak{U}=\begin{pmatrix}1 & 0\\ \mathfrak{a} & \mathfrak{E}\end{pmatrix},\quad \mathfrak{T}=\begin{pmatrix}b & \mathfrak{b}'\\ \mathfrak{b} & 0\end{pmatrix}$$

and integral $\mathfrak{a}, \mathfrak{b}, b$.

On the other hand, if $\mathfrak{U}$ and $\mathfrak{T}$ are defined by (119) with arbitrary integral $\mathfrak{a}, \mathfrak{b}, b$, then

(120) $$\mathfrak{M}_0=\begin{pmatrix}\mathfrak{U}' & \mathfrak{T}\mathfrak{U}^{-1}\\ 0 & \mathfrak{U}^{-1}\end{pmatrix}$$

is a modular matrix and $\mathfrak{M}=\mathfrak{M}_0\mathfrak{M}_2$ has again the form (118). It is obvious that the matrices $\mathfrak{M}_2$ constitute a group Δ_2 which is isomorphic to the homogeneous modular group of degree $n-1$ and is a subgroup of Δ_1. The left cosets of Δ_2 relative to Δ_1 are of the form $\mathfrak{M}_0\Delta_2$, where $\mathfrak{M}_0$ runs over all matrices defined by (119) and (120).

This result enables us to construct another fundamental domain of Δ_1. Let $\hat{\mathfrak{Z}}=(\mathfrak{A}_2\mathfrak{Z}+\mathfrak{B}_2)(\mathfrak{C}_2\mathfrak{Z}+\mathfrak{D}_2)^{-1}$ be the modular transformation with the matrix $\mathfrak{M}_2$. We decompose

$$\mathfrak{Z}=\begin{pmatrix}* & *\\ * & \mathfrak{Z}_1\end{pmatrix},\quad \hat{\mathfrak{Z}}=\begin{pmatrix}* & *\\ * & \hat{\mathfrak{Z}}_1\end{pmatrix}$$

and obtain

$$\mathfrak{A}_2\mathfrak{Z}+\mathfrak{B}_2=\begin{pmatrix}* & *\\ * & \mathfrak{A}_1\mathfrak{Z}_1+\mathfrak{B}_1\end{pmatrix},\quad \mathfrak{C}_2\mathfrak{Z}+\mathfrak{D}_2=\begin{pmatrix}1 & 0\\ * & \mathfrak{C}_1\mathfrak{Z}_1+\mathfrak{D}_1\end{pmatrix},$$

$$(\mathfrak{C}_2\mathfrak{Z}+\mathfrak{D}_2)^{-1}=\begin{pmatrix}1 & 0\\ * & (\mathfrak{C}_1\mathfrak{Z}_1+\mathfrak{D}_1)^{-1}\end{pmatrix};$$

consequently $\hat{\mathfrak{Z}}_1 = (\mathfrak{A}_1\mathfrak{Z}_1 + \mathfrak{B}_1)(\mathfrak{C}_1\mathfrak{Z}_1 + \mathfrak{D}_1)^{-1}$ is the image of $\mathfrak{Z}_1$ under the modular transformation with the matrix $\mathfrak{M}_1$, of degree $n-1$. For any $\mathfrak{Z}$ in H, we determine $\mathfrak{M}_1$, such that $\hat{\mathfrak{Z}}_1$ lies in the fundamental domain F_1 of the modular group of degree $n-1$. Since we may replace $\mathfrak{M}_1$ by $-\mathfrak{M}_1$, there are always two different possibilities for the choice of the corresponding element $\mathfrak{M}_2$ of Δ_2.

We replace $\hat{\mathfrak{Z}}$ again by $\mathfrak{Z}$ and consider now the modular transformation $\hat{\mathfrak{Z}} = \mathfrak{Z}[\mathfrak{U}] + \mathfrak{T}$ with the matrix $\mathfrak{M}_0$. By (119),

$$\hat{\mathfrak{Z}}_1 = \mathfrak{Z}_1, \quad \hat{\mathfrak{Y}}^{-1} = \mathfrak{Y}^{-1}\begin{bmatrix}1 & -\mathfrak{a}' \\ 0 & \mathfrak{E}\end{bmatrix}; \quad \hat{\mathfrak{X}} = \mathfrak{X}\begin{bmatrix}1 & 0 \\ \mathfrak{a} & \mathfrak{E}\end{bmatrix} + \begin{pmatrix}b & \mathfrak{b}' \\ \mathfrak{b} & 0\end{pmatrix},$$

where $\hat{\mathfrak{X}} + i\hat{\mathfrak{Y}} = \hat{\mathfrak{Z}}$. On account of the definition (77), the matrices $\mathfrak{Y}^{-1}$ and $\mathfrak{S}$ have the same first diagonal element p. Putting $\mathfrak{Y}^{-1} = (Y_{kl})$ and $\hat{\mathfrak{Y}}^{-1} = (\hat{Y}_{kl})$, we obtain $\hat{Y}_{1l} = Y_{1l} - pa_l$ $(l = 2, \cdots, n)$, where $a_2, \cdots, a_n$ denote the elements of $\mathfrak{a}$. We determine first $\mathfrak{a}$, such that $-(p/2) \leq \hat{Y}_{1l} \leq p/2$ $(l = 2, \cdots, n)$, and then $b, \mathfrak{b}$, such that the elements $\hat{x}_{1l}$ of the first row of $\hat{\mathfrak{X}}$ satisfy $-\frac{1}{2} \leq \hat{x}_{1l} \leq \frac{1}{2}$ $(l = 1, \cdots, n)$.

It follows that any point of H has, relative to Δ_1, an equivalent point in the domain defined by the conditions

$$\begin{array}{lllr} (121) & \mathfrak{Z}_1 \subset F_1, & -(p/2) \leq Y_{1l} \leq p/2 & (l = 2, \cdots, n), \\ & & -\frac{1}{2} \leq x_{1l} \leq \frac{1}{2} & (l = 1, \cdots, n). \end{array}$$

On account of the ambiguity in the choice of $\mathfrak{M}_2$, this domain is not yet a fundamental domain of Δ_1. It is transformed into itself by the particular mapping $z_{1l} \to -z_{1l}$ $(l = 2, \cdots, n)$, $z_{11} \to z_{11}$, $\mathfrak{Z}_1 \to \mathfrak{Z}_1$, obtained from $\mathfrak{M}_1 = -\mathfrak{E}$, $\mathfrak{M}_0 = \mathfrak{E}$. By the additional condition

$$(122) \qquad Y_{12} \geq 0$$

together with (121), we obtain now a fundamental domain F^* of Δ_1.

In the special case $n = 1$, the condition (122) does not exist, and (121) reduces to $-\frac{1}{2} \leq x_{11} \leq \frac{1}{2}$.

44. By Lemma 6, the first element p of $\mathfrak{S}$ is invariant under all transformations of the group Δ_1. Since F_0 and F^* are both fundamental domains of Δ_1, we obtain

$$(123) \qquad \int_{F_0} f(\epsilon^2 l^2 p)\,dv = \int_{F^*} f(\epsilon^2 l^2 p)\,dv.$$

We use now the decomposition

$$\mathfrak{Y}^{-1} = \begin{pmatrix}p^{-1} & 0 \\ 0 & \mathfrak{Y}_1^{-1}\end{pmatrix}\begin{bmatrix}p & \mathfrak{y}' \\ 0 & \mathfrak{E}\end{bmatrix} = \begin{pmatrix}p & \mathfrak{y}' \\ \mathfrak{y} & \mathfrak{Y}_0\end{pmatrix},$$

where

(124) $$\mathfrak{Y}_1^{-1} = \mathfrak{Y}_0 - p^{-1}\mathfrak{y}\mathfrak{y}'$$

and $\mathfrak{Y}_1$ denotes the imaginary part of $\mathfrak{Z}_1$. Introducing as new variables the elements η_{kl} $(1 \leq k \leq l \leq n-1)$ of $\mathfrak{Y}_1^{-1} = (\eta_{kl})$ instead of the elements Y_{kl} $(2 \leq k \leq l \leq n)$ of $\mathfrak{Y}_0$, we obtain, by (124),

(125) $$\begin{aligned} dv &= \prod_{1 \leq k \leq l \leq n} (dx_{kl} dY_{kl}) \\ &= \prod_{2 \leq k \leq l \leq n} (dx_{kl} d\eta_{kl}) \prod_{l=1}^{n} (dx_{1l} dY_{1l}) = dv_1 \prod_{l=1}^{n} (dx_{1l} dY_{1l}), \end{aligned}$$

where dv_1 is the symplectic volume element for $\mathfrak{Z}_1$ instead of $\mathfrak{Z}$; moreover $Y_{11} = p$.

Define $\eta_n = 1$ for $n = 1$ and $= 2$ for $n > 1$. By (111), (121), (122) and (125),

(126) $$\begin{aligned} \eta_n \int_{F^*} f(\epsilon^2 l^2 p)\, dv &= V_{n-1} \int_0^\infty p^{n-1} f(\epsilon^2 l^2 p)\, dp \\ &= V_{n-1} (\epsilon l)^{-2n} \int_0^1 x^{n-1} f(x)\, dx = V_{n-1} (\epsilon l)^{-2n}, \end{aligned}$$

where V_{n-1} is the volume $V(\Gamma_{n-1})$ of the fundamental domain F_1 for the modular group of degree $n-1$ and $V_0 = 1$. By (117), (123) and (126),

$$\psi(\epsilon) = 2\eta_n^{-1} V_{n-1} \zeta(2n),$$

independent of ϵ. Lemma 14 leads now to the recursion formula

$$V_n = 2\eta_n^{-1} (n-1)!\, \pi^{-n} \zeta(2n) V_{n-1},$$

whence

(127) $$V_n = 2 \prod_{k=1}^{n} \{(k-1)!\, \pi^{-k} \zeta(2k)\},$$

and Theorem 11 is proved.

45. By a well-known result of Euler,

$$2(2\pi)^{-2k} \zeta(2k) = \frac{B_{2k}}{(2k)!} \qquad (k = 1, 2, \cdots),$$

where B_{2k} is the absolute value of the Bernoulli numbers

$$B_2 = \frac{1}{6}, \quad -B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad -B_8 = -\frac{1}{30}, \cdots.$$

Thus we obtain the expression

$$V_n = 2^{n^2+1} \pi^{n(n+1)/2} \prod_{k=1}^{n} \left\{ \frac{(k-1)!}{(2k)!} B_{2k} \right\}.$$

and in particular

$$V_1 = \frac{\pi}{3}, \quad V_2 = \frac{\pi^3}{270}, \quad V_3 = \frac{\pi^6}{127575}, \quad V_4 = \frac{\pi^{10}}{200930625}.$$

Consider now the congruence subgroup $\Gamma_n(p)$ of the modular group of degree n, defined by $\mathfrak{M} \equiv \mathfrak{E} \pmod{p}$, where p is a prime number. Its index has the value

$$j_n(p) = \prod_{k=1}^{n} p^{2k-1}(p^{2k}-1). \tag{128}$$

For $p \neq 2$, this group has no fixed point in H, by Theorem 10. *If* Theorem 5 still holds good for the non-compact fundamental domain of $\Gamma_n(p)$, then this open manifold has the Euler number

$$\chi = c_n(-\pi)^{-n(n+1)/2} V_n j_n(p).$$

We denote by $\chi_n(p)$ the right-hand side of this hypothetic formula. Using the values of c_1, c_2, c_3 given in Theorem 5, we find

$$c_1 V_1 = -\frac{\pi}{3\,!}, \quad c_2 V_2 = -\frac{\pi^3}{6\,!}, \quad c_3 V_3 = \frac{\pi^6}{2 \cdot 9\,!}$$

and consequently

$$\chi_1(p) = -\frac{1}{6} p(p^2-1), \quad \chi_2(p) = -\frac{1}{720} p^4(p^2-1)(p^4-1),$$
$$\chi_3(p) = \frac{1}{2 \cdot 9\,!} p^9(p^2-1)(p^4-1)(p^6-1).$$

It is easily proved that these rational numbers are integers for all prime numbers p; in particular $\chi_1(2) = -1$, $\chi_2(2) = -1$, $\chi_3(2) = 2$.

46. Let $q = p^a$ $(a \geq 1)$ be a power of a prime number p. For any integral skew-symmetric matrix $\mathfrak{S}^{(2n)}$, we denote by $A_q(\mathfrak{S})$ the number of integral solutions $\mathfrak{L}$, modulo q, of the congruence

$$\mathfrak{J}[\mathfrak{L}] \equiv \mathfrak{S} \pmod{q}. \tag{129}$$

In particular, $A_q(\mathfrak{J}) = E_q$ is the order of the homogeneous modular group modulo q, namely the number of incongruent solutions $\mathfrak{M}$ of

$$\mathfrak{J}[\mathfrak{M}] \equiv \mathfrak{J} \pmod{q}. \tag{130}$$

It is known that $E_p = j_n(p)$ has the value (128), and more generally

$$E_q = q^{n(2n+1)} \prod_{k=1}^{n} (1 - p^{-2k}).$$

By (129) and (130), $\mathfrak{M}\mathfrak{L}$ is also a solution of (129); we call it equivalent to $\mathfrak{L}$, relative to the group of the $\mathfrak{M}$. Since there are exactly E_q matrices in each class of equivalent solutions, the number of different classes of solutions of (129) is $A_q(\mathfrak{S})/E_q$. On the other hand, the number of incongruent skew-symmetric matrices $\mathfrak{S}$ is $q^{n(2n-1)}$, and

$$\sum_{\mathfrak{S}} A_q(\mathfrak{S}) = q^{4n^2}$$

is the number of incongruent $\mathfrak{L}$. The average number of classes of solutions is therefore

$$q^{-n(2n-1)} \sum_{\mathfrak{S}} \frac{A_q(\mathfrak{S})}{E_q} = \prod_{k=1}^{n} (1 - p^{-2k})^{-1} = d_p, \tag{131}$$

independent of the exponent a.

By Euler's formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}; \qquad (s > 1),$$

the result (127) may be written

$$d_0 = \prod_p d_p,$$

where p runs over all prime numbers and

$$d_0 = \tfrac{1}{2} V_n \prod_{k=1}^{n} \frac{\pi^k}{(k-1)!}. \tag{132}$$

Now the main formula in the analytic theory of quadratic forms suggests that d_0 can be defined independently as a density connected with the real solutions $\mathfrak{L}$ of the equation $\mathfrak{J}[\mathfrak{L}] = \mathfrak{S}$. We shall prove that d_0 has the value defined in (4); this is the statement of Theorem 12.

47. Let

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & \mathfrak{S}_2 \\ -\mathfrak{S}'_2 & \mathfrak{S}_3 \end{pmatrix}$$

be a real skew-symmetric matrix and

$$\mathfrak{F} = \begin{pmatrix} \mathfrak{E} & \mathfrak{E} \\ -i\mathfrak{E} & i\mathfrak{E} \end{pmatrix}, \qquad \frac{1}{2i}\,\mathfrak{S}\{\mathfrak{F}\} = \begin{pmatrix} \mathfrak{H}_1 & -\bar{\mathfrak{A}} \\ \mathfrak{A} & -\bar{\mathfrak{H}}_1 \end{pmatrix} = \mathfrak{K}_1; \tag{133}$$

then $\mathfrak{H}_1 = \frac{1}{2}(\mathfrak{S}_2 + \mathfrak{S}'_2) + \frac{1}{2i}(\mathfrak{S}_1 + \mathfrak{S}_3)$ is hermitian and $\mathfrak{A} = \frac{1}{2}(\mathfrak{S}_2 - \mathfrak{S}'_2) + \frac{1}{2i}(\mathfrak{S}_1 - \mathfrak{S}_3)$ is complex skew-symmetric. For $\mathfrak{S} = \mathfrak{J}$, we have $\mathfrak{H}_1 = \mathfrak{E}$ and $\mathfrak{A} = 0$. We choose a neighborhood G of $\mathfrak{S} = \mathfrak{J}$, such that $|\mathfrak{H}_1| \neq 0$ and that the characteristic roots $b_1, \cdots, b_n$ of the matrix

$$\mathfrak{B} = -\bar{\mathfrak{A}}\bar{\mathfrak{H}}_1^{-1}\mathfrak{A}\mathfrak{H}_1^{-1} \tag{134}$$

are of absolute value < 1, for all $\mathfrak{S} \subset G$.

The series

$$\mathfrak{W} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \mathfrak{B}^k \tag{135}$$

converges and satisfies the equation

$$\mathfrak{W}^2 = \mathfrak{E} + \mathfrak{B}.$$

The characteristic roots of the matrix $\mathfrak{E} + \mathfrak{W}$ are $1 + (1 + b_k)^{\frac{1}{2}} \neq 0$ $(k = 1, \cdots, n)$. We define

$$\mathfrak{H} = \tfrac{1}{2}(\mathfrak{E} + \mathfrak{W})\mathfrak{H}_1; \tag{136}$$

then $|\mathfrak{H}| \neq 0$ and $|2\mathfrak{H} - \mathfrak{H}_1| \neq 0$.

Lemma 16. *The matrix $\mathfrak{H}$ is hermitian and*

$$\mathfrak{H} + \tfrac{1}{4}\bar{\mathfrak{A}}\bar{\mathfrak{H}}^{-1}\mathfrak{A} = \mathfrak{H}_1. \tag{137}$$

We have

$$\mathfrak{B}\bar{\mathfrak{A}} = -\bar{\mathfrak{A}}\bar{\mathfrak{H}}_1^{-1}\mathfrak{A}\mathfrak{H}_1^{-1}\bar{\mathfrak{A}} = \bar{\mathfrak{A}}\bar{\mathfrak{B}}' \quad \text{and} \quad \mathfrak{B}\mathfrak{H}_1 = -\bar{\mathfrak{A}}\bar{\mathfrak{H}}_1^{-1}\mathfrak{A} = \mathfrak{H}_1\bar{\mathfrak{B}}',$$

whence

$$\bar{\mathfrak{H}}' = \tfrac{1}{2}\mathfrak{H}_1(\mathfrak{E} + \bar{\mathfrak{W}}') = \tfrac{1}{2}(\mathfrak{E} + \mathfrak{W})\mathfrak{H}_1 = \mathfrak{H}$$

and

$$\begin{aligned}\tfrac{1}{2}\bar{\mathfrak{A}}\bar{\mathfrak{H}}^{-1}\mathfrak{A} = \bar{\mathfrak{A}}(\mathfrak{E} + \mathfrak{W}')\bar{\mathfrak{H}}_1^{-1}\mathfrak{A} &= -(\mathfrak{E} + \mathfrak{W})^{-1}\mathfrak{B}\mathfrak{H}_1 \\ &= (\mathfrak{E} - \mathfrak{W})\mathfrak{H}_1 = 2\mathfrak{H}_1 - 2\mathfrak{H},\end{aligned}$$

q. e. d.

We define now

$$\mathfrak{K} = \begin{pmatrix} \mathfrak{H} & 0 \\ 0 & -\bar{\mathfrak{H}} \end{pmatrix}, \qquad \mathfrak{C} = \begin{pmatrix} \mathfrak{E} & -\tfrac{1}{2}\bar{\mathfrak{H}}^{-1}\mathfrak{A} \\ -\tfrac{1}{2}\mathfrak{H}^{-1}\bar{\mathfrak{A}} & \mathfrak{E} \end{pmatrix}$$

and obtain, by (133) and (137), $\mathfrak{K}\{\mathfrak{C}\} = \mathfrak{K}_1$ and

$$\mathfrak{C}\begin{pmatrix} \mathfrak{E} & \tfrac{1}{2}\bar{\mathfrak{H}}^{-1}\mathfrak{A} \\ \tfrac{1}{2}\mathfrak{H}^{-1}\bar{\mathfrak{A}} & \mathfrak{E} \end{pmatrix} = \begin{pmatrix} 2\mathfrak{E} - \bar{\mathfrak{H}}^{-1}\bar{\mathfrak{H}}_1 & 0 \\ 0 & 2\mathfrak{E} - \mathfrak{H}^{-1}\mathfrak{H}_1 \end{pmatrix},$$

whence in particular $|\mathfrak{C}| \neq 0$.

48. We consider now the set L of real matrices $\mathfrak{L}$ satisfying $\mathfrak{J}[\mathfrak{L}] = \mathfrak{S} \subset G$. Putting

$$\mathfrak{L}\mathfrak{F} = \mathfrak{B}_1 = \begin{pmatrix} \mathfrak{P}_1 & \bar{\mathfrak{P}}_1 \\ \mathfrak{Q}_1 & \bar{\mathfrak{Q}}_1 \end{pmatrix}, \qquad \mathfrak{B}_1\mathfrak{C}^{-1} = \begin{pmatrix} \mathfrak{P} & \bar{\mathfrak{P}} \\ \mathfrak{Q} & \bar{\mathfrak{Q}} \end{pmatrix} = \mathfrak{B}, \tag{138}$$

we have

$$\frac{1}{2i}\,\mathfrak{J}\{\mathfrak{B}\} = \frac{1}{2i}\,\mathfrak{S}\{\mathfrak{F}\mathfrak{C}^{-1}\} = \mathfrak{K}_1\{\mathfrak{C}^{-1}\} = \mathfrak{K}$$

and consequently

$$\frac{1}{2i}\,(\mathfrak{P}'\bar{\mathfrak{Q}} - \mathfrak{Q}'\bar{\mathfrak{P}}) = \mathfrak{H}, \qquad \mathfrak{P}'\mathfrak{Q} - \mathfrak{Q}'\mathfrak{P} = 0; \tag{139}$$

moreover

$$\mathfrak{P}_1 = \mathfrak{P} - \tfrac{1}{2}\bar{\mathfrak{P}}\mathfrak{H}^{-1}\bar{\mathfrak{A}}, \qquad \mathfrak{Q}_1 = \mathfrak{Q} - \tfrac{1}{2}\bar{\mathfrak{Q}}\mathfrak{H}^{-1}\bar{\mathfrak{A}}.$$

By (133), (134), (135), (136) and (137), the neighborhood G of $\mathfrak{S} = \mathfrak{J}$ is mapped onto a neighborhood G^* of $\mathfrak{H} = \mathfrak{E}$, $\mathfrak{A} = 0$ in the $(\mathfrak{H}, \mathfrak{A})$ space. Since $|\mathfrak{H}| \neq 0$, we have $\mathfrak{H} > 0$. By (139), $|\mathfrak{Q}| \neq 0$ and

$$\mathfrak{P}\mathfrak{Q}^{-1} = \mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$$

is a point of H with the imaginary part $\mathfrak{Y} = \mathfrak{H}\{\mathfrak{Q}^{-1}\}$. Then

(140) $\quad \mathfrak{P}_1 = (\mathfrak{Z} - \tfrac{1}{2}\bar{\mathfrak{Z}}\mathfrak{Y}^{-1}\bar{\mathfrak{A}}[\mathfrak{Q}^{-1}])\mathfrak{Q}, \quad \mathfrak{Q}_1 = (\mathfrak{E} - \tfrac{1}{2}\mathfrak{Y}^{-1}\bar{\mathfrak{A}}[\mathfrak{Q}^{-1}])\mathfrak{Q}.$

It follows, from (138) and (140), that L is mapped onto the set L^* of the $(\mathfrak{Z}, \mathfrak{Q}, \mathfrak{A})$ space defined by the conditions $\mathfrak{Z} \subset H$, $(\mathfrak{Y}\{\mathfrak{Q}\}, \mathfrak{A}) \subset G^*$.

If

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{M}_1 & \mathfrak{M}_2 \\ \mathfrak{M}_3 & \mathfrak{M}_4 \end{pmatrix}$$

is any modular matrix, then the mapping

(141) $\quad \mathfrak{Z} \to (\mathfrak{M}_1\mathfrak{Z} + \mathfrak{M}_2)(\mathfrak{M}_3\mathfrak{Z} + \mathfrak{M}_4)^{-1}, \qquad \mathfrak{Q} \to (\mathfrak{M}_3\mathfrak{Z} + \mathfrak{M}_4)\mathfrak{Q}$

transforms L^* into itself and leaves every point of G^* invariant. We restrict now $\mathfrak{Z}$ to the fundamental domain F of the modular group Γ. Since the particular mapping with $\mathfrak{M} = -\mathfrak{E}$ leaves $\mathfrak{Z}$ invariant and changes $\mathfrak{Q}$ into $-\mathfrak{Q}$, we obtain in L^* a fundamental domain L^*_0 for all mappings (141), if we impose on $\mathfrak{Q}$ a linear homogeneous condition, e. g., $\sigma(\mathfrak{Q} + \bar{\mathfrak{Q}}) \geq 0$; then L^*_0 is defined by $\mathfrak{Z} \subset F$, $(\mathfrak{Y}\{\mathfrak{Q}\}, \mathfrak{A}) \subset G^*$, $\sigma(\mathfrak{Q} + \bar{\mathfrak{Q}}) \geq 0$. Let L_0 be the corresponding domain in L. Obviously L_0 is a fundamental domain in L relative to the homogeneous modular group Γ_0, such that the images of L_0 under the mappings $\mathfrak{L} \to \mathfrak{M}\mathfrak{L}$ cover L completely without gaps and overlappings.

Denote by $v(L_0)$ the euclidean volume of L_0, the elements of $\mathfrak{L}$ being considered as rectangular cartesian coördinates, and by $v(G)$ the euclidean volume of G, where the elements of the skew-symmetric matrix $\mathfrak{S} = \mathfrak{J}[\mathfrak{L}]$ above the diagonal are the coördinates. Theorem 12 asserts that

(142) $$\lim_{G \to \mathfrak{J}} \frac{v(L_0)}{v(G)} = d_0,$$

if G runs over any sequence of neighborhoods tending to the single point $\mathfrak{J}$, with the value d_0 defined in (132). Obviously the left-hand side of (142) is the analogue of the expression in (131), for the real valuation instead of the p-adic valuation.

49. In order to simplify the notation, we introduce cartesian coördinates for a matrix $\mathfrak{T}^{(m)} = (t_{kl})$ in the following way: If $\mathfrak{T}$ is arbitrarily real, we choose the m^2 coördinates t_{kl} $(k, l = 1, \cdots, m)$; if $\mathfrak{T}$ is symmetric, we take the $\frac{1}{2}m(m+1)$ coördinates t_{kl} $(1 \leq k \leq l \leq m)$; if $\mathfrak{T}$ is skew-symmetric, we take the $\frac{1}{2}m(m-1)$ coördinates t_{kl} $(1 \leq k < l \leq m)$. If $\mathfrak{T}$ is complex, we split

$$\mathfrak{T} = \mathfrak{T}_1 + i\mathfrak{T}_2, \qquad \mathfrak{T}_1 = \tfrac{1}{2}(\mathfrak{T} + \bar{\mathfrak{T}}), \qquad \mathfrak{T}_2 = \frac{1}{2i}(\mathfrak{T} - \bar{\mathfrak{T}}),$$

and proceed in the same manner with the real part $\mathfrak{T}_1$ and the imaginary part $\mathfrak{T}_2$. In particular, a hermitian matrix $\mathfrak{T}$ has then the m^2 coördinates

$$t_{kk}\ (k = 1, \cdots, m), \quad \tfrac{1}{2}(t_{kl} + \bar{t}_{kl}), \quad \frac{1}{2i}(t_{kl} - \bar{t}_{kl}) \qquad (1 \leq k < l \leq m).$$

In all these cases, we denote by $d\mathfrak{T}$ the euclidean volume element in the space of the coördinates of $\mathfrak{T}$.

By (133), (138) and (140), we have

$$v(L_0) = \int_{L_0} d\mathfrak{L}, \qquad d\mathfrak{L} = d\mathfrak{P}_1 d\mathfrak{Q}_1 = |\,\mathfrak{Q}_1\bar{\mathfrak{Q}}_1\,|^n\, d\mathfrak{R}\, d\mathfrak{Q}_1$$

with

$$\mathfrak{R} = \mathfrak{P}_1\mathfrak{Q}_1^{-1} = (\mathfrak{Z} + \tfrac{1}{2}\bar{\mathfrak{Z}}\mathfrak{Y}^{-1}\bar{\mathfrak{A}}[\mathfrak{Q}^{-1}])(\mathfrak{E} - \tfrac{1}{2}\mathfrak{Y}^{-1}\bar{\mathfrak{A}}[\mathfrak{Q}^{-1}])^{-1}.$$

In a sufficiently small neighborhood of $\mathfrak{A} = 0$, the power series

$$\mathfrak{R} = \mathfrak{Z} - i\bar{\mathfrak{A}}[\mathfrak{Q}^{-1}] + \cdots$$

converges, whence

$$d\mathfrak{R} \sim 2^{n(n-1)}\, |\,\mathfrak{Q}\bar{\mathfrak{Q}}\,|^{1-n}\, d\mathfrak{Z}\, d\mathfrak{A}, \qquad d\mathfrak{Q}_1 \sim d\mathfrak{Q} \qquad (\mathfrak{A} \to 0).$$

It follows that

$$v(L_0) \sim 2^{n(n-1)} \int_{L^*_0} |\,\mathfrak{Q}\bar{\mathfrak{Q}}\,|\, d\mathfrak{X}\, d\mathfrak{Y}\, d\mathfrak{Q}\, d\mathfrak{A} \qquad (G \to \mathfrak{J}).$$

We choose a real matrix $\mathfrak{C}_1^{(n)}$, such that $\mathfrak{C}'_1\mathfrak{C}_1 = \mathfrak{Y}$. Then the condition $\mathfrak{Y}\{\mathfrak{Q}\} = \mathfrak{H}$ is replaced by $\mathfrak{Q}'_2\bar{\mathfrak{Q}}_2 = \mathfrak{H}$ with $\mathfrak{Q}_2 = \mathfrak{C}_1\mathfrak{Q}$. Since

$$d\mathfrak{Q}_2 = |\,\mathfrak{C}_1\,|^{2n}\, d\mathfrak{Q} = |\,\mathfrak{Y}\,|^n\, d\mathfrak{Q}, \qquad |\,\mathfrak{Q}\bar{\mathfrak{Q}}\,| = |\,\mathfrak{Q}_2\bar{\mathfrak{Q}}_2\,|\ |\,\mathfrak{Y}\,|^{-1},$$

we obtain

$$(143) \quad v(L_0) \sim 2^{n(n-1)} \int_F |\,\mathfrak{Y}\,|^{-n-1}\, d\mathfrak{X}\, d\mathfrak{Y} \cdot \tfrac{1}{2} \int_{(\mathfrak{Q}'_2\bar{\mathfrak{Q}}_2,\, \mathfrak{A}) \subset G^*} |\,\mathfrak{Q}_2\bar{\mathfrak{Q}}_2\,|\, d\mathfrak{Q}_2\, d\mathfrak{A} \quad (G \to \mathfrak{J}).$$

On the other hand, by (133) and (136),

$$v(G)=\int_G d\mathfrak{G},\qquad d\mathfrak{G}=d\mathfrak{G}_1 d\mathfrak{G}_2 d\mathfrak{G}_3=2^{n(n-1)}d\mathfrak{H}_1 d\mathfrak{A},$$
$$d\mathfrak{H}_1\sim d\mathfrak{H}\qquad (G\to\mathfrak{J});$$

hence

$$v(G)\sim 2^{n(n-1)}\int_{(\mathfrak{H},\mathfrak{A})\subset G^*} d\mathfrak{H}d\mathfrak{A}. \tag{144}$$

The first integral in (143) has the value $V_n=V(\Gamma)$. By (132), (143) and (144), the proof of (142) is reduced to the proof of the following lemma.

LEMMA 17. *Let H_1 be a domain in the space of the positive hermitian matrices and $\mathfrak{H}^*>0$; then*

$$\int_{\mathfrak{H}^*\{\mathfrak{Q}\}\subset H_1} d\mathfrak{Q}=c_n\,|\,\mathfrak{H}^*\,|^{-n}\int_{\mathfrak{H}\subset H_1} d\mathfrak{H},\qquad c_n=\prod_{k=1}^{n}\frac{\pi^k}{(k-1)\,!}. \tag{145}$$

We determine a matrix $\mathfrak{C}^*$ satisfying $\mathfrak{H}^*\{\mathfrak{C}^*\}=\mathfrak{E}$ and replace $\mathfrak{Q}$, $d\mathfrak{Q}$ by $\mathfrak{C}^*\mathfrak{Q}$, $|\,\mathfrak{C}^*\bar{\mathfrak{C}}^*\,|^n\,d\mathfrak{Q}$. Then we have only to prove (145) in the special case $\mathfrak{H}^*=\mathfrak{E}$.

We apply induction and assume first $n>1$. Let $\mathfrak{Q}_0$ be a matrix having the same first column $\mathfrak{q}$ as $\mathfrak{Q}$ and $|\,\mathfrak{Q}_0\,|\neq 0$; then

$$\mathfrak{Q}=\mathfrak{Q}_0\begin{pmatrix}1&\mathfrak{t}'\\0&\mathfrak{T}\end{pmatrix},$$
$$d\mathfrak{Q}=|\,\mathfrak{Q}_0\bar{\mathfrak{Q}}_0\,|^{n-1}\,d\mathfrak{q}\,d\mathfrak{t}\,d\mathfrak{T}.$$

Introducing

$$\mathfrak{Q}'_0\bar{\mathfrak{Q}}_0=\mathfrak{H}_0=\begin{pmatrix}h&\bar{\mathfrak{d}}'_0\\\mathfrak{d}_0&\mathfrak{D}_0\end{pmatrix}=\begin{pmatrix}h&0\\0&\mathfrak{R}_0\end{pmatrix}\left\{\begin{matrix}1&h^{-1}\bar{\mathfrak{d}}'_0\\0&\mathfrak{E}\end{matrix}\right\},$$
$$\mathfrak{Q}'\,\bar{\mathfrak{Q}}\;=\mathfrak{H}\;=\begin{pmatrix}h&\bar{\mathfrak{d}}'\\\mathfrak{d}&\mathfrak{D}\end{pmatrix}=\begin{pmatrix}h&0\\0&\mathfrak{R}\end{pmatrix}\left\{\begin{matrix}1&h^{-1}\bar{\mathfrak{d}}'\\0&\mathfrak{E}\end{matrix}\right\}, \tag{146}$$

we have

$$\begin{gathered}h=\mathfrak{q}'\bar{\mathfrak{q}},\quad \mathfrak{R}=\mathfrak{D}-h^{-1}\mathfrak{d}\bar{\mathfrak{d}}',\\ h\,|\,\mathfrak{R}_0\,|=|\,\mathfrak{H}_0\,|=|\,\mathfrak{Q}_0\bar{\mathfrak{Q}}_0\,|,\quad \mathfrak{R}=\mathfrak{R}_0\{\mathfrak{T}\},\quad \mathfrak{d}=\mathfrak{T}'\mathfrak{d}_0+h\mathfrak{t}.\end{gathered} \tag{147}$$

We replace $\mathfrak{t}$ by the new variable $\mathfrak{d}$ and obtain

$$d\mathfrak{t}=h^{2-2n}d\mathfrak{d},$$
$$d\mathfrak{Q}=|\,\mathfrak{Q}_0\bar{\mathfrak{Q}}_0\,|^{n-1}h^{2-2n}d\mathfrak{q}\,d\mathfrak{d}\,d\mathfrak{T}.$$

Using (145) with $n-1$, $\mathfrak{R}_0$, $\mathfrak{T}$, $\mathfrak{R}$ instead of n, $\mathfrak{H}^*$, $\mathfrak{Q}$, $\mathfrak{H}$, we find

$$\int_{\mathfrak{Q}'\bar{\mathfrak{Q}}\subset H_1} d\mathfrak{Q}=c_{n-1}\int|\,\mathfrak{Q}_0\bar{\mathfrak{Q}}_0\,|^{n-1}h^{2-2n}\,|\,\mathfrak{R}_0\,|^{1-n}d\mathfrak{q}\,d\mathfrak{d}\,d\mathfrak{R}, \tag{148}$$

where the domain of integration for the variables $\mathfrak{q}$, $\mathfrak{h}$, $\mathfrak{R}$ is given by (146), (147) and the condition $\mathfrak{H} \subset H_1$. We take the new variable $\mathfrak{D}$ instead of $\mathfrak{R}$, with $d\mathfrak{D} = d\mathfrak{R}$, and perform the integration over $\mathfrak{q}$, for fixed values of $\mathfrak{h}$ and $\mathfrak{D}$. Now

$$\int\limits_{\mathfrak{q}'\bar{\mathfrak{q}} \leq g} d\mathfrak{q} = J(g) = \frac{\pi^n}{n!} g^n; \qquad (g > 0),$$

whence

$$(149) \qquad \int\limits_{h_1 \leq h \leq h_2} h^{1-n} d\mathfrak{q} = \int_{h_1}^{h_2} h^{1-n} \frac{dJ(h)}{dh}\, dh = \frac{\pi^n}{(n-1)!} \int_{h_1}^{h_2} dh.$$

But

$$| \mathfrak{Q}_0 \bar{\mathfrak{Q}}_0 |^{n-1} h^{2-2n} | \mathfrak{R}_0 |^{1-n} = h^{1-n}, \qquad c_{n-1} \frac{\pi^n}{(n-1)!} = c_n$$

and consequently, by (148) and (149),

$$(150) \qquad \int\limits_{\mathfrak{Q}'\bar{\mathfrak{Q}} \subset H_1} d\mathfrak{Q} = c_n \int\limits_{\mathfrak{H} \subset H_1} d\mathfrak{H}.$$

It remains to prove (150) in the case $n = 1$. Then it is contained in formula (149) which holds good also in this case.

50. The relationship

$$d_0 = \prod_p d_p$$

constitutes another example of the "integral formulae" of analytic number theory which appear in the theory of class fields and in the theory of quadratic forms. The formula also holds good in the case of an arbitrary group $\Delta(\mathfrak{G}, \mathfrak{H})$, if the densities d_0 and d_p are defined in an analogous manner. But we do not go into detail since the proof of this statement depends upon the analytic theory of hermitian forms, of which no complete account has hitherto been given.

We proved in Section **31**, that the matrix $\mathfrak{Y}^{-1}$ is bounded in the fundamental domain F of the modular group. Theorem 11 may be used for an estimation of the maximum μ_n of $| \mathfrak{Y} |^{-1}$ in F. It follows from (80), (90) and (91), that F is contained in the domain F_1 defined by the conditions

$$| \mathfrak{Y} |^{-1} \leq \mu_n, \quad \mathfrak{Y}^{-1} \subset R, \quad -\tfrac{1}{2} \leq x_{kl} \leq \tfrac{1}{2} \qquad (1 \leq k \leq l \leq n),$$

where R is the Minkowski domain of Section **29**. Consequently the symplectic volume of F_1 is equal to the euclidean volume of that part of R which is defined by $| \mathfrak{X} | \leq \mu_n$, $\mathfrak{X} \subset R$. By an important result of Minkowski, this volume is

$$(151) \qquad \frac{2}{n+1} \mu_n{}^{(n+1)/2} \prod_{k=2}^{n} \left\{ \pi^{-k/2} \Gamma\left(\frac{k}{2}\right) \zeta(k) \right\}.$$

Hence this number is an upper bound of V_n. Using (127), we obtain the estimation

$$(152) \qquad \mu_n \geq \nu_n, \qquad \nu_n^{(n+1)/2} = \frac{n+1}{6}\,\pi \prod_{k=2}^{n} \left\{ \pi^{-k/2} \frac{\Gamma(k)\zeta(2k)}{\Gamma(k/2)\zeta(k)} \right\}$$

and in particular

$$\mu_1 \geq \frac{\pi}{3}, \qquad \mu_2^{3/2} \geq \frac{\pi^2}{30}, \qquad \mu_3^2 \geq \frac{8\pi^6}{42525\zeta(3)},$$

the exact value of μ_1 being $2/3^{1/2}$. By Stirling's formula, for $n \to \infty$,

$$\log \nu_n = \frac{n}{2}\left(\log \frac{2n}{\pi} - \frac{3}{2}\right) + O(1); \quad \log \mu_n > \frac{n}{2} \log n + O(n).$$

This proves that μ_n increases rapidly, as a function of n.

Let now $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$ be an arbitrary point of H and let $\mathfrak{Z}^* = \mathfrak{X}^* + i\mathfrak{Y}^* = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}$ be the image of $\mathfrak{Z}$ under any modular transformation. By Section **30**, the expression

$$|\mathfrak{Y}^*|^{-1} = |\mathfrak{Y}|^{-1} \operatorname{abs} (\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^2$$

has its minimum value, if and only if $\mathfrak{Z}^*$ lies in F. Hence μ_n is the maximum of these minima, for the set of all $\mathfrak{Z}$ in H. A fortiori, there exists a matrix $\mathfrak{Z}$ in H, such that the inequality

$$\operatorname{abs} (\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^2 \geq \nu_n |\mathfrak{Y}|$$

holds for all second matrix rows $(\mathfrak{C}\mathfrak{D})$ of modular matrices, where ν_n is the number defined in (152). Writing $\mathfrak{Z} = \mathfrak{P}\mathfrak{Q}^{-1}$, we have

$$\mathfrak{P}'\mathfrak{Q} - \mathfrak{Q}'\mathfrak{P} = 0, \quad \frac{1}{2i}(\mathfrak{P}'\bar{\mathfrak{Q}} - \mathfrak{Q}'\bar{\mathfrak{P}}) > 0$$

and

$$\operatorname{abs} (\mathfrak{C}\mathfrak{P} + \mathfrak{D}\mathfrak{Q})^2 \geq \nu_n \left| \frac{1}{2i}(\mathfrak{P}'\bar{\mathfrak{Q}} - \mathfrak{Q}'\bar{\mathfrak{P}}) \right|.$$

IX. COMMENSURABLE GROUPS.

51. We proved in Chapter V that the group $\Delta_0(\mathfrak{G}, \mathfrak{H})$ is commensurable with $\Delta_0(r, s)$. In order to demonstrate Theorem 13, we have now to discuss the necessary and sufficient conditions for the commensurability of two groups $\Delta_0(r, s)$ and $\Delta_0(r_1, s_1)$. We assumed that the numbers r and s are integers of the totally real field K; all conjugates of s except s itself are negative, whereas r is totally positive. Obviously s generates the field K. The numbers r_1 and s_1 have the same properties with respect to the totally real field K_1.

With the groups $\Delta_0(r, s)$ and $\Delta_0(r_1, s_1)$ we associate the two quadratic forms

$$Q(r, s) = rsx^2 - ry^2 + sz^2 \quad \text{and} \quad Q(r_1, s_1) = r_1 s_1 x_1{}^2 - r_1 y_1{}^2 + s_1 z_1{}^2$$

in 3 variables.

LEMMA 18. *If $Q(r, s)$ and $Q(r_1, s_1)$ are equivalent in K, then $K = K_1$ and the groups $\Delta_0(r, s)$, $\Delta_0(r_1, s_1)$ are commensurable.*

Under the assumption of the lemma, s_1 is certainly a number of K; therefore $K_1 \subset K$. Since all conjugates of $Q(r, s)$ are negative definite except $Q(r, s)$ itself, the same holds good for $Q(r_1, s_1)$ considered as a quadratic form in K; consequently s_1 is also a generating number of K and $K = K_1$.

Using the quaternion units $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$ defined in (74), we have $Q(r, s)\epsilon_0 = (x\epsilon_1 + y\epsilon_2 + z\epsilon_3)^2$. Let $\mathfrak{F} = (f_{kl})$ be the matrix of the linear substitution transforming $Q(r, s)$ into $Q(r_1, s_1)$ and let

$$\eta_k = \sum_{l=1}^{3} f_{lk}\epsilon_l \qquad (k = 1, 2, 3).$$

Then

$$\eta_1{}^2 = r_1 s_1 \epsilon_0, \quad \eta_2{}^2 = -r_1\epsilon_0, \quad \eta_3{}^2 = s_1\epsilon_0, \quad \eta_k\eta_l = -\eta_l\eta_k \quad (1 \leq k < l \leq 3)$$

and consequently there exists a real matrix $\mathfrak{L}^{(2)}$ satisfying

$$|\mathfrak{L}| = 1, \qquad \mathfrak{L}^{-1}\eta_k\mathfrak{L} = \omega_k \qquad (k = 1, 2, 3),$$

$$\omega_1 = \sqrt{r_1 s_1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \omega_2 = \pm\sqrt{r_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \omega_3 = \pm\sqrt{s_1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The elements of $\Delta_0(r, s)$ are defined by

$$\mathfrak{M} = \sum_{k=0}^{3} \mathfrak{A}_k \times \epsilon_k, \qquad \mathfrak{S}[\mathfrak{M}] = \mathfrak{S},$$

with integral $\mathfrak{A}_k$ $(k = 0, \cdots, 3)$ in K. Putting

$$(\mathfrak{E}^{(n)} \times \mathfrak{F}^{-1})(\mathfrak{A}_1\mathfrak{A}_2\mathfrak{A}_3)' = (\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}_3)', \qquad \mathfrak{E}^{(n)} \times \mathfrak{L} = \mathfrak{K}, \tag{153}$$

we obtain

$$\mathfrak{M} = \mathfrak{A}_0 \times \epsilon_0 + \sum_{k=1}^{3} \mathfrak{B}_k \times \eta_k = \mathfrak{K}\mathfrak{M}_1\mathfrak{K}^{-1}$$

with

$$\mathfrak{M}_1 = \mathfrak{A}_0 \times \epsilon_0 + \sum_{k=1}^{3} \mathfrak{B}_k \times \omega_k, \tag{154}$$

and $\mathfrak{K}$ is symplectic.

Let q be an even positive rational integer, such that the matrix $q\mathfrak{F}^{-1}/2rs$ is integral. If $\mathfrak{M} \equiv \mathfrak{E} \pmod{q}$, then $\mathfrak{A}_0 + \sqrt{rs}\mathfrak{A}_1 \equiv \mathfrak{E}$, $\mathfrak{A}_0 - \sqrt{rs}\mathfrak{A}_1 \equiv \mathfrak{E}$,

$\sqrt{r}\mathfrak{A}_2 + \sqrt{s}\mathfrak{A}_3 \equiv 0$, $-\sqrt{r}\mathfrak{A}_2 + \sqrt{s}\mathfrak{A}_3 \equiv 0$; hence $\mathfrak{A}_0$ is integral and $\mathfrak{A}_k \equiv 0$ $(\operatorname{mod} q/2rs)$ $(k = 1, 2, 3)$. By (153) and (154), $\mathfrak{B}_k$ is also integral and $\mathfrak{M}_1 = \mathfrak{K}^{-1}\mathfrak{M}\mathfrak{K}$ is an element of the group $\Delta_0(r_1, s_1)$. Consider now in $\Delta_0(r, s)$ the subgroup of all $\mathfrak{M}$, such that $\mathfrak{M}_1$ is contained in $\Delta_0(r_1, s_1)$. Since this subgroup Δ^*_0 contains the congruence subgroup of $\Delta_0(r, s)$, for the module q, it is of finite index in $\Delta_0(r, s)$. On the other hand, $\mathfrak{K}^{-1}\Delta^*_0\mathfrak{K}$ consists of all $\mathfrak{M}_1$ in $\Delta_0(r_1, s_1)$, such that $\mathfrak{M} = \mathfrak{K}\mathfrak{M}_1\mathfrak{K}^{-1}$ is an element of $\Delta_0(r, s)$; hence the same argument shows that $\mathfrak{K}^{-1}\Delta^*_0\mathfrak{K}$ is a subgroup of finite index in $\Delta_0(r_1, s_1)$. It follows that $\Delta_0(r, s)$ and $\Delta_0(r_1, s_1)$ are commensurable.

52. On account of Lemma 18, the condition of Theorem 13 is sufficient for the commensurability of $\Delta_0(r, s)$ and $\Delta_0(r_1, s_1)$. It remains to prove that this condition is also necessary, which is a little more difficult.

LEMMA 19. *Let a and b be two numbers of K, $ab \neq 0$ and K_0 an arbitrary algebraic number field. There exists an integral number t in K, such that $at^2 - b$ is not the square of a number of K_0 and $a(at^2 - b)$ is totally positive.*

We may obviously assume that a, b are integral and $K \subset K_0$. We choose in K a prime ideal λ having the following 3 properties: λ is not divisible by the square of a prime ideal of K_0, i. e., λ is not a factor of the relative discriminant of K_0 with respect to K; $2ab$ is not divisible by λ; ab is a quadratic residue modulo λ in K. Then there exists an integer t in K, such that $at^2 - b$ is divisible by λ, but not by λ^2, and that, moreover, $a(at^2 - b)$ is totally positive. Since $at^2 - b$ is divisible by exactly the first power of a prime ideal in K_0, it cannot be a square number in K_0.

LEMMA 20. *There exists a quadratic form $Q(r_0, s_0)$ equivalent with $Q(r, s)$ in K and a quadratic form $Q(r_2, s_2)$ equivalent with $Q(r_1, s_1)$ in K_1, such that the field $K^*(\sqrt{r_0}, \sqrt{s_0}, \sqrt{r_2}, \sqrt{s_2})$ has the degree 16 relative to K^*, where K^* is the union of the fields K and K_1.*

Using Lemma 19, we choose an integral number t in K, such that $rt^2 - s = r_0$ is totally positive and no square in K^*. Then the quadratic form $Q(r, s) = rsx^2 - ry^2 + sz^2$ is equivalent with $rsx^2 - r_0y^2 + r_0rsz^2$ in K. Applying again Lemma 19, we construct an integer u in K, such that $r_0rsu^2 + rs = s_0$ is no square in $K^*(\sqrt{r_0})$ and ss_0 is totally positive. Then $Q(r, s)$ is equivalent with $Q(r_0, s_0)$ in K and the field $K^*(\sqrt{r_0}, \sqrt{s_0})$ has the degree 4 relative to K^*. We complete the proof of the lemma by using the same argument for $Q(r_1, s_1)$, K_1, $K^*(\sqrt{r_0}, \sqrt{s_0})$ instead of $Q(r, s)$, K, K^*.

By Lemmata 18 and 20, we can assume for the rest of the proof of Theorem 13, that $K^*(\sqrt{r}, \sqrt{s}, \sqrt{r_1}, \sqrt{s_1})$ has the degree 16 relative to the union K^* of K and K_1.

LEMMA 21. *Let G_1 be a subgroup of another group G, of finite index j. For any element A of G, there exists a positive rational integer $g \leq j$, such that A^g is an element of G_1.*

Let $G_1, G_2, \cdots, G_j$ be the right cosets of G_1 relative to G and consider the $j+1$ powers A^k $(k = 0, \cdots, j)$. Then two of these powers A^k and A^l $(0 \leq k < l \leq j)$ lie in the same coset, and A^g, with $g = l - k$, is an element of G_1, q. e. d.

53. LEMMA 22. *Let Δ^* be a subgroup of $\Delta_0(r, s)$ of finite index. Then there exists in Δ^* a diagonal matrix $\mathfrak{P}$, such that all diagonal elements of $\mathfrak{P}$ are different one from another and that no conjugate of $\mathfrak{P}$, different from $\mathfrak{P}$ and $\mathfrak{P}^{-1}$, has a diagonal element in common with $\mathfrak{P}$.*

Let h be the degree of the field K with the conjugates $K^{(1)}, \cdots, K^{(h)}$ and $K^{(1)} = K$. The number of algebraic fundamental units in K is $h - 1$. The field $K(\sqrt{rs})$ has the degree $2h$; since it has 2 real conjugates and $h - 1$ pairs of conjugate complex conjugates, the number of algebraic fundamental units in $K(\sqrt{rs})$ is h. Consequently there exists an algebraic unit $\lambda = a + b\sqrt{rs}$, where a and b are numbers of K, such that no power λ^q $(q = \pm 1, \pm 2, \cdots)$ is a number of K.

Denoting by N the norm relative to K, we have $N(\lambda) = a^2 - rsb^2 = c$, where c is an algebraic unit in K, and $N(c^{-1}\lambda^2) = 1$. We replace $c^{-1}\lambda^2$ again by λ; then λ is an algebraic unit in $K(\sqrt{rs})$, no power λ^q $(q = \pm 1, \pm 2, \cdots)$ is a number of K, and $N(\lambda) = 1$, $\lambda^{-1} = a - b\sqrt{rs}$. By Fermat's theorem, we may, moreover, assume $\lambda \equiv 1 \pmod{2\sqrt{rs}}$; hence a and b are integers of K.

Let $q_1, \cdots, q_n$ be different positive rational integers and let

$$\mathfrak{P}_1 = [\lambda^{q_1}, \cdots, \lambda^{q_n}].$$

Then the matrix

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{P}_1 & 0 \\ 0 & \mathfrak{P}_1^{-1} \end{pmatrix} = \tfrac{1}{2}(\mathfrak{P}_1 + \mathfrak{P}_1^{-1}) \times \epsilon_0 + \frac{1}{2\sqrt{rs}}(\mathfrak{P}_1 - \mathfrak{P}_1^{-1}) \times \epsilon_1$$

is an element of the group $\Delta_0(r, s)$. By Lemma 21, a certain power $\mathfrak{P}^g$ with positive exponent g lies in the subgroup Δ^*. Replacing gq_k $(k = 1, \cdots, n)$ by q_k, we may assume that already $\mathfrak{P}$ is an element of Δ^*. The diagonal elements of $\mathfrak{P}$ are $\lambda^{q_k}, \lambda^{-q_k}$ $(k = 1, \cdots, n)$. Since the exponents $q_k, -q_k$ are all different and λ is no root of unity, these diagonal elements are all different.

For any number t of K, we denote by $t^{(l)}$ the conjugate of t in $K^{(l)}$. The $2h$ conjugates of $\lambda^{q_k} = a_k + b_k\sqrt{rs}$ are $a_k^{(l)} \pm b_k^{(l)}\sqrt{r^{(l)}s^{(l)}}$ $(l = 1, \cdots, h)$. Since $b_k \neq 0$ and $r^{(l)}s^{(l)} < 0$ for $l = 2, \cdots, h$, only the two conjugates $a_k + b_k\sqrt{rs} = \lambda^{q_k}$ and $a_k - b_k\sqrt{rs} = \lambda^{-q_k}$ are real. If a conjugate $\mathfrak{P}^*$ of $\mathfrak{P}$ has a diagonal element in common with $\mathfrak{P}$, then the substitution $\mathfrak{P} \rightarrow \mathfrak{P}^*$ arises either from the identical mapping $\lambda \rightarrow \lambda$ or from $\lambda \rightarrow \lambda^{-1}$, hence $\mathfrak{P}^* = \mathfrak{P}$ or $\mathfrak{P}^* = \mathfrak{P}^{-1}$, q. e. d.

We assume that the groups $\Delta_0(r, s)$ and $\Delta_0(r_1, s_1)$ are commensurable; we shall now first prove, that then $K = K_1$. There exists a subgroup Δ^* of $\Delta_0(r, s)$, of finite index, and a symplectic matrix $\mathfrak{R}$, such that $\mathfrak{R}^{-1}\Delta^*\mathfrak{R} = \Delta_1$ is a subgroup of $\Delta_0(r_1, s_1)$, of finite index. Henceforth we do not need the existence of a *symplectic* matrix $\mathfrak{R}$ with this property; we have only to assume that there is a non-singular matrix $\mathfrak{R}$ with real or complex elements satisfying $\Delta^*\mathfrak{R} = \mathfrak{R}\Delta_1$ where Δ^* and Δ_1 are subgroups of $\Delta_0(r, s)$ and $\Delta_0(r_1, s_1)$, of finite indices. Obviously we may then, moreover, assume that $\mathfrak{R}$ lies in the field $K^*(\sqrt{r}, \sqrt{s}, \sqrt{r_1}, \sqrt{s_1}) = K_0$, of degree 16 over the union K^* of K and K_1.

Let $\mathfrak{P}$ be the matrix of Lemma 22. Since $\mathfrak{P}$ belongs to the subgroup Δ^* of $\Delta_0(r, s)$, the matrix $\mathfrak{R}^{-1}\mathfrak{P}\mathfrak{R}$ is an element of Δ_1 and lies therefore in the field $K_1(\sqrt{r_1}, \sqrt{s_1})$. We consider any isomorphism A of K_0 which does not change the numbers of $K_1(\sqrt{r_1}, \sqrt{s_1})$. Denoting the image under A by the subscript A, we have $\mathfrak{R}_A{}^{-1}\mathfrak{P}_A\mathfrak{R}_A = \mathfrak{R}^{-1}\mathfrak{P}\mathfrak{R}$; hence the matrix $\mathfrak{B} = \mathfrak{R}_A\mathfrak{R}^{-1}$ satisfies

$$\mathfrak{P}_A = \mathfrak{B}\mathfrak{P}\mathfrak{B}^{-1}. \tag{155}$$

This proves that the diagonal matrices $\mathfrak{P}_A$ and $\mathfrak{P}$ have the same diagonal elements, perhaps in different order. By Lemma 22, either $\mathfrak{P}_A = \mathfrak{P}$ or $\mathfrak{P}_A = \mathfrak{P}^{-1}$. If $a + b\sqrt{rs}$ is a diagonal element of $\mathfrak{P}$, then $b \neq 0$ and $(a + b\sqrt{rs})_A = a \pm b\sqrt{rs}$; whence $(rs)_A = rs$.

On the other hand, rs generates K, since all conjugates of rs except rs itself are negative. Consequently all numbers of K are invariant under A. This proves $K \subset K_1(\sqrt{r_1}, \sqrt{s_1})$. Since $K^*(\sqrt{r_1}, \sqrt{s_1})$ has the degree 4 relative to K^*, the intersection of $K_1(\sqrt{r_1}, \sqrt{s_1})$ and K^* is K_1. Therefore

$K \subset K_1$. Interchanging K and K_1, we have also $K_1 \subset K$; consequently $K = K_1$.

54. We use the abbreviation

$$\mathfrak{C} = \begin{pmatrix} \mathfrak{E} & \mathfrak{E} \\ -\mathfrak{E} & \mathfrak{E} \end{pmatrix}.$$

LEMMA 23. *There exists an element $\mathfrak{M}$ of Δ^*, such that $\mathfrak{C}\mathfrak{M}\mathfrak{C}^{-1} = \mathfrak{Q}$ is a diagonal matrix with different diagonal elements.*

Analogously to the proof of Lemma 22, we construct an algebraic unit $\mu = a + b\sqrt{s}$ in $K(\sqrt{s})$, such that a, b are integers of K, $a^2 - sb^2 = 1$ and all powers μ^q $(q = \pm 1, \pm 2, \cdots)$ are different one from another. Let $q_1, \cdots, q_n$ be different positive rational integers, $\mathfrak{Q}_1 = [\mu^{q_1}, \cdots \mu^{q_n}]$ and

$$\mathfrak{Q} = \begin{pmatrix} \mathfrak{Q}_1 & 0 \\ 0 & \mathfrak{Q}_1^{-1} \end{pmatrix}.$$

Then the matrix

$$\mathfrak{M} = \mathfrak{C}^{-1}\mathfrak{Q}\mathfrak{C} = \tfrac{1}{2}\begin{pmatrix} \mathfrak{Q}_1 + \mathfrak{Q}_1^{-1} & \mathfrak{Q}_1 - \mathfrak{Q}_1^{-1} \\ \mathfrak{Q}_1 - \mathfrak{Q}_1^{-1} & \mathfrak{Q}_1 + \mathfrak{Q}_1^{-1} \end{pmatrix}$$

$$= \tfrac{1}{2}(\mathfrak{Q}_1 + \mathfrak{Q}_1^{-1}) \times \epsilon_0 + \frac{1}{2\sqrt{s}}(\mathfrak{Q}_1 - \mathfrak{Q}_1^{-1}) \times \epsilon_3$$

is an element of $\Delta_0(r, s)$. By Lemma 3, a power $\mathfrak{M}^g = \mathfrak{C}^{-1}\mathfrak{Q}^g\mathfrak{C}$ lies in Δ^*. Replacing $\mathfrak{M}^g, \mathfrak{Q}^g$ by $\mathfrak{M}, \mathfrak{Q}$, we obtain the proof of Lemma 23.

Let again A denote an isomorphism of $K(\sqrt{r}, \sqrt{s}, \sqrt{r_1}, \sqrt{s_1}) = K_0$ which leaves invariant all numbers of $K(\sqrt{r_1}, \sqrt{s_1})$. Analogous to (155), we obtain

$$\mathfrak{Q}_A = (\mathfrak{C}\mathfrak{B}\mathfrak{C}^{-1})\mathfrak{Q}(\mathfrak{C}\mathfrak{B}\mathfrak{C}^{-1})^{-1} \tag{156}$$

with the matrix $\mathfrak{Q}$ of Lemma 23 and $\mathfrak{B} = \mathfrak{R}_A\mathfrak{R}^{-1}$. Consider in particular the isomorphism A_s defined by $\sqrt{s} \to -\sqrt{s}$, $\sqrt{r} \to \sqrt{r}$ and write more shortly the subscript s instead of A_ε; then

$$\mathfrak{P}_s = \mathfrak{P}^{-1}, \qquad \mathfrak{Q}_s = \mathfrak{Q}^{-1}.$$

By (155) the matrix $\mathfrak{R}_s\mathfrak{R}^{-1} = \mathfrak{B}$ has the form

$$\mathfrak{B} = \begin{pmatrix} 0 & \mathfrak{D}(1) \\ \mathfrak{D}(0) & 0 \end{pmatrix},$$

where $\mathfrak{D}(0)$ and $\mathfrak{D}(1)$ denote diagonal matrices; by (156), the matrix $\mathfrak{C}\mathfrak{B}\mathfrak{C}^{-1}$ has the same form, whence

$$\mathfrak{R}_s = \begin{pmatrix} 0 & \mathfrak{D}(1) \\ -\mathfrak{D}(1) & 0 \end{pmatrix} \mathfrak{R}, \tag{157}$$

$$\begin{gathered} \mathfrak{R} = \mathfrak{R}_{ss} = -(\mathfrak{D}_s(1)\mathfrak{D}(1) \times \epsilon_0)\mathfrak{R}, \\ \mathfrak{D}_s(1)\mathfrak{D}(1) = -\mathfrak{E}. \end{gathered} \tag{158}$$

Consider now the isomorphism A_r defined by $\sqrt{r} \to -\sqrt{r}$, $\sqrt{s} \to \sqrt{s}$; then

$$\mathfrak{P}_r = \mathfrak{P}^{-1}, \qquad \mathfrak{Q}_r = \mathfrak{Q},$$

and we obtain from (155) and (156) the formula

$$\mathfrak{R}_r = \begin{pmatrix} 0 & \mathfrak{D}(2) \\ \mathfrak{D}(2) & 0 \end{pmatrix} \mathfrak{R} \tag{159}$$

with a diagonal matrix $\mathfrak{D}(2)$, whence

$$\mathfrak{D}_r(2)\mathfrak{D}(2) = \mathfrak{E}. \tag{160}$$

Interchanging r, s and r_1, s_1, we find in an analogous manner

$$\mathfrak{R}_{s_1} = \mathfrak{R}\begin{pmatrix} 0 & \mathfrak{D}(3) \\ -\mathfrak{D}(3) & 0 \end{pmatrix}, \qquad \mathfrak{R}_{r_1} = \mathfrak{R}\begin{pmatrix} 0 & \mathfrak{D}(4) \\ \mathfrak{D}(4) & 0 \end{pmatrix}; \tag{161}$$

$$\mathfrak{D}(3)\mathfrak{D}_{s_1}(3) = -\mathfrak{E}, \qquad \mathfrak{D}(4)\mathfrak{D}_{r_1}(4) = \mathfrak{E} \tag{162}$$

with diagonal matrices $\mathfrak{D}(3)$ and $\mathfrak{D}(4)$. Moreover $\mathfrak{R}_{rr_1} = \mathfrak{R}_{r_1 r}$, whence

$$(\mathfrak{D}_{r_1}(2)\mathfrak{D}^{-1}(2) \times \epsilon_0)\mathfrak{R} = \mathfrak{R}(\mathfrak{D}_r(4)\mathfrak{D}^{-1}(4) \times \epsilon_0). \tag{163}$$

Since $|\mathfrak{R}| \neq 0$ at least one element r_{1l} of the first row of $\mathfrak{R} = (r_{kl})$ is $\neq 0$. We assume $r_{1l} \neq 0$ for $l = p_0$ and define $p = p_0$, if $p_0 \leq n$, and $p = p_0 - n$, if $p > n$. Let a be the first diagonal element of $\mathfrak{D}(2)$ and b be the p-th diagonal element of $\mathfrak{D}(4)$. By (160), (162) and (163),

$$aa_r = 1, \qquad bb_{r_1} = 1, \qquad ab_r = ba_{r_1}.$$

55. LEMMA 24. *There exists a number* $c \neq 0$ *in* K_0 *satisfying*

$$ac = c_r, \qquad bc = c_{r_1}. \tag{164}$$

If $a \neq 1$, we define $d = \dfrac{\sqrt{r}}{a-1}$, whence

$$d_r = \frac{-\sqrt{r}}{a_r - 1} = \frac{-a\sqrt{r}}{1-a} = ad;$$

if $a = 1$, we define $d = 1$. In both cases $ad = d_r$. Putting $\dfrac{bd}{d_{r_1}} = f$, we find $ff_{r_1} = 1$,

$$f_r = \frac{b_r d_r}{d_{r_1 r}} = \frac{b_r a d}{(ad)_{r_1}} = \frac{bd}{d_{r_1}} = f.$$

Let $g = \frac{\sqrt{r_1}}{f-1}$ for $f \neq 1$, and $g = 1$ for $f = 1$. In both cases

$$fg = g_{r_1}, \qquad g_r = g,$$

and $c = dg$ has the required propetries.

Since there is an arbitrary scalar factor $\neq 0$ in $\mathfrak{R}$, we may replace $\mathfrak{R}$ by $c\mathfrak{R}$, where c is defined in Lemma 24. It follows from (159), (161) and (164), that then a and b are both replaced by 1.

Let α be the first diagonal element of $\mathfrak{D}(1)$ and β the p-th diagonal element of $\mathfrak{D}(3)$. By (158) and (162),

$$\alpha\alpha_s = -1, \qquad \beta\beta_{s_1} = -1. \tag{165}$$

Calculating

$$\mathfrak{R}_{rs} = \mathfrak{R}_{sr}, \quad \mathfrak{R}_{sr_1} = \mathfrak{R}_{r_1 s}, \quad \mathfrak{R}_{rs_1} = \mathfrak{R}_{s_1 r}, \quad \mathfrak{R}_{r_1 s_1} = \mathfrak{R}_{s_1 r_1} \quad \text{and} \quad \mathfrak{R}_{ss_1} = \mathfrak{R}_{s_1 s}$$

by (157), (159) and (161), we obtain, moreover,

$$\alpha_r = -\alpha, \quad \alpha_{r_1} = \alpha, \quad \beta_r = \beta, \quad \beta_{r_1} = -\beta, \quad \beta\alpha_{s_1} = \alpha\beta_s. \tag{166}$$

By (165) and (166), the numbers $u = \alpha\sqrt{r}$ and $v = \beta\sqrt{r_1}$ lie in $K(\sqrt{s}, \sqrt{s_1})$ and satisfy

$$uu_s = -r, \qquad vv_{s_1} = -r_1, \qquad vu_{s_1} = uv_s.$$

Defining $w = u/u_{s_1} = v/v_s$, we have $w_{s_1} = w^{-1} = w_s$, and consequently w is a number of the field $K(\sqrt{ss_1})$. Let $\tau = \sqrt{ss_1}/(w-1)$ for $w \neq 1$, and $\tau = 1$ for $w = 1$; in both cases τ lies in $K(\sqrt{ss_1})$ and $\tau_s = \tau_{s_1} = w\tau$. Then the numbers $\rho = u\tau$ and $\sigma = v\tau$ satisfy

$$\rho_{s_1} = u_{s_1}\tau_{s_1} = \frac{u}{w} w\tau = u\tau = \rho, \qquad \sigma_s = v_s\tau_s = \frac{v}{w} w\tau = v\tau = \sigma,$$

$$\rho\rho_s = u\tau u_s\tau_s = -r\tau\tau_s, \qquad \sigma\sigma_{s_1} = v\tau v_{s_1}\tau_{s_1} = -r_1\tau\tau_{s_1},$$

whence

$$\rho = \xi + \eta\sqrt{s}, \quad \sigma = \xi_1 + \eta_1\sqrt{s_1}, \quad \tau = \zeta + \omega\sqrt{ss_1},$$

$$\xi^2 - s\eta^2 = -r(\zeta^2 - ss_1\omega^2), \quad \xi_1^2 - s_1\eta_1^2 = -r_1(\zeta^2 - ss_1\omega^2), \tag{167}$$

where $\xi, \eta, \xi_1, \eta_1, \zeta, \omega$ are numbers of K which are not all 0.

We consider first the case $\omega \neq 0$. Performing the 3 linear substitutions

$$x = \eta x_3 + \xi y_3, \qquad y = \xi x_3 + s\eta y_3, \qquad z = z_3$$

$$x_3 = \frac{x_2}{s\eta^2 - \xi^2}, \qquad y_3 = \frac{\zeta y_2}{s\omega(s\eta^2 - \xi^2)} + \frac{z_2}{rs\omega}, \qquad z_3 = \frac{y_2 + \zeta z_2}{s\omega}$$

$$x_2 = s_1\eta_1 x_1 + \xi_1 y_1, \qquad y_2 = \xi_1 x_1 + \eta_1 y_1, \qquad z_2 = z_1,$$

we obtain, by (167),

$$\begin{aligned} Q(r,s) &= rsx^2 - ry^2 + sz^2 = r(s\eta^2 - \xi^2)(x_3^2 - sy_3^2) + sz_3^2 \\ &= \frac{r}{s\eta^2 - \xi^2}(x_2^2 - s_1 y_2^2) + s_1 z_2^2 = r_1 s_1 x_1^2 - r_1 y_1^2 + s_1 z_1^2 = Q(r_1, s_1); \end{aligned}$$

hence $Q(r,s)$ and $Q(r_1,s_1)$ are equivalent in K.

In the remaining case $\omega = 0$, we have

$$\xi^2 - s\eta^2 = -r\zeta^2 \quad \text{and} \quad \xi_1^2 - s_1\eta_1^2 = -r_1\zeta^2.$$

Consequently the diophantine equation $Q(r,s) = 0$ has the non-trivial solution $x = \xi$, $y = s\eta$, $z = r\zeta$. Moreover it follows immediately from the signs of the conjugates of r and s, that K is the field of rational numbers. Applying the 2 linear substitutions

$$x = \frac{\eta x_2 + \xi y_2}{r\zeta}, \qquad y = \frac{\xi x_2 + s\eta y_2}{r\zeta}, \qquad z = z_2$$

$$x_2 = x_1, \qquad y_2 = \frac{s+1}{2s}y_1 + \frac{s-1}{2s}z_1, \qquad z_2 = \frac{s-1}{2s}y_1 + \frac{s+1}{2s}z_1,$$

we obtain

$$Q(r,s) = x_2^2 - sy_2^2 + sz_2^2 = x_1^2 - y_1^2 + z_1^2 = Q(1,1);$$

hence $Q(r,s)$ and $Q(1,1)$ are equivalent in K. Since the same holds for $Q(r_1,s_1)$ instead of $Q(r,s)$, $Q(r,s)$ and $Q(r_1,s_1)$ are also equivalent in K.

Theorem 13 is now completely proved. In the particular case

$$\mathfrak{S}_1 = \mathfrak{J}, \quad \mathfrak{H}_1 = i\mathfrak{J}, \quad r_1 = 1, \quad s_1 = 1,$$

we have

$$\Delta(\mathfrak{S}_1, \mathfrak{H}_1) = \Gamma \quad \text{and} \quad Q(r_1, s_1) = x_1^2 - y_1^2 + z_1^2.$$

It follows that the group $\Delta(\mathfrak{S}, \mathfrak{H})$ is then and only then commensurable with the modular group, when the diophantine equation $x^2 + ry^2 = s$ has a solution x, y in K.

X. UNIT GROUPS OF QUINARY QUADRATIC FORMS.

56. For $n > 2$, the groups $\Delta(\mathfrak{G}, \mathfrak{H})$ and their subgroups are the only known examples of non-trivial discontinuous subgroups of the symplectic group. However, in the case $n = 2$, another set of examples is provided by the unit groups of certain quinary quadratic forms.

Consider the special quinary quadratic form

$$\mathfrak{S}[\mathfrak{w}] = w_1 w_2 - w_3^2 + w_4 w_5$$

and a complex column $\mathfrak{w}$ satisfying

$$(168) \qquad \mathfrak{S}[\mathfrak{w}] = 0, \qquad \mathfrak{S}\{\mathfrak{w}\} > 0.$$

If $w_5 = 0$, then $w_1 w_2 = w_3^2$ and $w_1 \bar{w}_2 + w_2 \bar{w}_1 - 2 w_3 \bar{w}_3 > 0$, whence $w_1 \neq 0$,

$$(w_1 \bar{w}_3 - w_3 \bar{w}_1)^2 = w_1 \bar{w}_1 (w_1 \bar{w}_2 + w_2 \bar{w}_1 - 2 w_3 \bar{w}_3) > 0,$$

which is impossible, the left-hand side being the square of a pure imaginary quantity; consequently $w_5 \neq 0$. Introducing inhomogeneous coördinates $w_5^{-1}\mathfrak{w} = \mathfrak{z} = \mathfrak{x} + i\mathfrak{y}$, we infer from

$$\mathfrak{S}[\mathfrak{z}] = 0 \quad \text{and} \quad \mathfrak{S}[\mathfrak{y}] = \mathfrak{S}\left[\frac{\mathfrak{z} - \bar{\mathfrak{z}}}{2i}\right] = \tfrac{1}{2}\mathfrak{S}\{\mathfrak{z}\} > 0$$

the relationships

$$(169) \qquad z_1 z_2 - z_3^2 = z_4, \qquad y_1 y_2 - y_3^2 > 0,$$

whence, in particular, $y_1 \neq 0$. On the other hand, (168) follows again from (169), if we define $\mathfrak{w} = w_5 \mathfrak{z}$ with arbitrary $w_5 \neq 0$.

The matrices

$$\mathfrak{Z} = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$$

with $y_1 y_2 - y_3^2 > 0$ and $y_1 > 0$ form the space H for $n = 2$; let $\bar{H}$ be the space defined by $y_1 y_2 - y_3^2 > 0$ and $y_1 < 0$.

Consider now a real linear transformation $\hat{\mathfrak{w}} = \mathfrak{V}\mathfrak{w}$, such that $\mathfrak{S}[\hat{\mathfrak{w}}] = 0$, $\mathfrak{S}\{\hat{\mathfrak{w}}\} > 0$ follows from (168). Since the equation $\mathfrak{S}[\mathfrak{w}] = 0$ is irreducible, we conclude that $\mathfrak{S}[\mathfrak{V}] = \lambda\mathfrak{S}$, with a scalar factor $\lambda \neq 0$. Consequently $|\mathfrak{V}| \neq 0$, and these transformations form a group. Since $w_1, \cdots, w_5$ are homogeneous coördinates, we may replace $\mathfrak{V}$ by $\mu\mathfrak{V}$, for any scalar $\mu \neq 0$. But $|\mu\mathfrak{V}| = \mu^5 |\mathfrak{V}|$; hence we may assume $|\mathfrak{V}| = 1$ and $\mathfrak{S}[\mathfrak{V}] = \mathfrak{S}$.

Put $\mathfrak{V} = (v_{kl})$ and $\hat{w}_5^{-1}\hat{\mathfrak{w}} = \hat{\mathfrak{z}} = \hat{\mathfrak{x}} + i\hat{\mathfrak{y}}$. Then the fractional linear substitution

$$\hat{z}_k = \frac{\sum_{l=1}^{5} v_{kl} z_l}{\sum_{l=1}^{5} v_{5l} z_l} \qquad (k = 1, 2, 3) \tag{170}$$

with $z_4 = z_1 z_2 - z_3^2$ and $z_5 = 1$ transforms the space H either into itself or into $\bar{H}$. We are only interested in the substitutions which leave H invariant; they form an invariant subgroup of index 2 in the whole group. We denote this subgroup by $\Omega(\mathfrak{S})$.

Obviously we obtain the solution of (170) with respect to z_1, z_2, z_3, if we replace $\mathfrak{V}$ by $\mathfrak{V}^{-1}$ and interchange $\mathfrak{z}, \hat{\mathfrak{z}}$. Hence (170) is a birational analytic mapping of H onto itself. A simple calculation gives the functional determinant

$$\frac{d(\hat{z}_1, \hat{z}_2, \hat{z}_3)}{d(z_1, z_2, z_3)} = \hat{w}_5^{-3} w_5^3 \neq 0. \tag{171}$$

On account of Theorem 1, the transformation (170) is symplectic:

$$\begin{pmatrix} \hat{z}_1 & \hat{z}_3 \\ \hat{z}_3 & \hat{z}_2 \end{pmatrix} = \hat{\mathfrak{Z}} = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1}, \quad \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} = \mathfrak{M}, \quad \mathfrak{J}[\mathfrak{M}] = \mathfrak{J}. \tag{172}$$

57. Let us now start from an arbitrary symplectic transformation (172). Defining

$$\mathfrak{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have $\mathfrak{J}_2[\mathfrak{F}] = |\mathfrak{F}|\,\mathfrak{J}_2$ for any matrix $\mathfrak{F}^{(2)}$, and consequently

$$(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} = |\mathfrak{C}\mathfrak{Z} + \mathfrak{D}|^{-1}\mathfrak{J}_2(\mathfrak{Z}\mathfrak{C}' + \mathfrak{D}')\mathfrak{J}_2^{-1}. \tag{173}$$

Moreover

$$|\mathfrak{F}_1 + \mathfrak{F}_2| = |\mathfrak{F}_1| + |\mathfrak{F}_2| + \sigma(\mathfrak{F}_1\mathfrak{J}_2\mathfrak{F}'_2\mathfrak{J}_2^{-1})$$

for any two matrices $\mathfrak{F}_1^{(2)}$ and $\mathfrak{F}_2^{(2)}$, whence

$$|\mathfrak{A}\mathfrak{Z} + \mathfrak{B}| = |\mathfrak{A}\mathfrak{Z}| + |\mathfrak{B}| + \sigma(\mathfrak{A}\mathfrak{Z}\mathfrak{J}_2\mathfrak{B}'\mathfrak{J}_2^{-1}) \tag{174}$$

$$|\mathfrak{C}\mathfrak{Z} + \mathfrak{D}| = |\mathfrak{C}\mathfrak{Z}| + |\mathfrak{D}| + \sigma(\mathfrak{C}\mathfrak{Z}\mathfrak{J}_2\mathfrak{D}'\mathfrak{J}_2^{-1}). \tag{175}$$

We apply (173), (174) and (175) for the calculation of

$$\hat{\mathfrak{Z}} = (\mathfrak{A}\mathfrak{Z} + \mathfrak{B})(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})^{-1} \quad \text{and} \quad |\hat{\mathfrak{Z}}| = |\mathfrak{A}\mathfrak{Z} + \mathfrak{B}|\,|\mathfrak{C}\mathfrak{Z} + \mathfrak{D}|^{-1}.$$

It follows that

$$\hat{\mathfrak{W}} = (\mathfrak{A}\mathfrak{W}\mathfrak{J}_2\mathfrak{D}' + \mathfrak{B}\mathfrak{J}_2\mathfrak{W}\mathfrak{C}' + w_4\mathfrak{A}\mathfrak{J}_2\mathfrak{C}' + w_5\mathfrak{B}\mathfrak{J}_2\mathfrak{D}')\mathfrak{J}_2^{-1} \tag{176}$$

$$\hat{w}_4 = \sigma(\mathfrak{A}\mathfrak{W}\mathfrak{J}_2\mathfrak{B}'\mathfrak{J}_2^{-1}) + |\mathfrak{A}|\, w_4 + |\mathfrak{B}|\, w_5 \tag{177}$$

$$\hat{w}_5 = \sigma(\mathfrak{C}\mathfrak{W}\mathfrak{J}_2\mathfrak{D}'\mathfrak{J}_2^{-1}) + |\mathfrak{C}|\, w_4 + |\mathfrak{D}|\, w_5 \tag{178}$$

with

$$\mathfrak{W} = \begin{pmatrix} w_1 & w_3 \\ w_3 & w_2 \end{pmatrix}, \qquad \hat{\mathfrak{W}} = \begin{pmatrix} \hat{w}_1 & \hat{w}_3 \\ \hat{w}_3 & \hat{w}_2 \end{pmatrix}$$

is a real linear substitution $\hat{\mathfrak{w}} = \mathfrak{V}\mathfrak{w}$ mapping the domain $\mathfrak{S}[\mathfrak{w}] = 0$, $\mathfrak{S}\{\mathfrak{w}\} > 0$ into itself. Hence $|\mathfrak{V}| = \nu \neq 0$, $|\nu^{-1/5}\mathfrak{V}| = 1$. Moreover, $\mathfrak{V}$ is not changed if we replace $\mathfrak{M}$ my $-\mathfrak{M}$. This proves the identity of $\Omega(\mathfrak{S})$ and the symplectic group Ω, for $n = 2$. By (171) and (178), the functional determinant of the transformation (172) has the value

$$(\nu^{-1/5}\hat{w}_5)^{-3} w_5^{\,3} = \nu^{3/5}\, |\mathfrak{C}\mathfrak{Z} + \mathfrak{D}|^{-3};$$

on the other hand, by (24), $d\mathfrak{Z} = d\hat{\mathfrak{Z}}[\mathfrak{C}\mathfrak{Z} + \mathfrak{D}]$, which leads to the value $|\mathfrak{C}\mathfrak{Z} + \mathfrak{D}|^{-3}$ of that functional determinant; consequently $\nu = 1$, $|\mathfrak{V}| = 1$ and $\mathfrak{S}[\mathfrak{V}] = \mathfrak{S}$.

58. The formulae (176), (177), (178) and the identity of the groups $\Omega(\mathfrak{S})$ and Ω can be demonstrated in another way, without using Theorem 1.

Let us first determine all skew-symmetric symplectic matrices $\mathfrak{G}^{(4)}$. We have the conditions

$$\mathfrak{G} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} = -\mathfrak{G}' \tag{179}$$

$$\mathfrak{A}\mathfrak{B}' = \mathfrak{B}\mathfrak{A}', \qquad \mathfrak{C}\mathfrak{D}' = \mathfrak{D}\mathfrak{C}', \qquad \mathfrak{A}\mathfrak{D}' - \mathfrak{B}\mathfrak{C}' = \mathfrak{E}. \tag{180}$$

By (179),

$$\mathfrak{A} = w_4\mathfrak{J}_2, \quad \mathfrak{D} = w_5\mathfrak{J}_2, \quad \mathfrak{B} = \mathfrak{W}\mathfrak{J}_2, \quad \mathfrak{C} = \mathfrak{J}_2\mathfrak{W}', \quad \mathfrak{W} = \begin{pmatrix} w_1 & w_3 \\ w_6 & w_2 \end{pmatrix}$$

with arbitrary $w_1, \cdots, w_6$; by (180), $w_4\mathfrak{W}$ and $w_5\mathfrak{W}$ are symmetric and $\mathfrak{W}\mathfrak{J}_2\mathfrak{W} = (w_4 w_5 - 1)\mathfrak{J}_2$. Hence either $\mathfrak{W}$ is non-symmetric and then, necessarily, $\mathfrak{G} = \pm\mathfrak{J}$ or $\mathfrak{W}$ is symmetric and

$$w_1 w_2 - w_3^{\,2} = w_4 w_5 - 1. \tag{181}$$

Omitting in the second case the condition (181), we obtain the identity

$$(\mathfrak{J}\mathfrak{G})^2 = -\mathfrak{S}[\mathfrak{w}]\mathfrak{E} \tag{182}$$

with

$$\mathfrak{G} = \begin{pmatrix} w_4\mathfrak{J}_2 & \mathfrak{W}\mathfrak{J}_2 \\ \mathfrak{J}_2\mathfrak{W} & w_5\mathfrak{J}_2 \end{pmatrix}, \qquad \mathfrak{W} = \begin{pmatrix} w_1 & w_3 \\ w_3 & w_2 \end{pmatrix}$$

and arbitrary $w_1, \cdots, w_5$.

We substitute

$$(183) \qquad \begin{matrix} w_1 = -v_1 - v_2, & w_2 = v_1 - v_2, & w_3 = -v_3, \\ w_4 = -v_4 - v_5, & w_5 = v_4 - v_5; & \end{matrix}$$

then

$$(184) \qquad \mathfrak{J}\mathfrak{G} = \begin{pmatrix} \mathfrak{J}_2\mathfrak{W} & w_5\mathfrak{J}_2 \\ -w_4\mathfrak{J}_2 & -\mathfrak{W}\mathfrak{J}_2 \end{pmatrix}$$

$$= \begin{bmatrix} -v_3 & v_1 - v_2 & 0 & v_4 - v_5 \\ v_1 + v_2 & v_3 & -v_4 + v_5 & 0 \\ 0 & v_4 + v_5 & -v_3 & v_1 + v_2 \\ -v_4 - v_5 & 0 & v_1 - v_2 & v_3 \end{bmatrix} = \sum_{k=1}^{5} v_k\mathfrak{F}_k,$$

$$\mathfrak{S}[\mathfrak{w}] = \sum_{k=1}^{5} (-1)^k v_k^2,$$

$$(185) \qquad \begin{matrix} \mathfrak{F}_k^2 = (-1)^{k-1}\mathfrak{E} & (k = 1, \cdots, 5), \\ \mathfrak{F}_k\mathfrak{F}_l = -\mathfrak{F}_l\mathfrak{F}_k & (1 \leq k < l \leq 5), \end{matrix}$$

by (182). The 16 products $\mathfrak{F}_1^{e_1}\mathfrak{F}_2^{e_2}\mathfrak{F}_3^{e_3}\mathfrak{F}_4^{e_4}$ ($e_k = 0, 1$; $k = 1, \cdots, 4$) form a basis for the four-dimensional complete matric representation of the well-known Clifford-Lipschitz algebra of order 16. The matrix $\mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3\mathfrak{F}_4\mathfrak{F}_5$ is permutable with all matrices of the algebra and has the square $\mathfrak{E}$, hence it is equal to $\mathfrak{E}$ or $-\mathfrak{E}$; by direct calculation we find the value $\mathfrak{E}$ and therefore

$$(186) \qquad \mathfrak{F}_5 = \mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3\mathfrak{F}_4.$$

Let now $\hat{\mathfrak{w}} = \mathfrak{V}\mathfrak{w}$ be a real linear substitution with $\mathfrak{S}[\hat{\mathfrak{w}}] = \mathfrak{S}[\mathfrak{w}]$ and $|\mathfrak{V}| = 1$. We denote the matrix of the substitution (183) by $\mathfrak{L}$ and put $\mathfrak{L}^{-1}\mathfrak{V}\mathfrak{L} = \mathfrak{R} = (r_{kl})$. By (182) and (184), the matrices

$$(187) \qquad \hat{\mathfrak{F}}_k = \sum_{l=1}^{5} r_{lk}\mathfrak{F}_l \qquad (k = 1, \cdots, 5)$$

satisfy again (185), and consequently their product is $\pm\mathfrak{E}$. It follows from (185), (186) and the linear independence of the 16 products that

$$(188) \qquad \hat{\mathfrak{F}}_1\hat{\mathfrak{F}}_2\hat{\mathfrak{F}}_3\hat{\mathfrak{F}}_4\hat{\mathfrak{F}}_5 = |\mathfrak{R}|\,\mathfrak{E} = \mathfrak{E};$$

hence (186) holds also for $\hat{\mathfrak{F}}_k$ instead of $\mathfrak{F}_k$.

Since the two representations generated by $\mathfrak{F}_k$ and $\hat{\mathfrak{F}}_k$ are necessarily equivalent, there exists a real matrix $\mathfrak{M}$, such that

$$\hat{\mathfrak{F}}_k = \mathfrak{M}^{-1}\mathfrak{F}_k\mathfrak{M} \qquad (k = 1, \cdots, 5). \tag{189}$$

Moreover, by (184) and (187), the matrices $\mathfrak{J}\mathfrak{F}_k$ and $\mathfrak{J}\hat{\mathfrak{F}}_k$ are skew-symmetric, and consequently (189) leads to

$$\mathfrak{J}\mathfrak{M}^{-1}\mathfrak{F}_k\mathfrak{M} = \mathfrak{M}'\mathfrak{J}\mathfrak{F}_k\mathfrak{J}\mathfrak{M}'^{-1}\mathfrak{J}^{-1} \qquad (k = 1, \cdots, 5).$$

This proves that $\mathfrak{M}\mathfrak{J}\mathfrak{M}'\mathfrak{J}^{-1}$ is a scalar multiple of $\mathfrak{E}$. There is an arbitrary real scalar factor in $\mathfrak{M}$; hence we may assume $\mathfrak{J}[\mathfrak{M}] = \pm \mathfrak{J}$. Putting

$$\hat{\mathfrak{G}} = \begin{pmatrix} \hat{w}_4\mathfrak{J}_2 & \hat{\mathfrak{W}}\mathfrak{J}_2 \\ \mathfrak{J}_2\hat{\mathfrak{W}} & \hat{w}_5\mathfrak{J}_2 \end{pmatrix} = \sum_{k=1}^{5} v_k\mathfrak{J}^{-1}\hat{\mathfrak{F}}_k, \tag{190}$$

we obtain

$$\hat{\mathfrak{G}} = \pm\, \mathfrak{G}[\mathfrak{M}] \tag{191}$$

or more explicitly

$$\begin{pmatrix} \hat{w}_4\mathfrak{J}_2 & \hat{\mathfrak{W}}\mathfrak{J}_2 \\ \mathfrak{J}_2\hat{\mathfrak{W}} & \hat{w}_5\mathfrak{J}_2 \end{pmatrix} = \pm\, \mathfrak{M}' \begin{pmatrix} w_4\mathfrak{J}_2 & \mathfrak{W}\mathfrak{J}_2 \\ \mathfrak{J}_2\mathfrak{W} & w_5\mathfrak{J}_2 \end{pmatrix} \mathfrak{M}. \tag{192}$$

On the other hand, if $\mathfrak{M}$ is any real matrix satisfying $\mathfrak{J}[\mathfrak{M}] = \pm \mathfrak{J}$, then the matrix $\hat{\mathfrak{G}}$ in (191) is again a skew-symmetric solution of (182) and consequently of the form (190), and (192) defines a real linear substitution $\hat{\mathfrak{w}} = \mathfrak{V}\mathfrak{w}$ with $\mathfrak{S}[\mathfrak{V}] = \mathfrak{S}$ and $|\mathfrak{V}| = 1$, by (188). We may replace $\mathfrak{M}$ by $-\mathfrak{M}$ and obtain the same matrix $\mathfrak{V}$, but the pair $\mathfrak{M}, -\mathfrak{M}$ is uniquely determined by $\mathfrak{V}$.

Imposing on $\mathfrak{V}$ the condition $\mathfrak{J}[\mathfrak{M}] = + \mathfrak{J}$, we obtain a subgroup of index 2 in the group of all $\mathfrak{V}$; and this subgroup is isomorphic to the inhomogeneous symplectic group, by (192). If we write

$$\mathfrak{M}' = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$$

and calculate the single terms in (192), we find exactly the formulae (176), (177) and (178).

59. Let K be again a totally real algebraic number field of degree h. Let $\mathfrak{T}[\mathfrak{v}]$ be a quinary quadratic form with coefficients in K such that all conjugates except $\mathfrak{T}[\mathfrak{v}]$ itself are definite. We assume, moreover, that $\mathfrak{T}[\mathfrak{v}]$ has the signature 2, 3, i. e., that $\mathfrak{T}[\mathfrak{v}]$ can be transformed into $-v_1^2 + v_2^2 - v_3^2 + v_4^2 - v_5^2$ by a real linear substitution. Then there exists also a real matrix $\mathfrak{N}$, such that $\mathfrak{T}[\mathfrak{N}] = \mathfrak{S}$.

Consider now the units of $\mathfrak{T}$ in K, i. e., the integral matrices $\mathfrak{U}$ in K

satisfying $\mathfrak{T}[\mathfrak{U}] = \mathfrak{T}$. Since $|\mathfrak{U}| = \pm 1$ and $|-\mathfrak{U}| = -|\mathfrak{U}|$, we restrict ourselves to the case $|\mathfrak{U}| = +1$. Then the matrix $\mathfrak{N}^{-1}\mathfrak{U}\mathfrak{N} = \mathfrak{V}$ satisfies $\mathfrak{S}[\mathfrak{V}] = \mathfrak{S}$, $|\mathfrak{V}| = 1$, and the corresponding substitution (170) transforms H either into itself or into $\bar{H}$. We consider only the matrices $\mathfrak{U}$ with the first property; they form a subgroup of index 2 or 1 in the whole group of units with $|\mathfrak{U}| = 1$; we denote it by $\Lambda(\mathfrak{T})$, and by $\Delta(\mathfrak{T})$ the isomorphic subgroup of Ω.

Let us first prove that $\Delta(\mathfrak{T})$ is discontinuous. By Section **19**, it is sufficient to prove that $\Delta(\mathfrak{T})$ does not contain an infinite number of bounded elements. Otherwise $\Lambda(\mathfrak{T})$ would also contain infinitely many bounded matrices $\mathfrak{U}$, by (176), (177), (178) and $\mathfrak{U} = \mathfrak{N}\mathfrak{V}\mathfrak{N}^{-1}$. Since $\mathfrak{T}[\mathfrak{U}] = \mathfrak{T}$ and all conjugates of $\mathfrak{J}[\mathfrak{v}]$, except $\mathfrak{J}[\mathfrak{v}]$ itself, are definite, all conjugates of these matrices are bounded. Moreover their elements are integers, and this is a contradiction.

We apply now the following results from the theory of units of quadratic forms. Let Q be the space of real matrices $\mathfrak{Q}^{(32)}$ satisfying

$$\mathfrak{S}[\mathfrak{P}] > 0, \qquad \mathfrak{P}^{(52)} = \begin{pmatrix} \mathfrak{Q} \\ \mathfrak{E} \end{pmatrix}.$$

If

$$\mathfrak{V} = \begin{pmatrix} \mathfrak{V}_1 & \mathfrak{V}_2 \\ \mathfrak{V}_3 & \mathfrak{V}_4 \end{pmatrix}$$

is a real solution of $\mathfrak{S}[\mathfrak{V}] = \mathfrak{S}$, with $\mathfrak{V}_1 = \mathfrak{V}_1^{(3)}$, then the mapping

$$\mathfrak{Q} \to (\mathfrak{V}_1\mathfrak{Q} + \mathfrak{V}_2)(\mathfrak{V}_3\mathfrak{Q} + \mathfrak{V}_4)^{-1}$$

transforms Q into itself. We restrict $\mathfrak{V}$ to the matrices $\mathfrak{N}^{-1}\mathfrak{U}\mathfrak{N}$, where $\mathfrak{U}$ runs over the units of $\mathfrak{T}$ with $|\mathfrak{U}| = 1$. Then there exists in Q a fundamental domain Q_0, with respect to $\Lambda(\mathfrak{T})$, bounded by a finite number of algebraic surfaces and having only a finite number of neighbors. Moreover the integral

$$v(\mathfrak{T}) = \int_{Q_0} |\mathfrak{S}[\mathfrak{P}]|^{-5/2} d\mathfrak{Q} \tag{193}$$

has a finite value. As a matter of fact, a proof of these statements has been published only in the case of the field of rational numbers, instead of a totally real algebraic field K of arbitrary degree h; but the generalization of this proof offers no difficulties.

60. In order to derive the corresponding results for the group $\Delta(\mathfrak{T})$ and the space H, we have only to map Q onto H. If

$$(194) \qquad \mathfrak{Z} = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \subset H, \qquad z_4 = |\mathfrak{Z}|, \qquad z_5 = 1,$$

we define

$$\mathfrak{z} = \mathfrak{x} + i\mathfrak{y}, \qquad \mathfrak{P} = (\mathfrak{x}\mathfrak{y}) \begin{pmatrix} x_4 & y_4 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{2iy_4^2} (\mathfrak{z}\bar{\mathfrak{z}}) \begin{pmatrix} 1 & -\bar{z}_4 \\ -1 & z_4 \end{pmatrix};$$

then the elements of $\mathfrak{Q} = (q_{kl})$ are

$$q_{k1} = \frac{y_k}{y_4}, \qquad q_{k2} = \frac{x_k y_4 - y_k x_4}{y_4} \qquad (k = 1, 2, 3)$$

and

$$\mathfrak{S}[\mathfrak{P}] = \frac{1}{2y_4^2} \mathfrak{S}\{\mathfrak{z}\} \mathfrak{E} \begin{bmatrix} 0 & y_4 \\ 1 & -x_4 \end{bmatrix} > 0;$$

hence $\mathfrak{Q}$ is a point of Q. On the other hand, if $\mathfrak{Q}$ is an arbitrary point of Q, we determine z_4 from the quadratic equation

$$(195) \qquad \mathfrak{S}[\mathfrak{P}] \begin{bmatrix} z_4 \\ 1 \end{bmatrix} = 0$$

and put

$$(196) \qquad \mathfrak{P} \begin{pmatrix} z_4 \\ 1 \end{pmatrix} = \mathfrak{z};$$

then

$$\mathfrak{S}[\mathfrak{z}] = 0, \qquad \mathfrak{S}\{\mathfrak{z}\} = \mathfrak{S}[\mathfrak{P}] \left\{ \begin{matrix} z_4 \\ 1 \end{matrix} \right\} > 0.$$

If we replace z_4 by the other root $\bar{z}_4$ of (195), the column $\mathfrak{z}$ is replaced by $\bar{\mathfrak{z}}$; therefore we can choose the root z_4 of this quadratic equation, such that $y_1 > 0$, and then (194) is satisfied.

In this way, the fundamental domain Q_0 for $\Lambda(\mathfrak{T})$ in Q is mapped onto a fundamental domain F for $\Delta(\mathfrak{T})$ in H. It follows that F has only a finite number of neighbors $F_{\mathfrak{M}}$, for the elements $\mathfrak{M}$ of $\Delta(\mathfrak{T})$, and that any compact domain in H is covered by a finite number of images $F_{\mathfrak{M}}$; moreover F is bounded by a finite number of algebraic surfaces. Putting $\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$ and introducing the new variables $\mathfrak{X}, \mathfrak{Y}$ into (193), by (195) and (196), we obtain

$$v(\mathfrak{T}) = 4 \int_F |\mathfrak{Y}|^{-3} d\mathfrak{X} d\mathfrak{Y} = 4 \int_F d\mathfrak{X} d\mathfrak{Y}^{-1}.$$

This proves that F has a finite symplectic volume.

The latter result is not trivial in the case $h = 1$, since then H is not compact relative to $\Delta(\mathfrak{T})$; this may be derived from the theorem of A. Meyer, that an indefinite quinary quadratic form with rational coefficients is a zero form. On the other hand, $\mathfrak{T}[\mathfrak{b}]$ is not a zero form in K in the case $h > 1$, since we assumed that the conjugates of $\mathfrak{T}[\mathfrak{b}]$ are definite except $\mathfrak{T}[\mathfrak{b}]$ itself; it can be proved as a simple consequence that then F is compact.

This is a sketch of the proof of Theorem 14; the details may be completed according to the scheme of Chapter VII.

Consider now the invariant subgroup $\Delta_\kappa(\mathfrak{T})$ of $\Delta(\mathfrak{T})$, defined by the condition $\mathfrak{U} \equiv \mathfrak{E} \pmod{\kappa}$ for the units $\mathfrak{U}$ of $\mathfrak{T}$, where κ denotes, as in Theorem 10, a certain power of an arbitrary prime ideal. It follows from Section **39**, that $\Delta_\kappa(\mathfrak{T})$ has no fixed points in H. The subgroup $\Delta_\kappa(\mathfrak{T})$ is of finite index j in $\Delta(\mathfrak{T})$, and the union of j images $F\mathfrak{M}$ of F, for suitably chosen elements $\mathfrak{M}$ of $\Delta(\mathfrak{T})$, constitutes a fundamental domain F_κ of $\Delta_\kappa(\mathfrak{T})$. Since the domain F_κ is compact in the case $h > 1$, it gives another example of a closed manifold with the symplectic metric.

It is known that the volume $v(\mathfrak{T})$ appears in the formula for the measure of the genus of $\mathfrak{T}$. In this way an analogue of Theorem 12 might be found.

61. We proved in Section **27**, that $\Delta(\mathfrak{G}, \mathfrak{J})$ is commensurable with a group $\Delta(r, s)$ of symplectic quaternion matrices $\pm \mathfrak{M}$. We shall now derive a corresponding result for the group $\Delta(\mathfrak{T})$.

A matrix of rank r is called *primitive*, if its elements are algebraic integers and if the minors of degree r are relative prime. Let $\mathfrak{W}_0^{(24)}$ be a primitive matrix of rank 2. Then the matrix $\mathfrak{J}[\mathfrak{W}_0] = \mathfrak{G}^{(4)} = (g_{kl})$ is skew-symmetric and has the rank 2; its elements are the minors of $\mathfrak{W}_0$. It follows by an application of Laplace's theorem, that also $\mathfrak{G}$ is primitive.

LEMMA 25. *Let $\mathfrak{G}^{(4)}$ be a primitive skew-symmetric matrix of rank 2, with elements from an algebraic number field K_0. Then there exists in K_0 a primitive matrix $\mathfrak{W}_0$, such that $\mathfrak{J}_0[\mathfrak{W}_0] = \mathfrak{G}$.*

Since $\mathfrak{G}$ is a primitive matrix of rank 2 and degree 4, with elements from K_0, we can determine in K_0 a unimodular matrix $\mathfrak{U}_0$, such that the first two columns of $\mathfrak{G}\mathfrak{U}_0$ are zero. Moreover the matrix $\mathfrak{G}[\mathfrak{U}_0]$ is skew-symmetric and again primitive; hence

$$\mathfrak{G}[\mathfrak{U}_0] = \epsilon \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{J}_2 \end{pmatrix},$$

where ϵ is an algebraic unit in K_0. Obviously we may choose $\mathfrak{U}_0$, such that $\epsilon = 1$. Denoting by $\mathfrak{W}_0$ the matrix of the two last rows of $\mathfrak{U}_0^{-1}$, we obtain the statement of the lemma.

For any skew-symmetric $\mathfrak{G}^{(4)} = (g_{kl})$,

$$|\mathfrak{G}| = (g_{12}g_{34} - g_{13}g_{24} + g_{14}g_{23})^2$$

and consequently

$$g_{12}g_{34} - g_{13}g_{24} + g_{14}g_{23} = 0, \tag{197}$$

if the rank of $\mathfrak{G}$ is < 4. It follows from (197) that the 36 minors of $\mathfrak{G}$, of degree 2, have the values $\pm\, g_{kl}g_{pq}$ $(1 \leq k < l \leq 4;\ 1 \leq p < q \leq 4)$. This proves that the skew-symmetric matrix $\mathfrak{G}^{(4)}$ is a primitive matrix of rank 2, if and only if (197) holds and the six numbers g_{kl} $(1 \leq k < l \leq 4)$ are relative prime.

Let $\mathfrak{W}_0 = (\mathfrak{C}_1{}^{(2)}\mathfrak{D}_1{}^{(2)})$ be the matrix of Lemma 25; then $\mathfrak{C}_1\mathfrak{D}'_1 - \mathfrak{D}_1\mathfrak{C}'_1 = (g_{13} + g_{24})\mathfrak{J}_2$, and consequently the relationship $\mathfrak{C}_1\mathfrak{D}'_1 = \mathfrak{D}_1\mathfrak{C}'_1$ holds, if and only if the condition

$$g_{13} + g_{24} = 0 \tag{198}$$

is satisfied. Since $\mathfrak{W}_0\mathfrak{U}_0 = (0\mathfrak{E})$, the equation $\mathfrak{A}_0\mathfrak{D}'_1 - \mathfrak{B}_0\mathfrak{C}'_1 = \mathfrak{E}$ has an integral solution $\mathfrak{A}_0, \mathfrak{B}_0$ in K_0. Then the matrices $\mathfrak{A}_1 = \mathfrak{A}_0 - \mathfrak{B}_0\mathfrak{A}'_0\mathfrak{C}_1$ and $\mathfrak{B}_1 = \mathfrak{B}_0 - \mathfrak{B}_0\mathfrak{A}'_0\mathfrak{D}_1$ satisfy

$$\mathfrak{A}_1\mathfrak{B}'_1 - \mathfrak{B}_1\mathfrak{A}'_1 = \mathfrak{B}_0\mathfrak{A}'_0(\mathfrak{C}_1\mathfrak{D}'_1 - \mathfrak{D}_1\mathfrak{C}'_1)\mathfrak{A}_0\mathfrak{B}'_0,$$
$$\mathfrak{A}_1\mathfrak{D}'_1 - \mathfrak{B}_1\mathfrak{C}'_1 = \mathfrak{E} - \mathfrak{B}_0\mathfrak{A}'_0(\mathfrak{C}_1\mathfrak{D}'_1 - \mathfrak{D}_1\mathfrak{C}'_1),$$

and the matrix

$$\mathfrak{M}_1 = \begin{pmatrix} \mathfrak{A}_1 & \mathfrak{B}_1 \\ \mathfrak{C}_1 & \mathfrak{D}_1 \end{pmatrix}, \tag{199}$$

with integral elements in K_0 and the second matrix row $\mathfrak{W}_0$, is symplectic, if (198) is fulfilled.

62. We consider again the substitution $\hat{\mathfrak{w}} = \mathfrak{V}\mathfrak{w}$ with $\mathfrak{S}[\hat{\mathfrak{w}}] = \mathfrak{S}[\mathfrak{w}]$ and $|\mathfrak{V}| = 1$. We assume now that the elements v_{kl} of $\mathfrak{V}$ are integers in a real algebraic field K_0 and that $\mathfrak{V} \equiv \mathfrak{E} \pmod 2$. We define

$$g_{14} = v_{51},\ g_{23} = -v_{52},\ g_{24} = \tfrac{1}{2}v_{53},\ g_{13} = -\tfrac{1}{2}v_{53},\ g_{12} = v_{54},\ g_{34} = v_{55}.$$

Then (198) is satisfied, and also (197), as a consequence of $\mathfrak{S}^{-1}[\mathfrak{V}'] = \mathfrak{S}^{-1}$. Since $|\mathfrak{V}| = 1$ and $v_{53} \equiv 0 \pmod 2$, the 6 numbers g_{kl} $(1 \leq k < l \leq 4)$ are relative prime. By the results of the last section, the matrix $\mathfrak{M}_1$ of (199) is symplectic with integral elements in K_0 and $\mathfrak{J}_2[\mathfrak{C}_1\mathfrak{D}_1] = \mathfrak{G} = (g_{kl})$, whence

$$\hat{w}_5 = \sum_{l=1}^{5} v_{5l}w_l = \sigma(\mathfrak{C}_1\mathfrak{W}\mathfrak{J}_2\mathfrak{D}'_1\mathfrak{J}_2{}^{-1}) + |\mathfrak{C}_1|\, w_4 + |\mathfrak{D}_1|\, w_5. \tag{200}$$

Let $\mathfrak{V}_1$ be the matrix of the linear transformation (176), (177), (178) with $\mathfrak{M}_1$ instead of $\mathfrak{M}$. By (178) and (200), the matrices $\mathfrak{V}$ and $\mathfrak{V}_1$ have the same fifth row, hence (0 0 0 0 1) is the fifth row of $\mathfrak{V}\mathfrak{V}_1{}^{-1} = \mathfrak{V}_2$. Putting

$$\mathfrak{M}\mathfrak{M}_1^{-1} = \mathfrak{M}_2 = \begin{pmatrix} \mathfrak{A}_2 & \mathfrak{B}_2 \\ \mathfrak{C}_2 & \mathfrak{D}_2 \end{pmatrix},$$

we infer from (168) that

$$|\mathfrak{D}_2| = 1, \quad |\mathfrak{C}_2| = 0 \quad \text{and} \quad \sigma(\mathfrak{C}_2\mathfrak{W}\mathfrak{Z}_2\mathfrak{D}'_2\mathfrak{Z}_2^{-1}) = \sigma(\mathfrak{W}\mathfrak{D}_2^{-1}\mathfrak{C}_2) = 0,$$

whence $\mathfrak{C}_2 = 0$ and $\mathfrak{A}_2\mathfrak{D}'_2 = \mathfrak{E}$. The corresponding linear transformation (176) takes the simpler form

$$\hat{\mathfrak{W}} = \mathfrak{W}[\mathfrak{A}'_2] + w_5\mathfrak{B}_2\mathfrak{A}'_2. \tag{201}$$

Obviously $\mathfrak{B}_1$ is unimodular in K_0, hence the same is true for the matrix $\mathfrak{B}_2$, and the coefficients in (201) are integers. Let

$$\mathfrak{A}_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then a^2, b^2, c^2, d^2 are necessarily integers, hence also a, b, c, d. Moreover $\mathfrak{B}_2\mathfrak{A}'_2$ is integral and $|\mathfrak{A}_2| = 1$; hence $\mathfrak{B}_2$ is integral. This proves that $\mathfrak{M} = \mathfrak{M}_2\mathfrak{M}_1$ has integral elements. Since $|\mathfrak{M}| = 1$, these elements are relative prime.

On the other hand, by (176), (177) and (178), the elements of $\mathfrak{M}$ satisfy a system of algebraic equations with coefficients in K_0, and the only solutions of these equations are $\mathfrak{M}$ and $-\mathfrak{M}$. It follows that $\sqrt{t}\mathfrak{M} = \mathfrak{F}$, where $\mathfrak{F}$ is a matrix in K_0 and t is a number $\neq 0$ in K_0.

Since $\sqrt{t}$ is the greatest common divisor of the elements of $\mathfrak{F}$, the principal ideal (t) is the square of an ideal $\mathfrak{a}$ in K_0. Let $A_1 = E, A_2, \cdots, A_g$ denote the ambiguous classes of ideals in K_0, i. e., the classes A satisfying $A^2 = E$, where E is the principal class. We choose an integral ideal $\mathfrak{a}_k$ in A_k $(k = 2, \cdots, g)$ and take $\mathfrak{a}_1 = (1)$; then $\mathfrak{a}_k^2 = (a_k)$ is a principal ideal and $a_1 = 1$. Let $u_1, \cdots, u_s$ be a complete system of fundamental units in K_0 and denote by f_l $(l = 1, \cdots, 2^{s+1})$ the 2^{s+1} products $(-1)^{e_0}u_1^{e_1} \cdots u_s^{e_s}$ $(e_k = 0, 1;\ k = 0, \cdots, s)$, in particular $f_1 = 1$. The products a_kf_l are all different; we denote them by $t_1, \cdots, t_m$ with $m = 2^{s+1}g$ and $t_1 = a_1f_1 = 1$. Obviously $t = t_qv^2$, where t_q is one of the numbers $t_1, \cdots, t_m$ and v is a number in K_0.

None of the numbers $t_2, \cdots, t_m$ is a square in K_0. We choose now an integral ideal ω_0 in K_0, such that none of those numbers is a quadratic residue modulo ω_0. If $\mathfrak{B} \equiv \mathfrak{E} \pmod{2\omega_0}$, then $|\mathfrak{D}| \equiv 1$, $|\mathfrak{A}| \equiv 1$, $\mathfrak{C} \equiv 0$, $\mathfrak{B} \equiv 0$, by (176), (177) and (178). Moreover $\mathfrak{A}\mathfrak{D}' \equiv \mathfrak{E}$, and the coefficients of $\mathfrak{W}[\mathfrak{A}'] - \mathfrak{W}$ are $\equiv 0$. Putting

$$\mathfrak{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad \mathfrak{D} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix},$$

we obtain

$$a_1^2 \equiv 1, \quad a_1 \equiv a_4 \equiv d_1 \equiv d_4, \quad a_2 \equiv a_3 \equiv d_2 \equiv d_3 \equiv 0 \pmod{2\omega_0}. \tag{202}$$

But $\sqrt{t_q}\,a_1 = x$ is an integer in K_0; hence $x^2 \equiv t_q \pmod{\omega_0}$, $t_q = t_1 = 1$.

We have proved that $\mathfrak{M}$ is an integral matrix in K_0, if $\mathfrak{B}$ is an integral matrix in K_0 satisfying $\mathfrak{S}[\mathfrak{B}] = \mathfrak{S}$, $|\,\mathfrak{B}\,| = 1$ and $\mathfrak{B} \equiv \mathfrak{E} \pmod{2\omega_0}$.

63. Let $\mathfrak{T}[\mathfrak{v}]$ be the quinary quadratic form of Section **59.** There exists in K a matrix $\mathfrak{L}_0$, such that

$$\mathfrak{T}[\mathfrak{L}_0\mathfrak{v}] = m(-pv_1^2 + v_2^2 - qv_3^2 + rsv_4^2 - rv_5^2)$$

with integral positive m, p, q, r, s. Putting

$$(pqs)^{-1} = m_2, \quad pm_2 = m_1, \quad qm_2 = m_3, \quad rsm_2 = m_4, \quad rm_2 = m_5,$$

we have

$$\begin{gathered} \prod_{k=1}^{5} m_k = r^2 m_2^4, \\ -m^{-1}m_2\mathfrak{T}[\mathfrak{L}_0\mathfrak{v}] = \sum_{k=1}^{5} (-1)^{k-1} m_k v_k^2. \end{gathered} \tag{203}$$

Let $\mathfrak{L}_1$ be the matrix of the linear substitution

$$\begin{gathered} w_1 = -v_1\sqrt{m_1} - v_2\sqrt{m_2}, \quad w_2 = v_1\sqrt{m_1} - v_2\sqrt{m_2}, \quad w_3 = -v_3\sqrt{m_3}, \\ w_4 = -v_4\sqrt{m_4} - v_5\sqrt{m_5}, \quad w_5 = v_4\sqrt{m_4} - v_5\sqrt{m_5} \end{gathered} \tag{204}$$

and let $\mathfrak{L}_0\mathfrak{L}_1^{-1} = \mathfrak{N}_1$; then $m^{-1}m_2\mathfrak{T}[\mathfrak{N}_1] = \mathfrak{S}$.

By (185) and (186), the matrices

$$\mathfrak{Q}_k = \sqrt{m_k}\,\mathfrak{F}_k \qquad (k = 1, \cdots, 5)$$

satisfy the conditions

$$\begin{gathered} \mathfrak{Q}_k\mathfrak{Q}_l = -\mathfrak{Q}_l\mathfrak{Q}_k \ (1 \leq k < l \leq 5), \quad \mathfrak{Q}_k^2 = (-1)^{k-1}m_k\mathfrak{E} \ (k = 1, \cdots, 5), \\ \prod_{k=1}^{5} \mathfrak{Q}_k = rm_2^2\mathfrak{E}. \end{gathered} \tag{205}$$

Using the abbreviations

$$\mathfrak{J}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathfrak{J}_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sqrt{p} = \rho_1, \qquad \sqrt{s} = \rho_2, \qquad \sqrt{q} = \rho_3, \qquad \sqrt{r} = \rho_4,$$

we introduce the 16 linearly independent matrices

$$\mathfrak{P}_1 = \begin{pmatrix} \mathfrak{E} & 0 \\ 0 & \mathfrak{E} \end{pmatrix}; \qquad \mathfrak{P}_2 = m_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2 = \rho_1 \begin{pmatrix} -\mathfrak{J}_3 & 0 \\ 0 & \mathfrak{J}_3 \end{pmatrix},$$

$$\mathfrak{P}_3 = m_2^{-1}s\mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3 = \rho_2 \begin{pmatrix} -\mathfrak{E} & 0 \\ 0 & \mathfrak{E} \end{pmatrix}, \qquad \mathfrak{P}_4 = ps\mathfrak{Q}_3 = \rho_1\rho_2 \begin{pmatrix} \mathfrak{J}_3 & 0 \\ 0 & \mathfrak{J}_3 \end{pmatrix},$$

$$\mathfrak{P}_5 = m_2^{-1}\mathfrak{Q}_2\mathfrak{Q}_3 = \rho_3 \begin{pmatrix} -\mathfrak{J}_1 & 0 \\ 0 & \mathfrak{J}_1 \end{pmatrix}, \qquad \mathfrak{P}_6 = m_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_3 = \rho_1\rho_3 \begin{pmatrix} \mathfrak{J}_2 & 0 \\ 0 & \mathfrak{J}_2 \end{pmatrix},$$

$$\mathfrak{P}_7 = qs\mathfrak{Q}_1 = \rho_2\rho_3 \begin{pmatrix} \mathfrak{J}_1 & 0 \\ 0 & \mathfrak{J}_1 \end{pmatrix}, \qquad \mathfrak{P}_8 = m_2^{-1}\mathfrak{Q}_2 = \rho_1\rho_2\rho_3 \begin{pmatrix} -\mathfrak{J}_2 & 0 \\ 0 & \mathfrak{J}_2 \end{pmatrix},$$

$$\mathfrak{P}_9 = m_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_3\mathfrak{Q}_4 = \rho_4 \begin{pmatrix} 0 & -\mathfrak{E} \\ -\mathfrak{E} & 0 \end{pmatrix}, \qquad \mathfrak{P}_{10} = m_2^{-1}p\mathfrak{Q}_2\mathfrak{Q}_3\mathfrak{Q}_4 = \rho_1\rho_4 \begin{pmatrix} 0 & -\mathfrak{J}_3 \\ \mathfrak{J}_3 & 0 \end{pmatrix},$$

$$\mathfrak{P}_{11} = m_2^{-1}\mathfrak{Q}_2\mathfrak{Q}_4 = \rho_2\rho_4 \begin{pmatrix} 0 & \mathfrak{E} \\ -\mathfrak{E} & 0 \end{pmatrix}, \qquad \mathfrak{P}_{12} = m_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_4 = \rho_1\rho_2\rho_4 \begin{pmatrix} 0 & \mathfrak{J}_3 \\ \mathfrak{J}_3 & 0 \end{pmatrix},$$

$$\mathfrak{P}_{13} = m_2^{-1}q\mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_4 = \rho_3\rho_4 \begin{pmatrix} 0 & \mathfrak{J}_1 \\ -\mathfrak{J}_1 & 0 \end{pmatrix}, \qquad \mathfrak{P}_{14} = pq\mathfrak{Q}_4 = \rho_1\rho_3\rho_4 \begin{pmatrix} 0 & \mathfrak{J}_2 \\ \mathfrak{J}_2 & 0 \end{pmatrix},$$

$$\mathfrak{P}_{15} = m_2^{-1}\mathfrak{Q}_3\mathfrak{Q}_4 = \rho_2\rho_3\rho_4 \begin{pmatrix} 0 & -\mathfrak{J}_1 \\ -\mathfrak{J}_1 & 0 \end{pmatrix}, \qquad \mathfrak{P}_{16} = m_2^{-2}\mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3\mathfrak{Q}_4 = \rho_1\rho_2\rho_3\rho_4 \begin{pmatrix} 0 & -\mathfrak{J}_2 \\ \mathfrak{J}_2 & 0 \end{pmatrix}$$

Then any real matrix $\mathfrak{M}^{(4)}$ can be expressed in the form

$$\mathfrak{M} = \sum_{k=1}^{16} x_k \mathfrak{P}_k \tag{206}$$

with uniquely determined real scalar factors x_k.

We denote the real algebraic number field $K(\sqrt{p}, \sqrt{q}, \sqrt{r}, \sqrt{s})$ by K_0. Let ω_0 be the ideal of the preceding section and choose in K an ideal ω having the corresponding properties with respect to K instead of K_0. We put $4pqrs\omega\omega_0 = \mu$.

On account of (203) and (204), the elements of the matrix $\mathfrak{N}_1$ lie in K_0. Let f be an integer $\neq 0$, such that $f\mathfrak{N}_1$ and $f\mathfrak{N}_1^{-1}$ are integral, and choose in K an ideal ν which is divisible by $f^2\mu$. We consider now the elements $\mathfrak{U}$ of the congruence subgroup $\Lambda_\nu(\mathfrak{T})$ of $\Lambda(\mathfrak{T})$ defined by condition $\mathfrak{U} \equiv \mathfrak{E} \pmod{\nu}$. Then $\mathfrak{B} = \mathfrak{N}_1^{-1}\mathfrak{U}\mathfrak{N}_1$ is an integral matrix in K_0 satisfying $\mathfrak{S}[\mathfrak{B}] = \mathfrak{S}$, $|\mathfrak{B}| = 1$ and $\mathfrak{B} \equiv \mathfrak{E} \pmod{\mu}$. The corresponding symplectic matrices $\pm\mathfrak{M}$ form a subgroup $\Delta_\nu(\mathfrak{T})$ of $\Delta(\mathfrak{T})$.

By Section **58**, the pair $\mathfrak{M}, -\mathfrak{M}$ is uniquely determined by the conditions

$$\hat{\mathfrak{Q}}_k = \mathfrak{M}^{-1}\mathfrak{Q}_k\mathfrak{M} \ (k = 1, \cdots, 5), \qquad \mathfrak{J}[\mathfrak{M}] = \mathfrak{J} \tag{207}$$

with

$$\hat{\mathfrak{Q}}_k = \sum_{l=1}^{5} r_{lk}\mathfrak{Q}_l \ (k = 1, \cdots, 5), \qquad (r_{kl}) = \mathfrak{L}_0^{-1}\mathfrak{U}\mathfrak{L}_0. \tag{208}$$

On account of the result of Section **62**, the coefficients x_k in (206) are numbers of K_0. We apply any isomorphism of K_0 which leaves all numbers of K

invariant and denote by $\mathfrak{P}^*_k$, $\mathfrak{Q}^*_k$, $\hat{\mathfrak{Q}}^*_k$, x^*_k the images of $\mathfrak{P}_k, \mathfrak{Q}_k, \hat{\mathfrak{Q}}_k, x_k$. Then (205) and (208) hold good with $\mathfrak{Q}^*_k$, $\hat{\mathfrak{Q}}^*_k$ instead of $\mathfrak{Q}_k$, $\hat{\mathfrak{Q}}_k$, and the same is true for the relationship

$$\mathfrak{Q}'_k = \mathfrak{J}^{-1}\mathfrak{Q}_k\mathfrak{J} = (\mathfrak{Q}_2\mathfrak{Q}_4)^{-1}\mathfrak{Q}_k(\mathfrak{Q}_2\mathfrak{Q}_4) \qquad (k = 1, \cdots, 5).$$

It follows that both matrices

$$\mathfrak{M}^* = \sum_{k=1}^{16} x^*_k\mathfrak{P}^*_k, \qquad \mathfrak{M}_0 = \sum_{k=1}^{16} x_k\mathfrak{P}^*_k$$

are solutions of (207), with $\mathfrak{Q}^*_k$, $\hat{\mathfrak{Q}}^*_k$ instead of $\mathfrak{Q}_k$, $\hat{\mathfrak{Q}}_k$, and consequently

$$\mathfrak{M}^* = \pm\mathfrak{M}_0, \qquad x^*_k = \pm x_k \qquad (k = 1, \cdots, 16).$$

Putting

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix},$$

we have

$$\mathfrak{A} = \begin{pmatrix} x_1 + x_2\rho_1 - x_3\rho_2 - x_4\rho_1\rho_2 & (-x_5 + x_3\rho_1 + x_7\rho_2 - x_8\rho_1\rho_2)\rho_3 \\ (-x_5 - x_6\rho_1 + x_7\rho_2 + x_8\rho_1\rho_2)\rho_3 & x_1 - x_2\rho_1 - x_3\rho_2 + x_4\rho_1\rho_2 \end{pmatrix},$$

$$\mathfrak{D} = \begin{pmatrix} x_1 - x_2\rho_1 + x_3\rho_2 - x_4\rho_1\rho_2 & (x_5 + x_6\rho_1 + x_7\rho_2 + x_8\rho_1\rho_2)\rho_3 \\ (x_5 - x_6\rho_1 + x_7\rho_2 - x_8\rho_1\rho_2)\rho_3 & x_1 + x_2\rho_1 + x_3\rho_2 + x_4\rho_1\rho_2 \end{pmatrix};$$

if we replace x_k by $(-1)^k x_{k+8}\rho_4$ $(k = 1, \cdots, 8)$, we obtain the expressions for $\mathfrak{B}$, $\mathfrak{C}$. By (176), (177) and (178), $\mathfrak{B} \equiv 0 \pmod{\mu}$ and $\mathfrak{C} \equiv 0 \pmod{\mu}$, hence $x_9, \cdots, x_{16}$ are integers. Moreover (202) holds now for the module μ instead of $2\omega_0$; consequently $x_1, \cdots, x_8$ are also integers and $x_1{}^2 \equiv 1 \pmod{\omega}$. Since $|\mathfrak{M}| = 1$, the numbers $x_1, \cdots, x_{16}$ are relative prime. We apply the argument of Section **62** and conclude that the coefficients x_k are numbers of the field K.

Let $\Delta(p, q, r, s)$ denote the group of all symplectic matrices $\pm\mathfrak{M}$ with integral $x_1, \cdots, x_{16}$ in K. We have proved that $\Delta_\nu(\mathfrak{T})$ is a subgroup of $\Delta(p, q, r, s)$. On the other hand, if $\mathfrak{M}$ is an element of $\Delta(p, q, r, s)$ satisfying $x_1 \equiv 1$, $x_k \equiv 0 \pmod{f^2}$ for $k = 2, \cdots, 16$, then $\mathfrak{B} \equiv \mathfrak{C} \pmod{f^2}$, by (176), (177) and (178), and $\mathfrak{U} = \mathfrak{N}_1\mathfrak{B}\mathfrak{N}_1{}^{-1}$ is an element of the group $\Lambda(\mathfrak{T})$. Consequently the groups $\Delta(\mathfrak{T})$ and $\Delta(p, q, r, s)$ are commensurable.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J.

THE THEORY OF SIMPLE RINGS.*

By Emil Artin and George Whaples.

Introduction. In this paper we discuss the theory of simple associative rings; for such rings we prove structure theorems which are generalizations of the well known theorems concerning commutators, automorphisms, etc. of simple algebras.[1] We use a new method—namely, the study of the transformations which the functions

$$\sigma(x) = \sum_{\nu=1}^{n} a_\nu x b_\nu$$

produce on the elements of the ring. The reader will be able, if he wishes, to extract from our discussion proofs of the structure theorems for algebras which are shorter than those usually given. For his convenience, we have occasionally inserted known proofs, in order to make the theory complete.

1. Vector spaces over rings.

If R is a ring, we call a set V of elements an R-vector space if its elements form an abelian group under addition and if R is an operator domain on this abelian group, satisfying the usual axioms—namely the distributive law and the condition that multiplication in the ring is the same as combination of operators. Usually R operates on the left side, and in that case the product of operators corresponds to the function-theoretic product $(\sigma\tau(x) = \sigma(\tau(x)))$; but occasionally we write the operators on the other side, and then this is not true. In the following, we shall usually assume that no non-zero vector is annihilated by every element of R. The only spaces we shall consider will be R-subspaces.

A space is said to satisfy the *minimal condition* if any set of R-subspaces contains a smallest one. This is equivalent to the descending chain condition; that is, to the condition that every descending chain of subspaces must be finite. The minimal condition on spaces is occasionally easy to verify by the following:

Theorem 1. *If V is the sum (not necessarily direct sum) of the subspaces V_1 and V_2, both of which satisfy the minimal condition, then V satisfies the minimal condition.*

---

* Received January 26, 1942.

[1] For these theorems, and for complete references, see M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, no. 1, Berlin, 1935, or A. A. Albert, *Structure of Algebras*, New York, 1939.

Proof. Let W be any subspace of V, W_2 the intersection of W and V_2, and W_1 the set of all elements v_1 which appear in any possible equation $w = v_1 + v_2$. (These elements will be called the V_1-components of W.) Then both W_1 and W_2 are subspaces of V. Let W' be a subspace of W. We contend:

LEMMA. *If W' contains W_2, and if all of W_1 occurs among the V_1-components of W', then $W' = W$.*

Proof. Let $w = v_1 + v_2$ be any element of W. Then v_1 lies in W_1, so that there is an element w' with the same V_1-component: $w' = v_1 + v'_2$. Since $w - w' = v_2 - v'_2$ and belongs to W, it belongs to W_2. Therefore $w - w'$ belongs to W' and so w itself belongs to W'.

Now suppose we have a set of subspaces W of $V_1 + V_2$. Consider first all those of our set for which the corresponding subset W_2 of V_2 is minimal, and, among these, one for which W_1 is minimal. There obviously cannot be a smaller W in our set.

A space V is said to be irreducible if $RV \neq 0$ and if V contains no proper subspaces other than 0.

LEMMA. *If the R-space V is the sum of r irreducible subspaces $V_1, V_2, \cdots V_r$, and if $x_1, x_2, \cdots x_s$ are any $s > r$ vectors of V, then one of them can be written as a linear combination of the others, with coefficients in R.*

Proof. The lemma is evident for $r = 0$, because that means that V is the null-space. Assume that it is true for $r - 1$ and use induction. Let

$$\begin{aligned} x_1 &= x_{11} + x_{12} + \cdots + x_{1r} \\ x_2 &= x_{21} + x_{22} + \cdots + x_{2r} \\ &\cdot \quad \cdot \quad \cdot \\ x_s &= x_{s1} + x_{s2} + \cdots + x_{sr} \end{aligned}$$

be a splitting of our vectors according to the covering $V = V_1 + V_2 + \cdots + V_r$. If the first term of each of these sums is 0, then $x_1, x_2, \cdots x_s$ are contained in the subspace $V_2 + V_3 + \cdots + V_r$, and the lemma follows from our assumption. Otherwise we can assume $x_{11} \neq 0$. Then Rx_{11} is a non-zero subspace of V_1 and so it follows that $Rx_{11} = V_1$. We can, therefore, find elements $a_2, a_3, \cdots a_s$ in R, such that $a_i x_{11} = x_{i1}$. The new elements

$$y_2 = x_2 - a_2 x_1; \qquad y_3 = x_3 - a_3 x_1; \cdots; \qquad y_s = x_s - a_s x_1$$

are then obviously contained in the subspace $V_2 + V_3 + \cdots + V_s$. Since their number $s - 1 > r - 1$, one of them, say y_s, can be expressed in terms of the others. From this expression, we easily get an expression for x_s in terms of $x_1, x_2, \cdots x_{s-1}$.

A sum $V = V_1 + V_2 + \cdots + V_n$ is said to be direct whenever every element of V is uniquely expressible as a sum of elements of the V_i, or, what is the same thing, whenever $v_1 + v_2 + \cdots + v_n = 0$ implies that each $v_i = 0$. In case our sum is not direct and V_n is irreducible, there is obviously only this alternative: the sum of the spaces V_n and $(V_1 + V_2 + \cdots + V_{n-1})$ is direct, or else V_n has a vector in common with the sum of the others; then V_n, since it is irreducible, is included in this sum, and $V = V_1 + V_2 + \cdots + V_{n-1}$. Should all the V_i be irreducible, we should therefore get a direct sum by removing as many terms as possible from our expression $V = V_1 + V_2 + \cdots + V_n$; the sum which remains will be direct.

The following assumption gives a useful class of special R-spaces: Let every vector of V be contained in a finite sum of irreducible subspaces. We shall call such spaces *regular.*

If V is a regular space and contains a direct sum $V_1 + V_2 + \cdots + V_r$ of irreducible subspaces, we have the following alternative: either V is equal to that sum, or there is a vector of V not contained in it. Since the space is regular, there must, then, be an irreducible space V_{r+1} not contained in our previous sum; then $V_1 + V_2 + \cdots + V_{r+1}$ is direct. Should this process never come to an end by leading to the full space, we should obtain an infinite sequence $V_1, V_2, V_3, \cdots$ of irreducible subspaces with a direct sum. The following descending chain

$$\begin{aligned} W_1 &= V_1 + V_2 + V_3 + \cdots \\ W_2 &= \phantom{V_1 + {}} V_2 + V_3 + \cdots \\ W_3 &= \phantom{V_1 + V_2 + {}} V_3 + \cdots \\ &\quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

shows that in such a case the minimal condition cannot be fulfilled. A regular space with minimal condition is consequently the direct sum of a finite number of irreducible spaces. Conversely, the direct sum of a finite number of irreducible spaces is, by Theorem 1, a space with minimal condition. The number r of components in such a direct splitting is obviously an invariant, for if V is the direct sum of $W_1, W_2, \cdots W_s$, it follows from our lemma that s cannot be larger than r; if it were, we could select vectors different from 0 in $W_1, W_2, \cdots W_s$ such that one of them is expressible in terms of the others, contradicting the directness of the sum.

If V and W are two R-spaces, a homomorphism of V to W is a mapping of each vector of V onto some vector of W, which is a homomorphism for the group operation and is connected with R by the assumption that if $\sigma(v) = w$, then $\sigma(rv) = rw$, for all r in R. The set of all images of V need not fill out the whole space W, but it is always a subspace. Should this subspace be all of W, then we may say that W is the homomorphic image of V. On the other hand, many vectors of V might be mapped onto 0 of W. The set of all these forms a subspace of V; should this subspace be 0, and should W be the homomorphic image of V, then the mapping is really a one to one correspondence and the two spaces are said to be isomorphic.

We now readily see that in case V and W are irreducible, we have only this alternative: either all of V is mapped onto 0, or the mapping is an isomorphism. Let $V = V_1 + V_2 + \cdot \cdot \cdot V_r$ be a direct splitting of V into irreducible subspaces, and let W be a subspace of V. The following procedure will give us a homomorphic mapping of W onto V_1: split any element w according to the direct sum into components, and map w onto its V_1-component v_1. If we, furthermore, assume W to be irreducible, we see that if V_1 and W are not isomorphic, our homomorphic mapping must map W onto 0, and consequently the V_1-component of any element of W is 0. So in this case W is already contained in the direct sum of those among $V_1, V_2, \cdot \cdot \cdot V_r$ which are isomorphic to W. This suggests a grouping of the components V_i in such a way that isomorphic V_i are united. Should $V_1 + V_2 + \cdot \cdot \cdot + V_s$ be the space generated by such a group, then we see that it contains all irreducible subspaces W of V which are isomorphic to V_1, and no others. We can, therefore, describe this space in a way not related to any special splitting, by saying that it is the sum of all subspaces isomorphic to V_1. That its multiplicity s is invariant follows from our previous discussion.

Every ring R can always be considered as an R-space, by defining the operator product of the operator a and the vector x (where of course both of them are in R) as the ordinary product ax. A subspace, under this concept of R as left R-space, is then a left ideal. R can also be considered as right R-space; the subspaces of R are then right ideals. Since it is very useful to study these spaces, it is clear that we shall have to make some finiteness assumption about R: we usually assume, therefore, that R satisfies the minimal condition for left ideals, or, what is the same thing, that R is a left R-space with minimal condition.

2. Simple rings. A simple ring is a ring R containing no two sided ideal other than itself and 0, and such that $R^2 \neq 0$. A vector space over a simple ring is either annihilated by the whole ring, or it is not annihilated by any element other than 0; for the annihilators of an R-space V form a two sided ideal, since $aV = 0$ implies both $RaV = 0$ and $aRV = a(RV) \subset aV = 0$. This shows that a non-zero left ideal $\mathfrak{l}$ of a simple ring has no left annihilator other than 0; for since the set of annihilators of $\mathfrak{l}$ is a two sided ideal, the only other possibility is that $\mathfrak{l}$ is annihilated by the whole ring R. But then

R is annihilated on the right by $\mathfrak{l}$, and then, for the same reason, the set of right annihilators of R would be R itself, contradicting $R^2 \neq 0$. The same reasoning applies to right ideals.

Now assume that R contains minimal left ideals (ideals containing no subideal other than 0). Let V be an irreducible space and $\mathfrak{l}$ a minimal left ideal. Since $\mathfrak{l}V \neq 0$, we can find a vector x_0 such that $\mathfrak{l}x_0 = V$. If we map the element λ of $\mathfrak{l}$ onto λx_0, we obviously get a homomorphism; since both spaces are irreducible, it is an isomorphism. Hence all irreducible spaces are isomorphic, and the isomorphism with any minimal left ideal can be produced by the process described. Let us especially identify V with $\mathfrak{l}$. x_0 is then an element of $\mathfrak{l}$ such that $\lambda \rightarrow \lambda x_0$ is a one to one correspondence of $\mathfrak{l}$ onto itself. Since x_0 must appear among the images, there is an element e of $\mathfrak{l}$ such that $ex_0 = x_0$. Hence $ex_0 = e^2x_0 \neq 0$, and the fact that it is a one to one correspondence shows that $e^2 = e \neq 0$. If $\mathfrak{l}$ is a minimal left ideal, then $\mathfrak{l}a$ is a homomorphic image of $\mathfrak{l}$, and hence is either 0 or isomorphic to $\mathfrak{l}$. Let us now consider the sum of all minimal left ideals;[2] from the preceding sentence, it follows that this sum is a two-sided ideal and consequently is the whole ring R.

We can now prove a theorem concerning the existence of a unit element.

THEOREM 2. *If R is simple and contains minimal ideals, it contains a unit element if and only if it satisfies the minimal condition for left ideals.*

Proof. R is the sum of its minimal left ideals. Suppose it contains a unit element. Then this element 1 can be expressed as the sum $\lambda_1 + \lambda_2 + \cdots \lambda_r$ of a finite number of components, each of which is contained in a minimal ideal. The sum $R\lambda_1 + R\lambda_2 + \cdots + R\lambda_r$ therefore contains $R \cdot 1 = R$, and each $R\lambda_i$ is a minimal ideal. So R is covered by a finite number of irreducible spaces and hence satisfies the minimal condition.

To show the other part of the theorem, we first prove a more general statement:

LEMMA. *If R is a simple ring with left minimal condition and $\mathfrak{l}$ is any left ideal, then there is an idempotent e in $\mathfrak{l}$ such that $\mathfrak{l} = Re$.*

Proof. Since every ideal contains a minimal ideal, it also contains some idempotent. Among all the idempotents of $\mathfrak{l}$ select one, e_1, such that the set of elements of $\mathfrak{l}$ annihilated by e_1 from the right is as small as possible. (This set always forms a left ideal $\mathfrak{m}$.) We contend that $\mathfrak{m} = 0$. If $\mathfrak{m}$ were not 0, it would contain some idempotent e_2 different from 0; then $e_2e_1 = 0$. Put $e = e_1 - e_1e_2 + e_2$; then $ee_1 = e_1$, $ee_2 = e_2$ and so $ee = ee_1 - ee_1e_2 + ee_2 = e$. Now $xe = 0$ implies $xee_1 = xe_1 = 0$, so that e can never annihilate more than e_1. But e really annihilates less than e_1, because $e_2e = e_2$, whereas $e_2e_1 = 0$.

---

[2] A sum of an infinite number of sets is defined to be the set of all finite sums of elements of the components.

Thus I contains an idempotent which annihilates no non-zero element from the right; let e be such an element. If x is any element of I, $x - xe$ is also in I and is annihilated from the right by e; hence $x = xe$. We see that e acts as a right unit for all elements of I. If we apply the lemma to the case $\mathfrak{l} = R$, we get the existence of a right unit e in R. The solutions of the equation $ex = 0$ would form a right ideal $\mathfrak{r}$. Now $\mathfrak{r} \cdot \mathfrak{r} = (\mathfrak{r}e)\mathfrak{r} = \mathfrak{r}(e\mathfrak{r}) = 0$; since a right ideal is annihilated from the right only by 0, $\mathfrak{r} = 0$. But $x - ex$ is annihilated by e from the left; hence $x = ex$.

COROLLARY. *A simple ring with minimal condition for left ideals satisfies the minimal condition for right ideals. In fact, if Re is a minimal left ideal, eR is a minimal right ideal.*

This corollary will follow from Theorem 2 as soon as we have proved the following

LEMMA. *In any simple ring R, an ideal of the form Re (e idempotent) is minimal if and only if eRe is a quasi-field.*

Proof. Let R be a simple ring and Re a minimal ideal. For any non-zero element a of eRe, we know that $Ra \neq 0$, since R contains e and $ea = a$. So $Ra = Re$; this implies $eRa = eRe$ and, since $ea = a$, $eRea = eRe$. Hence the ring eRe has a unit element, and a left inverse of any non-zero element.

If, conversely, a is any non-zero element of Re, the set eRa is a left ideal in eRe, since $eRe \cdot eRa \subset eRa$. If, therefore, eRe is a quasi-field, eRa can be only 0 or eRe. In the first case the left ideal Ra would be annihilated by e on the left; this can happen in a simple ring only if Ra is 0. In the other case, we conclude that $ReRa = ReRe$. Since $ReR = R$, this gives us $Ra = Re$; Re is minimal.

Thus if R is simple with minimal condition for left ideals, R contains a minimal right ideal and has a unit element; hence, by Theorem 2, (applied to right ideals instead of left ideals) R has the minimal condition for right ideals.

THEOREM 3. *If $\mathfrak{l} = Re$ is a minimal ideal, then its R-homomorphisms are exactly the set of transformations obtained by multiplying I on the right by elements of eRe; hence, if these homomorphisms are written as right operators, they form a ring isomorphic to eRe.*

Proof. $\mathfrak{l} \cdot eRe = Re \cdot eRe = Re$, so that multiplication of $\mathfrak{l}$ by any element of eRe gives a homomorphic mapping of $\mathfrak{l}$ on itself. If, conversely, a is the image of e under any homomorphic mapping, the image of xe will be xa. Setting x equal to e, we get $ea = a$; since a is in $\mathfrak{l}$, we have also $a = ae$. Hence $a = eae \in eRe$.

THEOREM 4. *To every simple ring R, with minimal condition, we can find a quasi-field k and a right space of finite dimension over that quasi-field, such that R is isomorphic to the set of all k-linear transformations of that space, if we write them on the left side. The space will be any minimal left ideal and the field the set of all R-homorphisms of that left ideal.*

Conversely, the set of all k-linear transformations of any right space of finite dimension over a quasi-field k is a simple ring R with minimal condition; the space is irreducible, and therefore isomorphic to any minimal left ideal of R; and the field is isomorphic to the set of R-homomorphisms of the space.

Thus the structure of k and the dimension of the space are uniquely determined by R.[3]

Proof. We split R into a direct sum of minimal right ideals: $R = \mathfrak{r}_1 + \mathfrak{r}_2 + \cdots + \mathfrak{r}_n$. If

$$1 = e_1 + e_2 + \cdots + e_n \tag{1}$$

is the corresponding splitting of the unit element 1, we multiply (1) by e_i from the right and compare the result with the obvious splitting of e_i into components: $e_i = 0 + \cdots + 0 + e_i + 0 + \cdots + 0$. This shows that e_i is idempotent and that $e_i e_k = 0$ if $i \neq k$. Multiplying (1) from the right by R shows that none of the e_i can be 0, or else we could split R into fewer right ideals.

We now take $\mathfrak{l} = Re_1$ as our left ideal, and let $k = e_1 R e_1$. e_i cannot annihilate $\mathfrak{l}$ from the left, because a left ideal has no left annihilator. Select $\alpha_i \neq 0$ out of $e_i\mathfrak{l}$. Then $\alpha_i e_1 R e_1 = \alpha_i R e_1$; but $\alpha_i R$ is part of $e_i R$ and not 0, so that $\alpha_i k = e_i R e_1 = e_i \mathfrak{l}$. Hence $\alpha_1 k + \alpha_2 + \cdots + \alpha_n k = e_1\mathfrak{l} + e_2\mathfrak{l} + \cdots + e_n\mathfrak{l} \supset \mathfrak{l}$ and is therefore equal to $\mathfrak{l}$.

If we multiply $\alpha_1, \alpha_2, \cdots \alpha_n$ on the left side by xe_i, α_k is taken into 0 if $k \neq i$, and α_i is taken into $x\alpha_i$. Since $R\alpha_i = \mathfrak{l}$, we see that α_i can be transformed into an arbitrary element of $\mathfrak{l}$ while the other α_k are taken into 0. This shows that the α_i are linearly independent over k and that we can find an element $x \in R$ which takes the n vectors α_i into arbitrarily given images in $\mathfrak{l}$.

---

[3] This theorem was first proved (for simple algebras) by J. H. M. Wedderburn: "On hypercomplex numbers," *Proceedings of the London Mathematical Society*, vol. 6 (1908), pp. 77-118.

To prove the second half of the theorem, let V be a right k-space of dimension n, and R the ring of all k-homomorphisms of V. Let $\mathfrak{a}$ be a two sided ideal of R and $a \neq 0$ an element of $\mathfrak{a}$. Select any n independent vectors $A_1, A_2, \cdots A_n$ of V. a cannot take all these vectors into 0; assume that it maps A_1 onto $B \neq 0$. Let σ be a transformation mapping A_i onto A_1 and all the other A_j onto 0; then $a\sigma$ will map A_i onto B and all the other basis vectors onto 0. Let τ be a transformation mapping B onto C; $\tau a \sigma$ will map A_i onto an arbitrary vector C and all the other A_j onto 0. But $\tau a \sigma$ also belongs to $\mathfrak{a}$; it follows that $\mathfrak{a} = R$.

That V is irreducible over R is clear, since every vector different from 0 can be mapped onto any other vector. It remains to be shown that if we consider V as R-space only, then the field k consists of all R-homomorphisms. Let A be a fixed vector different from 0 and let B be its image under a given R-homomorphism of V. If B were linearly independent of A, with respect to k, we could find an element σ of R such that $\sigma(A) = A$ and $\sigma(B) = 0$. Our homomorphism mapped A onto B, so it should map $\sigma(A)$ onto $\sigma(B)$, i. e. A onto 0, contrary to assumption. Consequently, A is taken into $A\kappa$. If C is any other vector, we can find a σ such that $\sigma(A) = C$; then $\sigma(A\kappa) = C\kappa$. Our homomorphism takes A into $A\kappa$, and therefore takes $\sigma(A)$ into $\sigma(A\kappa)$; hence it takes any vector C into $C\kappa$, with the same κ.

Any space over a simple ring with minimal condition is regular. Indeed, any vector x is contained in Rx and therefore in $\mathfrak{l}_1 x + \mathfrak{l}_2 x + \cdots + \mathfrak{l}_n x$, where the $\mathfrak{l}_i$ are minimal ideals. But every $\mathfrak{l}_i x$ is either 0 or irreducible.

If two spaces with minimal condition over a simple ring R with minimal condition split into the same number of irreducible components, they are isomorphic. The isomorphism consists, indeed, in mapping each irreducible component of V onto one of the irreducible components of V', and then extending that mapping to a mapping of V on V' by addition. If we have a slightly more general situation, namely, two rings R and R', that are isomorphic under the mapping $r \leftrightarrow r'$, and an R-space V and an R'-space V', which both split into the same number of irreducible components, we can do something similar: we first consider V' as an R-space, by defining the multiplication of V' by an element r of R to have the same effect as the multiplication by the image r' of r in R'. Hence there is a function σ mapping V onto V' in such a manner that for any vector v, and any element r of R, we have $\sigma(rv) = r\sigma(v)$. The multiplication of v by r is the original one; since $\sigma(v)$ is in V' the multiplication by r means really the multiplication by r'. We consequently have

THEOREM 5. *If R and R' are two simple, isomorphic rings which contain* *minimal ideals, and V and V' are spaces over R and R', respectively, splitting into the same number of irreducible components, we can find a mapping function $\sigma(v)$ of V onto V', which is a one to one correspondence and satisfies the identity $\sigma(rv) = r'\sigma(v)$ (where r' is the image of r under our isomorphism).*

COROLLARY. *Under the same conditions as before, except for the equality of the number of irreducible components of the two spaces with minimal condition, it is possible to map one of the spaces, say V, onto a part of the other, by a one to one correspondence $\sigma(v)$ such that the same equation holds.*

Proof. Should V' have more components than V, we can find a subspace of V' having the same number.

3. Direct products.

We give a definition of direct product of two spaces over a quasi-field k.

If k is a quasi-field and A a right, B a left k-space (such that the unit of k acts as unit on A and B), we consider the set of all finite formal sums:

$$\sum_{\nu} A_\nu B_\nu \tag{1}$$

with $A_\nu \varepsilon A$, $B_\nu \varepsilon B$. We allow at first no simplification whatever besides the commutative law of addition, and are neither allowed to cancel two terms nor to drop a term with a factor 0. Two such sums are added in the obvious way. These sums do not form an additive group, because we have neither a zero element nor a negative. The associative and commutative laws are obvious.

Such a sum shall be called a nullform if and only if there is a substitution

$$B_i = \sum_{\mu} \beta_{i\mu} B'_\mu \tag{2}$$

for the B_i by some other vectors B'_i of B $(\beta_{i\mu} \varepsilon k)$ such that we get 0 as coefficient of each B'_i if we formally substitute (2) in (1) and collect all the coefficients of B'_i—that is, if for all i:

$$\sum_{\nu} A_\nu \beta_{\nu i} = 0. \tag{3}$$

The set of all nullforms shall be denoted by N_B. If (1) is a nullform such that (2) and (3) hold, we select any linearly independent vectors A'_i, such that every A_i can be expressed by them:

$$A_i = \sum A'_\mu \alpha_{\mu i}. \tag{4}$$

Substituting this in (3), we have $\sum_{\mu,\nu} A'_\mu \alpha_{\mu\nu} \beta_{\nu i} = 0.$ Since the A'_i are linearly independent, we get:

$$\sum \alpha_{k\nu} \beta_{\nu i} = 0 \tag{5}$$

for all k and i. Multiplying (5) from the right by B'_i and taking the sum over i, we see that because of (2):

$$\sum \alpha_{k\nu} B_\nu = 0. \tag{6}$$

But (6) means that (4) substituted in (1) leads to a coefficient 0 of A'_i if we collect terms. This shows the symmetry of our definition of nullforms; namely, $N_B \subset N_A$ (where N_A stands for the nullforms defined by A-substitution.) Consequently, $N_A = N_B$, so that we denote this set merely by N. Furthermore, we have shown that the question whether or not (1) is a null form can be decided by expressing the A_i in terms of *any* set of linearly independent A'_i. Since our definition of N is now known to be symmetric with regard to the A's and B's, we have:

LEMMA. *The decision whether* (1) *is a nullform can be reached by expressing the B_i in terms of any set of linearly independent vectors B'_i:* (1) *is a nullform if and only if* (3) *holds.*

It is fairly obvious how to prove that the sum of two nullforms is a nullform, and that $\Sigma(-A_\nu)B_\nu$ is in N if (1) is in N. But we need also:

LEMMA. *If $\Sigma A_\nu B_\nu + \Sigma \bar{A}_\nu \bar{B}_\nu \varepsilon N$ and $\Sigma \bar{A}_\nu \bar{B}_\nu \varepsilon N$, then $\Sigma A_\nu B_\nu \varepsilon N$.*

Express both the B_ν and the $\bar{B}_\nu$ in terms of a set of linearly independent B'_ν. These expressions are uniquely determined: $B_i = \Sigma \beta_{i\nu} B'_\nu$, $\bar{B}_i = \Sigma \bar{\beta}_{i\nu} B'_\nu$. Since $\Sigma \bar{A}_\nu \bar{B}_\nu \varepsilon N$, we have $\Sigma \bar{A}_\nu \bar{\beta}_{\nu i} = 0$. From this and the first of our assumptions, we deduce $\Sigma A_\nu B_\nu \varepsilon N$.

Finally, we remark that $AB + (-A)B$ is always in N. Indeed, both the B-vectors can be expressed by one alone, namely B. The coefficient of this B after collecting is $A - A = 0$.

If X stands for a sum (1), we agree to let $-X$ stand for $\Sigma(-A_\nu)B_\nu$. We now introduce a congruence:

$$X \equiv Y \pmod{N}$$

means that $X + (-Y) \varepsilon N$. Reflexiveness and symmetry are fairly obvious. We concentrate on transitivity. Let

$$X + (-Y) \varepsilon N; \quad Y + (-Z) \varepsilon N.$$

Then

$$X + (-Z) + Y + (-Y) \varepsilon N.$$

Since $Y + (-Y) \varepsilon N$, we deduce $X + (-Z) \varepsilon N$ from the second of our lemmas.

If $X \equiv Y \pmod{N}$ and $Z \equiv W \pmod{N}$, we have $X + Z \equiv Y + W$ $(\bmod\ N)$. In the sense of congruence, our forms are now a group with every element of N as zero element and $-X$ as negative of X. The rules

$$\begin{aligned} A(B + B') &\equiv AB + AB' \pmod{N} \\ (A + A')B &\equiv AB + A'B \pmod{N} \\ A(\alpha B) &\equiv (A\alpha)B \qquad \pmod{N} \end{aligned}$$

show that in the sense of congruence any formal change is allowed with the vectors A and B.

We now substitute $=$ for $\equiv$ and call this abelian group. $A \times {}_kB$, the direct product of the spaces A and B relative to k.

If A and B are rings as well as k-spaces, and k is a field operating on both sides of A and B, we define the product of the elements $\sum_{\nu} a_\nu b_\nu$ and $\sum_{\mu} a'_\mu b'_\mu$ to be $\sum_{\nu,\mu} (a_\nu a'_\mu)(b_\nu b'_\mu)$. To show that this defines a ring, it suffices to show that this multiplication is well defined relative to equality in $A \times {}_kB$. To show this, we need only prove that if $\Sigma a_\nu b_\nu$ is a nullform, then $\Sigma(aa_\nu)(bb_\nu)$ is a nullform. But if the substitution $b_i = \Sigma\beta_{i\mu} b'_\mu$ makes the first sum obviously 0, the substitution $(bb_i) = \Sigma\beta_{i\mu}(bb'_\mu)$ makes the second obviously 0. So $A \times {}_kB$ is a ring. Clearly $A \times {}_kB$ is isomorphic to $B \times {}_kA$; we shall allow the order of the factors of the terms of (1) to be changed and say that $A \times {}_kB = B \times {}_kA$: the direct product is commutative. It is also easy to verify that $A \times {}_k(B \times {}_kC) = (A \times {}_kB) \times {}_kC$: direct product is associative. If A and B are of finite degree over k, then the degree of $A \times {}_kB$, over k, is the product of the degrees of A and of B, since if $a_1 \cdots a_n$ and $b_1 \cdots b_m$ are linearly independent bases for A and for B, the elements $a_i b_j$ are a linearly independent basis for $A \times {}_kB$.

If B has a unit element 1_B, the elements $a1_B$ of $A \times {}_kB$ form a subring isomorphic to A. Indeed, $a1_B = 0$ if and only if $a = 0$, since 1_B is linearly independent. The properties for addition and multiplication are obvious. We identify this subring with A itself. Should A have a unit element, we can in a similar way identify B with the subring of elements $1_A b$. Since $(a1_B)(1_A b) = ab$, we see furthermore that in that sense $A \times {}_kB$ is really the product of A and B.

If R is a simple ring, we call any function $l(x)$ of the form $\Sigma a_\nu x a'_\nu$, where x is a variable in R, an analytic linear function. If l and l' are two analytic linear functions, their iteration $l(l'(x))$ is obviously again analytic. The main theorem about analytic linear functions in a simple ring is the following:

THEOREM 6. *If R is a simple ring with unit element, and centrum k, and if $a_1, a_2, \cdots a_r$ are linearly independent with respect to k, we can find an analytic linear function mapping these elements on arbitrarily chosen elements $b_1, b_2, \cdots b_r$.*

Proof. It obviously suffices to prove the theorem in case one of the b_i is 1 and the others 0. If $r = 1$, the possible values $l(a_1)$ form the two sided ideal Ra_1R, which is R because R is simple: this proves our theorem for $r = 1$. Let it be true for $r - 1$ and assume that we have found, for each $i = 1 \cdots r - 1$, an analytic linear function $l_i(x)$ that maps a_i into 1 and all the others (except possibly a_r) into 0. We try to find an $l(x)$ which maps a_r into 1 and all the other a_i into 0. This is easily done as soon as any $l_i(a_r)$ is an element outside the centrum k; indeed, if c is not commutative with $l_i(a_r)$, the linear function $\lambda(x) = l_i(x)c - cl_i(x)$ will map $a_1, a_2, \cdots a_{r-1}$ into 0 and a_r into a non-zero element b. If $\mu(x)$ is a linear function mapping b into 1, we can take $\mu(\lambda(x))$ as our desired function $l(x)$.

We therefore assume that $l_i(a_r) = \kappa_i \varepsilon k$ for $i = 1 \cdots r - 1$. We now put $\lambda(x) = a_1 l_1(x) + a_2 l_2(x) + \cdots + a_{r-1} l_{r-1}(x) - x$. Obviously $\lambda(a_i) = 0$ for $i = 1, \cdots, r - 1$. $\lambda(a_r)$ cannot be 0 for else $0 = a_1\kappa_1 + a_2\kappa_2 \cdots + a_{r-1}\kappa_{r-1} - a_r$, contradicting the linear independence of the a_i with respect to k. If $\mu(x)$ maps $\lambda(a_r)$ into 1, the function $l(x) = \mu(\lambda(x))$ is the solution.

THEOREM 7. *If A is a simple ring with unit element and centrum k, and B is any ring with unit element, containing k in its centrum, then every two sided ideal of $A \times_k B$ is of the form $A \times_k \mathfrak{b}$, where $\mathfrak{b}$ is a two sided ideal of B.*

Proof. Let $\mathfrak{A}$ be any ideal of $A \times_k B$, and $\mathfrak{b}$ the intersection of $\mathfrak{A}$ with B. It is clear that $A \times_k \mathfrak{b}$ is part of $\mathfrak{A}$. Let, conversely, $\xi = \Sigma a_\nu b_\nu$ be an element of $\mathfrak{A}$. We can assume that all the a_ν are linearly independent with respect to k, since we could otherwise simplify the expression for ξ. If $l_i(x)$ is an analytic linear function with coefficients in A, mapping a_i onto 1 and all other a's onto 0, we find that $l_i(\xi) = b_i$. Since $l_i(\xi)$ belongs, with ξ, to the two sided ideal $\mathfrak{A}$, every b_i belongs to b, and hence $\mathfrak{A} = A \times_k \mathfrak{b}$.

COROLLARY. *If A and B are simple, with unit element, and the centrum k of A is contained in the centrum of B, then $A \times_k B$ is simple.*

With the help of the preceding theorem we can, in many cases, determine the intertwining relations of two subrings A and B of a bigger ring R.

THEOREM 8. *Let A and B be two subrings of any ring R, with the following properties:*

A and B both have the same unit element (which need not be unit element for R).

A is simple, and its centrum k is included in the centrum of B.

Every element of B is commutative with every element of A.

Then the subring $A \cdot B$ of R has the structure $A \times {}_kB$. This means that there are no intertwining relations between A and B other than the most trivial ones.

Proof. We construct two new rings $\bar{A}$ and $\bar{B}$, isomorphic to A and B, and having only the image $\bar{k}$ of k as common elements. We form the product $\bar{A} \times {}_{\bar{k}}\bar{B}$ and contend that the following mapping onto $A \cdot B$ is an isomorphism: let $\Sigma \bar{a}_\nu \bar{b}_\nu$ correspond to $\Sigma a_\nu b_\nu$. We have first to prove that the mapping is uniquely determined, but if $\Sigma \bar{a}_\nu \bar{b}_\nu = 0$, there is a transformation in the $\bar{b}_i$ that makes this obvious. The corresponding transformation in $\Sigma a_\nu b_\nu$ will show that this element is also 0, because the computation involves only the distributive and associative laws. That the mapping is a homomorphism under addition and multiplication is also obvious. It remains to be seen that it is a one to one correspondence. The set $\mathfrak{A}$ of all elements of $\bar{A} \times {}_{\bar{k}}\bar{B}$ that are mapped onto 0 forms a two sided ideal. According to our theorem, this ideal has the structure $\bar{A} \times {}_{\bar{k}}\bar{\mathfrak{b}}$, where $\bar{\mathfrak{b}}$ consists of those elements $\bar{b}$ of $\bar{B}$ which are mapped onto 0. Consequently $\bar{\mathfrak{b}} = 0$, and therefore $\mathfrak{A} = 0$.

This allows us to determine the structure of the set of all analytic linear functions of R, from another point of view. These functions form themselves a ring (with iteration used as multiplication). This ring, L, contains two subrings: First, the functions $l_a(x) = ax$. The set of all these functions is clearly isomorphic to R as soon as R contains a unit element, and we can, without misunderstanding, identify this subring with R itself. Second, the functions $l_a{}^*(x) = xa$. The set of all these functions is not isomorphic to R, because

$$l_a{}^*(l_b{}^*(x)) = xba = l_{ba}{}^*(x).$$

But the mapping of R onto the $l_a{}^*$ is an anti-isomorphism, interchanging the order in a product. This new ring will be called R^*; its structure is of course known if the structure of R is known. If R is simple, with unit element, R^* has the same properties. l_a and $l_b{}^*$ are commutative, since $l_a l_b{}^*(x)) = axb = l_b{}^*(l_a(x))$. If, finally, κ is in the centrum of R, $l_\kappa(x) = l_\kappa{}^*(x)$. This shows that the two rings satisfy all conditions of the preceding theorem, and therefore $R \times {}_kR^*$ is contained in L. Since, on the other hand, every element of L can be obtained from the special functions R and R^* by iteration and addition, we have $L = R \times {}_kR^*$. This shows that L is again simple.

As for the minimal condition in direct products, we prove:

THEOREM 9. *If A and B have unit elements included in the field k, the minimal condition in a ring $A \times {}_kB$ is satisfied if it is true for A and B is of finite degree over k.*

Proof. The minimal condition means that $A \times {}_kB$, considered as left space over $A \times {}_kB$, satisfies the minimal condition for subspaces. Now each $A \times {}_kB$-space is certainly an A-space. If, therefore, we prove the minimal condition for $A \times {}_kB$, considered as A-space, then our theorem follows. Let $w_1, w_2, \cdots w_n$ be a basis of B with respect to k. Then $A \times {}_kB = Aw_1 + Aw_2 + \cdots + Aw_n$. Each of the spaces Aw_i is isomorphic to A, so the minimal condition for $A \times {}_kB$ follows by use of Theorem 1.

Coming back to the linear functions $L = R \times {}_kR^*$ of a simple ring with centrum k, we can prove:

THEOREM 10. *The minimal condition for left ideals of L is satisfied if and only if R is of finite degree over k.*

It suffices to prove the second half. Let, therefore, R not be of finite degree over k, and $w_1, w_2, w_3, \cdots$ an infinite sequence of linearly independent elements (with respect to k).

Let $\mathfrak{l}_i$ stand for the set of all analytic linear functions $l_i(x)$ that map $w_1, w_2, \cdots w_i$ onto 0. Clearly $\mathfrak{l}_i \supset \mathfrak{l}_{i+1}$, and differs from it, because $\mathfrak{l}_i$ contains a linear function that maps $w_1, \cdots w_i$ onto 0 and w_{i+1} onto 1. Since these are left ideals, we have proved the theorem.

If R is of finite dimension over k, the structure of L is completely determined by Theorem 6:

THEOREM 11. *If R is simple, and of finite degree over its centrum k, then $L = R \times {}_kR^*$ consists of all linear transformations of the space R, with respect to k.*

Proof. Since every analytic linear transformation satisfies the conditions $l(x + y) = l(x) + l(y)$ and $l(\kappa x) = \kappa l(x)$, for every element κ of the centrum, L is part of the set of all linear transformations. If, on the other hand, $w_1, w_2, \cdots w_n$ are a linearly independent basis for B, with respect to k, we can find a function $l(x)$, mapping $w_1, w_2, \cdots w_n$ on arbitrarily selected images. But any linear transformation with respect to k is uniquely determined by the images of the w_i.

Another way to state the previous theorem is this:

If R is simple and of finite degree n over k, the ring $R \times {}_kR^*$ is isomorphic to the ring of all matrices of degree n over k.

4. Commutators of subrings. If R is a ring with minimal condition, and we want to study a subring of L that also satisfies the minimal condition, it seems natural to restrict one of the two factors of $L = R \times {}_k R^*$, by replacing it by a simple subring A of finite dimension over k. Such a ring A is called a simple algebra. We shall therefore study the subring $S = R \times {}_k A^*$. It again satisfies the minimal condition, and consists of the linear functions

$$l(x) = \Sigma r_\nu x a_\nu, \tag{1}$$

with $r_\nu \varepsilon R$ and $a_\nu \varepsilon A$. We might now consider R as a left S-space, in the sense that l is applied onto x, as left operator, by computing $l(x)$.

Since S contains the linear functions denoted by R, every S-space is also a left R-space; since R satisfies the minimal condition over R, the same is true when R is considered as space over S. Assume now that we have in R a second algebra A', isomorphic to A, under an isomorphism $a \leftrightarrow a'$ that leaves every element of k fixed. The set $S' = R \times {}_k A'^*$ is then isomorphic to S under an isomorphism that maps the function (1) onto $l'(x) = \Sigma r_\nu x a'_\nu$. We remark that the special function $l(x) = xa$ is mapped onto the function xa'. Let us now make use of the corollary to Theorem 5. R appears as space to the two isomorphic rings S and S', and we may therefore assume that a function $\sigma(x)$ can be found, which maps, by a one to one correspondence, R onto some part of itself, and satisfies, for every $l \varepsilon S$, the equation

$$\sigma(l(x)) = l'(\sigma(x)). \tag{2}$$

We first specialize to $l(x) = rx$ and get $\sigma(rx) = r\sigma(x)$. Putting $x = 1$, and writing c for $\sigma(1)$, we have $\sigma(r) = rc$. Since r can be any element of R, we have $\sigma(x) = xc$. This mapping is a one to one correspondence between R and the images, so that $xc = 0$ implies $x = 0$. c being no right divisor of 0, it has an inverse.[4] Specializing now in (2) to $l(x) = xa$, we get $xac = xca'$. Putting $x = 1$, we find:

$$a' = c^{-1}ac.$$

This gives us the following:

THEOREM 12. *Let R be a simple ring with minimal condition and A and A' two simple subalgebras of R containing the centrum k of R. Any isomorphism of A to A', that leaves every element of the centrum k fixed, can be extended to an inner automorphism of R.*

---

[4] The left ideals Rc^ν form a descending chain; hence, for some n, $Rc^{n+1} = Rc^n$. In the set on the right we have c^n and therefore, for some x, $xc^{n+1} = c^n$; since c is no divisor of zero, $xc = 1$.

COROLLARY. *If R is an algebra, every automorphism of R that leaves the centrum fixed is inner.*

Proof. Select $A = A' = R$.

In order to study the way in which a simple subring is embedded in a larger ring, we need to define the degree of a ring, not only over a subfield, but over a simple subring. To do this, we commence with the:

LEMMA. *Let S be a simple subring, with minimal condition, of any ring R, and let R and S have the same unit element. If $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two minimal ideals of S, then any isomorphism of $\mathfrak{l}_1$ onto $\mathfrak{l}_2$, considered as S-spaces, can be extended to an isomorphism of $R\mathfrak{l}_1$ to $R\mathfrak{l}_2$, considered as R-spaces.*

Proof. If e_1 and e_2 are idempotents of $\mathfrak{l}_1$ and $\mathfrak{l}_2$, let c_2 be the image of e_1 under the mapping of $\mathfrak{l}_1$ onto $\mathfrak{l}_2$. Then $x = xe_1$ of $\mathfrak{l}_1$ is mapped onto xc_2 of $\mathfrak{l}_2$. If we multiply, therefore, the set $Re_1 = RSe_1 = R\mathfrak{l}_1$ from the right by c_2, this mapping is an R-homomorphism and an extension of our mapping of $\mathfrak{l}_1$ onto $\mathfrak{l}_2$. The set of images is $Rc_2 = RSc_2 = R\mathfrak{l}_2$.

Let now c_1 be the image of e_2 under the inverse of our isomorphism, so that $x \in \mathfrak{l}_2$ is hereby mapped onto xc_1. Then $c_2c_1 = e_1$. If now $xe_1c_2 = 0$, we would have $xe_1c_2c_1 = xe_1 = 0$. Our mapping is therefore an isomorphism.

Now let $S = \mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_r$ be a splitting in minimal left ideals and $1 = \Sigma e_\nu$ be the corresponding splitting in orthogonal idempotents. We easily deduce $R = Re_1 + Re_2 + \cdots + Re_r$. This is a direct sum, since $x_1e_1 + x_2e_2 + \cdots + x_re_r = 0$ shows after right multiplication by e_i that $x_ie_i = 0$. We now consider R as a left S-space. Each of the components Re_i is then also a left S-space, and they are isomorphic, since they are even isomorphic if considered as R-spaces. Therefore we can break each of them into an equal number of irreducible S-spaces. This will lead, after regrouping, to a splitting of R into a finite or infinite sum of left S-spaces, each of which is a direct sum of precisely r components, and is thus isomorphic to S itself. An S-subspace of R that is isomorphic to S has the following structure: let w be the image of 1 if we map S onto the subspace. x of S is then mapped onto xw of the subspace, and since the mapping is an isomorphism, $xw = 0$ with x in S implies $x = 0$. Doing this with each of our spaces, we see that there exist elements w_i, such that $R = \Sigma Sw_i$. Any finite number of w_i is linearly independent with respect to S, in the sense that a linear combination of them, with coefficients in S, is 0 if and only if all coefficients (not merely all terms) of the sum are 0. The number of irreducible S-components of the S-space R is r times as large as the number of w_i; therefore the number of w_i is invariant; we shall call it the left degree of R over S and shall denote it by $(R:S)$. Now in precisely the same way as in case of fields, we can prove that $(R:S_1) = (R:S)(S:S_1)$, if S_1 is a simple subring of S.

If R is a ring and S a subset of R, we denote by R^S the set of all elements of R which are commutative with every element of S. If S is still a subset of R, and T another ring with unit element, we have $(R \times {}_kT)^S = R^S \times {}_kT$. For the right side is certainly contained in the left. Let $\Sigma r_\nu t_\nu$ be any element of the left side, with linearly independent t_ν. If s is any element of S, then $\Sigma(sr_\nu - r_\nu s)t_\nu = 0$. Since the t_ν are linearly independent, we see that every r_ν is in R^S. Similarly, if $S \subset R$ and $U \subset T$, and S and T are rings containing k, $(R \times {}_kT)^{S \times_k U} = R^S \times {}_kT^U$. R^R is of course the centrum of R.

Now let R be a ring and T be an algebra, both with centrum k. Then $(R \times {}_kT)^{R \times_k T} = k \times {}_kk = k$, so that k is also the centrum of $R \times {}_kT$. Assume now that R and T are simple rings with minimal condition, that each contains a simple algebra containing k, and that these two algebras are isomorphic under a correspondence leaving the elements of k invariant. We shall denote both these algebras, though different, by the same letter A, since it will be clear from the formulas which is meant. The simple ring with minimal condition $R \times {}_kT$ now contains the algebra A in two ways; consequently there is an inner automorphism of $R \times {}_kT$ that will map one onto the other. Computing $(R \times {}_kT)^A$ for both algebras A, we get the two sets $R^A \times {}_kT$ and $R \times {}_kT^A$. These are, therefore, isomorphic under the same inner automorphism. Computing further

$$(R \times {}_kT)^{(R^A \times_k T)} \text{ and } (R \times {}_kT)^{(R \times_k T^A)}$$

we get the two sets $R^{(R^A)}$ and $T^{(T^A)}$, which are therefore isomorphic under the same isomorphism.

Should we be able to prove that R^A is simple, we could compute the degree of $R \times {}_kT$ over $R^A \times {}_kT$; since an inner automorphism of $R \times {}_kT$ maps $R^A \times {}_kT$ onto $R \times {}_kT^A$, the degree of $R \times {}_kT$ over $R^A \times {}_kT$ should be the same as the degree of $R \times {}_kT$ over $R \times {}_kT^A$. The reader can easily deduce the equality $(R:R^A) = (T:T^A)$. The proof for the simplicity will be given soon.

If we only know that our algebra A is contained in R, we have to construct a T. Assume that the degree $(A:k)$ is n. Consider A as a k-space, and denote by M the set of all linear transformations of A over k; that is, the algebra of $n \times n$ matrices over k. The degree $(M:k)$ will be n^2. M contains a subalgebra isomorphic to A, namely the set of all linear transformations of A that are obtained by multiplying A from the left by an element of A. Calling this ring of linear transformations also A, we now ask for M^A, i. e. for the linear transformations σ that satisfy the condition $\sigma(ax) = a\sigma(x)$, for all a

and x in A. Putting $x = 1$, denoting $\sigma(1)$ by c, and writing x instead of a, we see: $\sigma(x) = xc$. It follows easily that the set of these linear transformations is isomorphic to A^*, so that we have $M^A = A^*$. (We did not assume that the centrum of A is k; the centrum might be larger, so that M would possibly not be equal to $A \times {}_kA^*$.)

Substituting M for T in our previous result, we can prove:

THEOREM 13. *If R is simple with minimal condition, and A is a simple subalgebra of R containing the centrum k of R, and if also $(A:k) = n$ and M is the set of all linear transformations of A over k, we have:*

1) *$R^A \times {}_kM$ and $R \times {}_kA^*$ are isomorphic under an inner automorphism of $R \times {}_kM$.*
2) *R^A is simple.*
3) *$R^{(R^A)} = A$.*
4) *$(R:R^A) = n$.*

Proof. The first statement is clear. Since A^* is simple, $R \times {}_kA^*$ is simple. This proves that $R^A \times {}_kM$, and consequently R^A, is simple. As for the third statement, $R^{(R^A)}$ is mapped by our inner automorphism onto $M^{(M^A)} = M^{A^*}$, which is the A contained in M. Since we know that our isomorphism maps precisely the A of R onto the A of M, we get our statement. The fourth statement follows from the fact that $(M:A^*) = n$. (For $(R:R^A) = (R \times {}_kM : R^A \times {}_kM) = (R \times {}_kM : R \times {}_kA^*) = (M:A^*)$). This follows in turn from $(M:k) = n^2$ and $(A^*:k) = n$.

There is one very general theorem concerning simple subalgebras of rings, that holds for any ring R whose centrum contains the centrum k of A and whose unit element is contained in k. The proof of this theorem depends on the following:

LEMMA. *Assume that an analytic linear function $l(x) = \Sigma a_\nu x a'_\nu$, with coefficients in A, is 0 for all $x \varepsilon A$. Then $l(x)$ is 0 for all $x \varepsilon R$.*

Proof. Since we can express the a'_ν by linearly independent elements of A and collect terms, we can from the beginning assume that the a'_ν are linearly independent. The necessary operation for collecting is the interchange of x with an element of the centrum k of A; this can be done, according to our assumption, for all $x \varepsilon R$.

We contend that now all the a_ν are 0. To show this, we consider $l(x)$ for values of A only. The linear functions of A form a ring isomorphic to $A \times {}_kA^*$, and under this isomorphism our linear function will correspond to

$\Sigma a_\nu a'^*_\nu$. Since $l(x) = 0$ for all $x \varepsilon A$, we have $\Sigma a_\nu a'^*_\nu = 0$, and the linear independence of the a' shows that all the a_ν are 0. But this means that $l(x) = 0$ for every x in R.

Our first aim is to compute R^A; we contend that this can be done in the following way:

THEOREM 14. *Let $l(x)$ be any analytic linear function with coefficients in A which is not identically zero and which maps every element of A onto an element of k. The values $l(x)$, for all x in R, will then range precisely over R^A.*

Proof. If a is any element of A, $l(x)a - al(x)$ is a linear function with coefficients in A that is 0 for every $x \varepsilon A$. According to our lemma, it is 0 for every x in R. Since a is arbitrary, this proves that $l(x) \varepsilon R^A$.

To prove that we get every element of R^A, take any element a such that $l(a) = \kappa \neq 0$. Since we can replace a by $\kappa^{-1}a$, we can assume that $l(a) = 1$. If z is any element of R^A, we get $l(za) = zl(a) = z$, because we can interchange z with every coefficient.

THEOREM 15. *Under the same assumptions as before, we have $R = R^A \times_k A$.*

Proof. Let $a_1, a_2, \cdots a_n$ be a k-basis for A. Let $l_i(x)$ be the analytic linear function with coefficients in A that takes a_i into 1 and all the other a's into 0. The linear function $l(x) = a_1 l_1(x) + a_2 l_2(x) + \cdots + a_n l_n(x) - x$ is 0 for $x = a_1, a_2, \cdots a_n$, and therefore for every x in A. This proves that for every x in R we have $x = a_1 l_1(x) + a_2 l_2(x) + \cdots + a_n l_n(x)$. Since the values of $l_i(x)$ range through R^A, we see that $R = A \cdot R^A$. That the dot can be replaced by the cross follows from Theorem 8.

COROLLARY. *If A is a simple algebra with unit element e and centrum k, contained in a ring R (for which we make no assumption), then $eR^k = A \times_k eR^A$.*

Proof. Consider first the ring $(eRe)^k$. Obviously the centrum of this ring contains k. e is unit element for this ring, and the previous lemma can be applied. But $((eRe)^k)^A = (eRe)^A = eR^A$.

THEOREM 16. *Let R be simple with chain condition, and let k be its centrum and T a subring containing k. Then if $R = w_1 T + w_2 T + \cdots w_n T$, R^T is an algebra A, and $(A : k) \leq n$.*

Furthermore, these two statements are equivalent:

1) *T is the commutator of a simple subalgebra of R.*
2) *T is simple, $(R:T)$ is finite, and all homomorphisms of R into itself, considering R as right T-space, are analytic.*

Proof. Let T be any subring of R such that $R = w_1T + w_2T + \cdots + w_nT$. Consider the set S of all homomorphisms of R, considered as right T-space. This ring S contains as subring the analytic transformations among them. Let $\sigma(x) = \Sigma r_\nu x r'_\nu$, and assume the r_ν linearly independent with respect to k. Since, for every $t \varepsilon T$, $\sigma(xt) - \sigma(x)t = 0$, we get the equations $tr'_\nu - r'_\nu t = 0$; so that all r'_ν belong to $A = R^T$. This subring is therefore to be denoted by $R \times {}_kA^*$. It contains as further subring the left multiplications of x by elements of R, and this ring of transformations we have always denoted by R. Since the degree $(A:k) = (R \times {}_kA^*:R)$, our inequality will be proved if we estimate the degree $(S:R)$. Every homomorphism $\sigma(x)$ of S will map $w_1, w_2, \cdots w_n$ onto certain elements $a_1, a_2, \cdots a_n$ of R. If these elements are known and $x = w_1t_1 + w_2t_2 + \cdots + w_nt_n$, then its image will be $a_1t_1 + a_2t_2 + \cdots + a_nt_n$. Let us denote $\sigma(x)$ by the n-dimensional vector $(a_1, a_2, \cdots a_n)$. We are interested in the behavior of S, considered as left R-space, where R means here the transformations $\rho(x) = rx$. Iterating our previous σ with that transformation would lead to $(ra_1, ra_2, \cdots ra_n)$. This shows that S is isomorphic to a certain part of the set of all n-dimensional vectors over R. The statement concerning the degrees can also be interpreted as a statement about the number of irreducible components: the degree has only to be multiplied by the number of irreducible components R has. This makes it clear that $(S:R) \leq n$. (From previous work, we know that this degree exists). Hence $(A:k) \leq n$. It is furthermore clear that $(S:R) = n$ if and only if S fills out the whole of our space of n-vectors. This means that there are linear transformations mapping the w_i onto arbitrary images. But then the w_i have to be linearly independent, because $w_1t_1 + w_2t_2 + \cdots + w_nt_n = 0$ would yield $t_i = 0$ if we applied the homomorphism mapping w_i onto 1 and the others onto 0. If, conversely, the w_i are linearly independent, and we map $w_1t_1 + w_2t_2 + \cdots + w_nt_n$ onto $a_1t_1 + a_2t_2 + \cdots + a_nt_n$, this clearly gives an element of S.

If A is a simple subalgebra of R and T is taken as R^A, we may apply Theorem 13. Hence the w_i can be taken as linearly independent and $n = (R:R^A) = (A:k) = (S:R)$. Since $(R \times {}_kA^*:R) = n$ and $R \times {}_kA^* \subset S$, we get $R \times {}_kA^* = S$, which means that all our homomorphisms are analytic.

Conversely, if T is simple, we can take the w's linearly independent. If we assume that all our homomorphisms are analytic, so that $S = R \times_k A^*$, we get first $n = (R \times_k A^* : R) = (A : k)$. We next get that S is simple, because it consists of all homomorphisms of the n-dimensional T-space R, and is therefore isomorphic to the set of all $n \times n$ matrices with elements in T, or $T \times_k M_n$. So $R \times_k A^*$ is simple, which implies that A is simple. Since R^A must be of degree n under R, and certainly contains T, we find that $T = R^A$, which finishes the proof.

Corollary. *If A is a simple subalgebra of R containing the centrum k of R, (R simple with minimal condition) and $\sigma(x)$ is a homomorphism of R satisfying the equations $\sigma(xt) = \sigma(x)t$ and $\sigma(tx) = t\sigma(x)$, for all elements t of $T = R^A$, then any expression of $\sigma(x)$ with linearly independent coefficients has the form $\Sigma a_\nu x a'_\nu$, where the a_ν as well as the a'_ν are in A.*

This is proved by the method used in the first part of the proof of the preceding theorem.

Theorem 17. *Under the same assumptions as in the preceding corollary, every automorphism of R that leaves every element of $T = R^A$ fixed consists of a transformation by an element of A.*

Proof. Since $\sigma(xt) = \sigma(x)\sigma(t) = \sigma(x)t$ and $\sigma(tx) = t\sigma(x)$, our corollary gives $\sigma(x) = \Sigma a_\nu x a'_\nu$. We can assume that the a_ν and the a'_ν are linearly independent with respect to k. Let $l(x) = \Sigma b_\mu x b'_\mu$ be an analytic linear function of A which takes a_1 into 1 and the other a_i into 0. Compute $\sum_\mu b_\mu \sigma(b'_\mu x)$. On the one hand, it is $\{\sum_\mu b_\mu \sigma(b'_\mu)\}\sigma(x)$, and therefore is of the form $a\sigma(x)$. On the other hand, it is $\sum_{\nu,\mu} b_\mu a_\nu b'_\mu x a'_\nu = \sum_\nu l(a_\nu) x a'_\nu = x a'_1$. Here a'_1 is not 0 because it is one of a set of linearly independent elements. We therefore get $a\sigma(x) = x a'_1$. Let $\lambda(x) = \sum c_\nu x c'_\nu$ be such that $\lambda(a'_1) = 1$. Compute $\sum_\nu a\sigma(x c_\nu) c'_\nu$. On one hand, it has the form $a\sigma(x)b$; on the other hand, it is $\sum x c_\nu a'_1 c'_\nu = x$. Hence $a\sigma(x)b = x$. Putting $x = 1$, we see that $b = a^{-1}$, and hence $\sigma(x) = a^{-1}xa$. That a belongs to A follows from the corollary.

Indiana University,
Institute for Advanced Study.

ON WEAKLY COMPACT SUBSETS OF A BANACH SPACE.*

R. S. PHILLIPS.

In the last few years use of the Banach space weak topology has proven exceedingly fruitful in the hands of Bourbaki (4), Smulian (24), Kakutani (14), Ghantmacher and Smulian (9), Sirvint (23), Dunford and Pettis (8), and others. These investigations have been concerned either with sequentially compact subsets or with bicompact subsets. In this paper an attempt has been made to bridge the gap between these two extremes by considering $\aleph_a$-compact subsets where $\aleph_a$ is any cardinal number.

One could define a subset to be $\aleph_a$-compact if every covering of power $\leq \aleph_a$ of the set's weak closure contained a finite subcovering. Bourgin has done this, treating the problem from a topological point of view (5). We have preferred an analysis approach, which, we believe, permits a much more direct and essentially simpler development.

A subset G of a Banach space X will be said to be $\aleph_a$-compact if for every subset $[x_\pi \mid \pi \epsilon$ directed set of power $\leq \aleph_a]$[1] *of G there exists an $x_0 \epsilon X$ such that*

$$\underline{\lim_{\pi}}\, \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim_{\pi}}\, \bar{x}(x_\pi)$$

for all $\bar{x}$ belonging to the adjoint space $\bar{X}$. The notion of limit is defined in the Moore-Smith sense (18). We do not insist that the limit inferior or limit superior be finite. If $\lim_{\pi} \bar{x}(x_\pi)$ exists for all $\bar{x} \epsilon \bar{X}$, it is clear that $\bar{x}(x_0) = \lim_{\pi} \bar{x}(x_\pi)$. A subset G will be called $\aleph_a$-*complete* if for every subset $[x_\pi \mid \pi \epsilon$ directed set of power $\leq \aleph_a]$ of G for which $\lim_{\pi} \bar{x}(x_\pi)$ exists for all $\bar{x} \epsilon \bar{X}$, there exists an x_0 such that $\bar{x}(x_0) = \lim_{\pi} \bar{x}(x_\pi)$ for all $\bar{x} \epsilon \bar{X}$. Such an x_0 is necessarily unique. If, then, G is $\aleph_a$-compact, it is likewise $\aleph_a$-complete. The converse is not true. An $\aleph$-*compact* subset G will be defined as a set which is $\aleph_a$-compact for all $\aleph_a$.

Many of our results are simply generalizations of known theorems for sequentially weak compact sets. For purposes of completeness we have given new proofs for some of these theorems. We have tried to free ourselves from the

---

* Received August 29, 1941.

[1] The class of elements s satisfying the property P will be designated by $[s \mid P]$.

analogues of the usual separability condition which has often accompanied the discussion of sequentially weak compact sets. This we have been able to do in the statement of our results, but unfortunately not always in our method of proof.

In Section 1 we have compared $\aleph_a$-compactness with related compactness notions. We find, for instance, that for weakly closed sets coverings of power $\leq \aleph_a$ have finite subcoverings if and only if the sets are $\aleph_a$-compact, while for convex sets $\aleph_a$-compactness is equivalent to transfinite closedness of order $\aleph_a$. If the unit sphere of X is $\aleph_a$-compact, then the unit sphere of $\bar{X}$ is also $\aleph_a$-compact. If in addition there exists a set of power $\leq \aleph_a$ total on X, then X is reflexive.

Section 2 deals with the obvious generalization of the weakly completely continuous transformations, namely, $\aleph_a$-compact transformations. If U is $\aleph_a$-compact, then the adjoint transformation $\bar{U}$ is likewise $\aleph_a$-compact. If a sequence of $\aleph_a$-compact transformations converges in norm to a transformation U, then U is found to be $\aleph_a$-compact.

By the principal theorem of Section 3, the closure of the set $[\Sigma_G a_x x \mid \Sigma_G \mid a_x \mid \leq 1]$ is $\aleph_a$-compact if and only if G is $\aleph_a$-compact. Hence the closed convex extension of an $\aleph_a$-compact set is $\aleph_a$-compact. An application of this theorem gives us a criterion for $\aleph_a$-compact sets.

Section 4 is concerned with determining conditions on the geometry of X under which $\aleph_a$-compactness implies $\aleph$-compactness. For example, all $\aleph_0$-compact subsets of a strictly convex space are $\aleph$-compact, and all $\aleph_a$-compact subsets of a space for which there exists a total set of power $\leq \aleph_a$ are $\aleph$-compact. We have as yet not been able to find an example of an $\aleph_0$-compact set which is not $\aleph$-compact.

An application of our results is given in the final section of the paper. Here we prove that if $x(\tau)$ is a completely additive, absolutely continuous set function of bounded variation such that $[(x(\tau)/|\tau|) \mid (\| x(\tau) \| / | \tau |) \leq n]$ is $\aleph_0$-compact for all n, then there exists a measurable function $x(t)$ such that $x(\tau) = \int_\tau x(t)\,d\alpha$. This leads to a general form of the weakly completely continuous transformation on Ω to an arbitrary Banach space and to the conclusion that this transformation is necessarily separable.

It will be convenient to make the following notational conventions: The weak topology $w(X)$ of X as elements has the generic neighborhood $U(x_0) \equiv [x \,\|\, \bar{x}_i(x - x_0) \,| < \epsilon \quad i = 1, 2, \cdots, n]$ where the $\bar{x}_i$ are arbitrary elements of $\bar{X}$. The weak topology $\bar{w}(\bar{X})$ of $\bar{X}$ as functionals has the generic neighborhood $U(\bar{x}_0) \equiv [\bar{x} \,\|\, (\bar{x} - \bar{x}_0)x_i \mid < \epsilon \quad i = 1, 2, \cdots, n]$ where the x_i are arbitrary elements of X. X is a regular Hausdorf space in the $w(X)$ topology; while $\bar{X}$ is a bicompact Hausdorf space in the $w(\bar{X})$ topology. If G is a subset of X, then G_L will designate the smallest linear extension of G; G_C will be the norm closure of G; and $cv(G)$ will be the convex extension of G. M_T will be the Banach space of bounded functions $a(t)$ on an abstract class $T \equiv [t]$ to real numbers having the norm $\| a \| = \text{l. u. b. } [| a(t) | \mid t \in T]$. l_T will be the Banach space of functions $a(t)$ on an abstract class $T \equiv [t]$ to real numbers for which $\| a \| = \sum_T | a(t) |$ exists. If $f(t)$ is a function, then $f(\cdot)$ will designate the function considered as an element of some class of functions. Finally, $X_C \equiv [x \mid x \in X, \| x \| \leq c]$ so that X_1 is the unit sphere.

1. Related compactness definitions. Before attempting a systematic study of the properties of $\aleph_\alpha$-compact sets, we shall illustrate the usefulness of this notion by giving a criterion for reflexive Banach spaces. A Banach space is called *reflexive* if for every $\bar{\bar{x}} \in \bar{\bar{X}}$, there exists an $x \in X$ such that $\bar{\bar{x}}(\bar{x}) = \bar{x}(x)$ for all $\bar{x} \in \bar{X}$.

1. 1. THEOREM. *A necessary and sufficient condition that X be reflexive is that X_1 be $\aleph_\alpha$-compact and that $\bar{X}$ contain a subset of power $\leq \aleph_\alpha$ total on X.*

Given $\bar{\bar{x}} \in \bar{\bar{X}}$ and Γ, an arbitrary subset of $\bar{X}$ of power $\leq \aleph_\alpha$, we first show that there exists an $x_\Gamma \in X$ such that $\bar{\bar{x}}(\bar{x}) = \bar{x}(x_\Gamma)$ for all $\bar{x} \in \Gamma$. The set of all finite subsets π of Γ is a directed set ($\leq$ means set inclusion) of power $\leq \aleph_\alpha$. By Helly's theorem (13), there exists an $x_\pi \in X$ such that $\bar{\bar{x}}(\bar{x}) = \bar{x}(x_\pi)$ for each $\bar{x} \in \pi$ and $\| x_\pi \| \leq \| \bar{\bar{x}} \| + 1$. As X_1 (and hence X_n) is $\aleph_\alpha$-compact, there exists an $x_\Gamma \in X$ satisfying the condition

$$\underline{\lim}_{\pi}\, \bar{x}(x_\pi) \leq x(x_\Gamma) \leq \overline{\lim}_{\pi}\, \bar{x}(x_\pi)$$

for all $\bar{x} \in \bar{X}$. Now $\lim_{\pi} \bar{x}(x_\pi) = \bar{\bar{x}}(\bar{x})$ for all $\bar{x} \in \Gamma$. Therefore $\bar{\bar{x}}(\bar{x}) = \bar{x}(x_\Gamma)$ for all $\bar{x} \in \Gamma$.

Let us now apply this first to the case where Γ is a set of power $\leq \aleph_\alpha$ total on X, and then to $\Gamma' = \Gamma \cup \bar{x}_0$ where $\bar{x}_0$ is an arbitrary element of $\bar{X}$. Then $\bar{x}(x_\Gamma) = \bar{\bar{x}}(\bar{x}) = \bar{x}(x_{\Gamma'})$ for all $\bar{x} \in \Gamma$. As Γ is total, $x_\Gamma = x_{\Gamma'}$. Thus $\bar{\bar{x}}(\bar{x}) = \bar{x}(x_\Gamma)$ for all $\bar{x}$ belonging to Γ' and hence, since $\bar{x}_0$ is arbitrary, for all $\bar{x} \in \bar{X}$. In other words, X is reflexive.

The necessity is quite evident. For, given any set $[x_\pi] \subset X_1$ where $[\pi]$ is a directed set, an argument similar to that used by Banach (1, p. 118) demonstrates the existence of an $\bar{\bar{x}} \in \bar{\bar{X}}$ such that

$$\underline{\lim}_{\pi}\, \bar{x}(x_\pi) \leq \bar{\bar{x}}(\bar{x}) \leq \overline{\lim}_{\pi}\, \bar{x}(x_\pi)$$

for all $\bar{x} \epsilon \bar{X}$. As X is reflexive we can replace $\bar{\bar{x}}$ by an element of X. As this holds for all sets $[\pi]$, it holds in particular for all sets of power less than or equal to the power of any subset of $\bar{X}$ total on X.

1.2 *Definition.* A set $G \subset X$ is said to be $\aleph_a$-*compact** if for any family of subsets $[A_t \mid t \epsilon T]$ of G such that 1). A_t is closed in the $w(X)$ topology; 2). the power of $T \leq \aleph_a$; 3). any finite subset has a non-vacuous intersection; then $\bigcap_T A_t$ is non-vacuous.

D. G. Bourgin (5) has shown that if X_1 is $\aleph_a$-compact* and if $\bar{X}$ contains a subset of power $\leq \aleph_a$ which is $\bar{w}(\bar{X})$ dense, then X is reflexive. In Theorem 1.9 we show that X_1 is $\aleph_a$-compact if and only if it is $\aleph_a$-compact*. Further if $\Gamma \subset \bar{X}$ is total on X then the set of all finite linear combinations with rational coefficients, which is of the same power, is $\bar{w}(\bar{X})$ dense in $\bar{X}$ (see Lemma 2.4). Hence the two theorems are equivalent.

In the statement of Theorem 1.1, the hypothesis that X be $\aleph_a$-compact can not be replaced by $\aleph_a$-completeness in the $w(X)$ topology. This breaks down for the space l which is $\aleph_0$-complete in the $w(l)$ topology and for which any dense subset of the separable space c_0 is total.

For purposes of orientation, we shall next compare $\aleph_a$-compactness with related compactness notions. In this it will be convenient to have the following lemmas.

1.3 Lemma. *If G is $\aleph_a$-compact, then given $[x_\pi \mid \pi \epsilon$ directed set of power $\leq \aleph_a] \subset G$, there exists an x_0 belonging to the closed convex extension of G such that*

$$\underline{\lim}_{\pi}\, \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim_{\pi}}\, \bar{x}(x_\pi),$$

for every $\bar{x} \epsilon \bar{X}$.

If x_0 did not belong to the closed convex extension of G, there would exist an $\bar{x}_0$ and a constant c such that $\bar{x}_0(x_0) > c$ while $\bar{x}_0(x) \leq c$ for all $x \epsilon G$ (8, p. 331). This clearly contradicts the inequalities satisfied by x_0.

1.4 Lemma. *If G is $\aleph_a$-compact, $[x_\pi \mid \pi \epsilon$ directed set of power $\leq \aleph_a] \subset G$, and if $\lim_{\pi} \bar{x}(x_\pi)$ exists for a set $\Gamma \subset \bar{X}$ total on G_{LC}, then x_π converges weakly to a unique element of X.*

As G is $\aleph_a$-compact, there exists by Lemma 1.3 an $x_0 \epsilon G_{LC}$ such that

$$\underline{\lim}_{\pi}\, \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim_{\pi}}\, \bar{x}(x_\pi)$$

for all $\bar{x} \epsilon \bar{X}$. Suppose that for a given $\bar{x}_0 \epsilon \bar{X}$, $\lim_{\pi} \bar{x}_0(x_\pi) \neq \bar{x}_0(x_0)$. Then there exists an $\epsilon > 0$ and a subset $[\pi']$ of $[\pi]$ cofinal with $[\pi]$ such that $\underline{\lim}_{\pi'} \bar{x}_0(x_{\pi'}) \geq \bar{x}_0(x_0) + \epsilon$ (or $\overline{\lim}_{\pi'} \bar{x}_0(x_{\pi'}) \leq \bar{x}(x_0) - \epsilon$). As before there exists an $x'_0 \in G_{LC}$ such that

$$\underline{\lim}_{\pi'} \bar{x}(x_{\pi'}) \leq \bar{x}(x'_0) \leq \overline{\lim}_{\pi'} \bar{x}(x_{\pi'})$$

for all $\bar{x} \in \bar{X}$. Now $\bar{x}(x_0) = \bar{x}(x'_0)$ for all $\bar{x} \in \Gamma$. Therefore $x_0 = x'_0$, contrary to $\bar{x}_0(x'_0) \geq \bar{x}_0(x\ ) + \epsilon$. Hence $\lim_{\pi} \bar{x}(x_\pi) = \bar{x}(x_0)$ for all $\bar{x} \in \bar{X}$. x_0 is clearly unique.

1.5 *Definition.* G is said to be *$\aleph_a$-closed* if, given any subset $[x_\mu]$ where $[\mu]$ is a *well-ordered* set of power $\leq \aleph_a$, there is an $x_0 \in X$ such that

$$\underline{\lim}_{\mu} \bar{x}(x_\mu) \leq \bar{x}(x_0) \leq \overline{\lim}_{\mu} \bar{x}(x_\mu)$$

for all $\bar{x} \in \bar{X}$.

1.6 THEOREM. *The following are equivalent:*

1). *G is $\aleph_0$-compact,*

2). *G is sequentially compact in the $w(X)$ topology,*

3). *G is $\aleph_0$-closed.*

Let $\{x_n\}$ be a subset of G. Then $X_0 = \{x_n\}_{LC}$ is separable. We can therefore choose a denumerable set of functionals on X_0, dense in the $\bar{w}(\bar{X}_0)$ topology (1, pp. 185-186). These functionals can be extended to be functionals on X (1, p. 55). The set of extended functionals, Γ, will clearly be total on X_0. Using the diagonal process, we can obtain a subsequence $\{x_{n'}\}$ for which $\lim \bar{x}(x_{n'})$ exists for all $\bar{x} \in \Gamma$. As $G \cap X_0$ is again $\aleph_0$-compact (Lemma 1.3) we can apply Lemma 1.4. Thus $\{x_{n'}\}$ converges weakly to an element of X. Hence 1) implies 2). 2) clearly implies 3). To prove that 3) implies 1), suppose we are given a denumerable subset $\{x_\pi\}$ of G. Let us order the π's $(\pi_1, \pi_2, \cdots)$ and choose $\pi^n \geq (\pi_1, \pi_2, \cdots, \pi_n)$. The set (π^n) is clearly cofinal with (π). By hypothesis there exists an $x_0 \in X$ such that $\underline{\lim}_{n} \bar{x}(x_{\pi^n}) \leq \bar{x}(x_0) \leq \overline{\lim}_{n} \bar{x}(x_{\pi^n})$ for all $\bar{x} \in \bar{X}$. It follows that

$$\underline{\lim}_{\pi} \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim}_{\pi} \bar{x}(x_\pi)$$

for all $\bar{x} \in \bar{X}$.

1.7 COROLLARY. *If G is $\aleph_0$-compact, then G is bounded.*

1.8 LEMMA. *Given a directed set $\Pi = [\pi]$, let S_a, for each $a \in A$, be a* *bicompact metric space containing the set* $[a_{\alpha\pi} \mid \pi \in \Pi]$. *Then there exists a directed set* $\Sigma = [\sigma]$ *and a function* π_σ *on* Σ *to* Π *for which*

1. $\lim_\sigma a_{\alpha\pi_\sigma}$ *exists in* S_α.
2. *Given* π_0, *there exists a* σ_0 *such that* $\pi_\sigma \geqq \pi_0$ *for all* $\sigma \geqq \sigma_0$.
3. *The power of* Σ *is less than or equal to the power of* $A + \Pi$.

Let P_A be the product space of the spaces S_α where the generic point is $p = [p(\alpha) \in S_\alpha]$ and the generic neighborhood is $U(p_0) \equiv [p \mid \rho[p_0(\alpha_i), p(\alpha_i)] < \epsilon \quad i = 1, 2, \cdots, k]$. Čech (6, p. 830) has shown that this is a bicompact Hausdorff space. Define $p_\pi = [a_{\alpha\pi} \mid \alpha \in A] \in P_A$. Let $F_{\pi_0} \equiv [p_\pi \mid \pi \geqq \pi_0]$. Since Π is a directed set, all finite sets of F_π's have a non-vacuous intersection. Therefore there exists a $p \in \bigcap \overline{F_\pi}$. We define Σ to consist of all combinations $[\pi, \alpha_1, \alpha_2, \cdots, \alpha_k]$ where $\pi \in \Pi$ and $\alpha \in A$. $\sigma_1 \geqq \sigma_2$ will mean that the α's in σ_2 are a subset of those in σ_1 and $\pi_1 \geqq \pi_2$. We define the function π_σ as follows: If $\sigma = [\pi_0, \alpha_1, \cdots, \alpha_k]$, then as $p \in \overline{F_{\pi_0}}$ there will be a $p_{\pi_1} \in F_{\pi_0}$ such that $\rho[p_{\pi_1}(\alpha_i), p(\alpha_i)] < 1/k$, $i = 1, 2, \cdots, k$, $\pi_1 \geqq \pi_0$. Define $\pi_\sigma = \pi_1$. Clearly $\lim_\sigma p_{\pi_\sigma} = p$. Conditions 2) and 3) are likewise satisfied.

1.9 THEOREM. *A set G, closed in the* $w(X)$ *topology is* $\aleph_\alpha$*-compact if and only if it is* $\aleph_\alpha$*-compact*.*

Suppose first that G is $\aleph_\alpha$-compact and that $[A_t \mid t \in T]$ is a family of $w(X)$ closed subsets of G such that any finite set has a non-vacuous intersection and T is of power $\leqq \aleph_\alpha$. Let $[\pi]$ be a directed family of finite subsets of T. The power of $[\pi]$ is then less than or equal to $\aleph_\alpha$. Choose $x_\pi \in \bigcap_{t \in \pi} A_t$. Let H be the set of all finite linear combinations of the x_π with rational coefficients and let $X_0 = H_C = [x_\pi]_{LC}$. Clearly H is of power $\leqq \aleph_\alpha$. For every $x \in H$, choose an $\bar{x} \in \bar{X}$ such that $\bar{x}(x) \geqq ||x||/2$ and $||\bar{x}|| = 1$. Then the set Γ of $\bar{x}$'s so chosen will be total on X_0 and of power $\leqq \aleph_\alpha$. Applying Lemma 1.8 to the set of numbers $[\bar{x}(x_\pi) \mid \bar{x} \in \Gamma, \pi \in [\pi]]$ we obtain a directed set $[\sigma]$ of power $\leqq \aleph_\alpha$ for which $\lim_\sigma \bar{x}(x_{\pi_\sigma})$ exists for all $\bar{x} \in \Gamma$. By Lemma 1.4, there exists an $x_0 \in X$ such that $\lim_\sigma \bar{x}(x_{\pi_\sigma}) = \bar{x}(x_0)$ for all $\bar{x} \in \bar{X}$. Now by Lemma 1.8, given $\pi_0 \equiv t_0$, there exists a σ_0 such that $\pi_\sigma \geqq \pi_0$ for all $\sigma \geqq \sigma_0$. In other words, $x_{\pi_\sigma} \in A_{t_0}$ for all $\sigma \geqq \sigma_0$. As A_t is $w(X)$ closed, x_0 belongs to A_t, i.e., $x_0 \in \bigcap_{t \in T} A_t$.

To prove the converse, suppose G to be $\aleph_\alpha$-compact*. Let $[x_\pi] \subset G$ where $[\pi]$ is a directed family of power $\leqq \aleph_\alpha$. Define $F_{\pi_0} \equiv [x_\pi \mid \pi \geqq \pi_0]$ and $A_\pi \equiv w(X)$ closure of F_π. As $[\pi]$ is a directed set, any finite set of F_π's (and hence A_π's) have a non-vacuous intersection. There will therefore exist an $x_0 \epsilon \bigcap A_\pi$. Finally given π_0, $\bar{x}_0$, and $\epsilon > 0$ there exists an $x_\pi \epsilon F_{\pi_0}$ for which $|\bar{x}_0(x_\pi - x_0)| < \epsilon$. As $x_\pi \epsilon F_{\pi_0}$ implies $\pi \geq \pi_0$, we have

$$\underline{\lim}_{\pi} \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim}_{\pi} \bar{x}(x_\pi)$$

for all $\bar{x} \epsilon \bar{X}$.

1.10 THEOREM. *A convex set G is $\aleph_a$-compact if and only if it is $\aleph_a$-closed.*

As any well-ordered set is a directed set, it is clear that $\aleph_a$-compact implies $\aleph_a$-closed. Suppose that G is convex and $\aleph_a$-closed. By the remark following Theorem 3.4, the norm closure of G is likewise $\aleph_a$-closed. For convex sets the norm closure and the $w(X)$ closure are identical (15). We may therefore suppose that G is a convex $w(X)$ closed set. Let $[x_\pi \mid \pi \epsilon \text{ directed set of power } \leq \aleph_a] \subset G$ and define F_{π_0} to be the convex extension of $[x_\pi \mid \pi \geq \pi_0]$. Let A_π be the $w(X)$ closure of F_π. It is clear that all finite sets of A_π's have a non-vacuous intersection. Assume that this is true for all cardinals less than $\aleph_b \leq \aleph_a$. Then given any set of A_π's of power $\aleph_b$, we well-order this set with ordinals μ less that the first ordinal O_b having $\aleph_b$ predecessors. For each $\lambda < O_b$, choose an $x_\lambda \epsilon \bigcap_{\mu < \lambda} A_\mu$. As G is $\aleph_a$-closed there exists an x_0 for which

$$\underline{\lim}_{\lambda \to O_b} \bar{x}(x_\lambda) \leq \bar{x}(x_0) \leq \overline{\lim}_{\lambda \to O_b} \bar{x}(x_\lambda)$$

for all $\bar{x} \epsilon \bar{X}$. If x_0 were not an element of A_{π_λ}, then we could find an $\bar{x} \epsilon \bar{X}$ and a constant c such that $\bar{x}(A_{\pi_\lambda}) \leq c$ and $\bar{x}(x_0) > c$ (8, p. 331). Thus $\bar{x}(x_\mu) \leq c$ for all $\mu \geq \lambda$, contrary to the above inequalities. This establishes the induction. Hence let x belong to $\bigcap A_\pi$. Then for a given ϵ, $\bar{x}$, π_0 there exist $\pi_1, \pi_2, \cdots, \pi_n \geq \pi_0$ such that $|\bar{x}(\sum_{i=1}^{n} c_i x_{\pi_i} - x)| < \epsilon$ where $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$. For one of these π_i's, $\bar{x}(x_\pi) \leq \bar{x}(x) + \epsilon$ while for another $\bar{x}(x_\pi) \geq \bar{x}(x) - \epsilon$. Hence

$$\underline{\lim}_{\pi} \bar{x}(x_\pi) \leq \bar{x}(x) \leq \overline{\lim}_{\pi} \bar{x}(x_\pi).$$

1.11 *Definition.* A space X is said to satisfy the *Helly property of order* $\aleph_a$, if whenever

$$\left|\sum_{i=1}^{n} a_i c_{a_i}\right| \leq M \left\|\sum_{i=1}^{n} a_i \bar{x}_{a_i}\right\|,$$

where the a_i are arbitrary real numbers, then the system $\bar{x}_a(x) = c_a$ consisting of at most $\aleph_a$ equations has a solution $x \epsilon X_M$.

1.12 LEMMA. *If X_1 is $\aleph_a$-compact, then X satisfies the Helly property of order $\aleph_a$.*

Let us suppose the conditions stated in Definition 1.11 are satisfied by the functionals $[\bar{x}_a]$ and the numbers $[c_a]$. Let π be a finite subset $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ of $[\alpha]$. Then $[\pi]$ is a directed set ($\leq$ means set inclusion) of power $\leq \aleph_a$. By Helly's theorem (14), we can find for each π an x_π such that $\bar{x}_a(x_\pi) = c_a$ for every $\alpha \in \pi$ and $|| x_\pi || \leq M + 1/k$. As X_{M+1} is $\aleph_a$-closed there exists an x_0 such that

$$\underline{\lim}_{\pi} \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim}_{\pi} \bar{x}(x_\pi)$$

for each $\bar{x} \in \bar{X}$. Since $\lim_{\pi} \bar{x}_a(x_\pi)$ exists, we have $\bar{x}_a(x_0) = c_a$. Finally $| \bar{x}(x_0) | \leq \overline{\lim}_{\pi} | \bar{x}(x_\pi) | \leq M \, || \bar{x} ||$ implies $|| x_0 || \leq M$.

1.13 LEMMA. *If X satisfies the Helly property of order $\aleph_a$, then $\bar{X}_1$ is $\aleph_a$-compact.*

Let $[\bar{x}_\pi \mid \pi \in \text{directed set of power} \leq \aleph_a]$ be bounded in the norm by one. By an argument similar to that used by Banach (1, p. 118) we can obtain an $\bar{x}_0 \in \bar{X}$ for which

$$\underline{\lim}_{\pi} \bar{x}_\pi(x) \leq \bar{x}_0(x) \leq \overline{\lim}_{\pi} \bar{x}_\pi(x)$$

for all $x \in X$. Now if $\bar{\bar{x}} \in \bar{\bar{X}}$, then

$$\left| \sum_{i=1}^{n} a_i \bar{\bar{x}}(\bar{x}_{a_i}) \right| \leq || \bar{\bar{x}} || \cdot || \sum_{i=1}^{n} a_i \bar{x}_{a_i} ||$$

where $\bar{x}_a \in [\bar{x}_\pi]$ or $\bar{x}_a = \bar{x}_0$. Hence the Helly property of order $\aleph_a$ implies the existence of an x_0 for which $\bar{x}_a(x_0) = \bar{\bar{x}}(\bar{x}_a)$ for all π and $\bar{x}_0(x_0) = \bar{\bar{x}}(\bar{x}_0)$. Consequently

$$\underline{\lim}_{\pi} \bar{\bar{x}}(\bar{x}_\pi) = \underline{\lim}_{\pi} \bar{x}_\pi(x_0) \leq \bar{x}_0(x_0) = \bar{\bar{x}}(\bar{x}_0) \leq \overline{\lim}_{\pi} \bar{x}_\pi(x_0) = \underline{\lim}_{\pi} \bar{\bar{x}}(\bar{x}_\pi),$$

so that $\bar{X}_1$ is $\aleph_a$-compact.

1.14 THEOREM. *The following statements are equivalent:*

1). *X_1 is $\aleph_a$-compact.*

2). *X satisfies the Helly property of order $\aleph_a$.*

3). *$\bar{\bar{X}}_1$ is $\aleph_a$-compact.*

Lemmas 1.12 and 1.13 show that 1) implies 2) implies 3). Reiterating this reasoning we find that 3) implies that $\bar{X}_1$ is $\aleph_{1a}$-compact. As X_1 is a closed convex subset of $\bar{\bar{X}}_1$, it follows that $\bar{X}_1$ is $\aleph_a$-compact (see Lemma 1.3).

A property will be said to be *hereditary* if whenever it holds for a Banach space, it also holds for every closed linear subspace. Theorem 1.14 and Lemma 1.3 give

1.15 COROLLARY. *The Helly property of order $\aleph_a$ is hereditary.*

2. On weakly completely continuous transformations.

We shall consider in this section a generalization of the weakly completely continuous transformations of a Banach space X to another Banach space Y.

Definition. A linear transformation U on X to Y will be called *$\aleph_a$-compact* if $U(X_1)$ is $\aleph_a$-compact.

This definition is an extension of the usual notion of a completely continuous transformation. In view of the following lemma, it would clearly not do to require only that the transformation take $\aleph_a$-compact sets into $\aleph_a$-compact sets.

2.1 LEMMA. *If G is an $\aleph_a$-compact subset of X and U is a linear limited transformation on X to Y, then $U(G)$ is $\aleph_a$-compact.*

Suppose $[y_\pi \mid \pi \in$ directed set of power $\leq \aleph_a] \subset U(G)$. Then y_π is the image of some $x_\pi \in G$. G being $\aleph_a$-compact, there exists an $x_0 \in X$ such that

$$\underline{\lim}_{\pi}\, \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \overline{\lim}_{\pi}\, \bar{x}(x_\pi)$$

for all $\bar{x} \in \bar{X}$. Consequently

$$\underline{\lim}_{\pi}\, \bar{y}(y_\pi) = \underline{\lim}_{\pi}\, \bar{U}\bar{y}(x_\pi) \leq \bar{U}\bar{y}(x_0) = \bar{y}(Ux_0) \leq \overline{\lim}_{\pi}\, \bar{U}\bar{y}(x_\pi) = \overline{\lim}_{\pi}\, \bar{y}(y_\pi)$$

for all $\bar{y} \in \bar{Y}$.

2.2 THEOREM. *If U is $\aleph_a$-compact, then the adjoint transformation $\bar{U}$ is $\aleph_a$-compact.*

Let $[\bar{y}_\pi \mid \pi \in$ directed set of power $\leq \aleph_a]$ be bounded by one. Then there exists a $\bar{y}_0 \in \bar{Y}$ such that

$$\underline{\lim}_{\pi}\, \bar{y}_\pi(y) \leq \bar{y}_0(y) \leq \overline{\lim}_{\pi}\, \bar{y}_\pi(y)$$

for all $y \in Y$. We shall show that for any $\bar{\bar{x}} \in \bar{\bar{X}}$

$$\underline{\lim}_{\pi}\, \bar{\bar{x}}\bar{U}(\bar{y}_\pi) \leq \bar{\bar{x}}\bar{U}(\bar{y}_0) \leq \overline{\lim}_{\pi}\, \bar{\bar{x}}\bar{U}(\bar{y}_\pi).$$

Define σ to be a finite subset of $[\bar{y}_\pi] \cup \bar{y}_0$. Then $[\sigma]$ is a directed set ($\leq$ means set inclusion) of power $\leq \aleph_a$. By Helly's theorem (13), for each σ there exists an $x_\sigma \in X$ such that both $\bar{U}(\bar{y})x_\sigma = \bar{\bar{x}}\bar{U}(\bar{y})$ for $\bar{y} \in \sigma$ and $\| x_\sigma \| \leq \| \bar{\bar{x}} \| + 1$. Hence $\lim_{\sigma} \bar{y}U(x_\sigma) = \bar{\bar{x}}\bar{U}(\bar{y})$ exists for any $\bar{y}$ of the set $[\bar{y}_\pi] \cup \bar{y}_0$. As $U(X_1)$ is $\aleph_\alpha$-compact, there exists a $y_0 \epsilon Y$ for which

$$\varliminf_{\sigma} \bar{y}U(x_\sigma) \leq \bar{y}(y_0) \leq \varlimsup_{\sigma} \bar{y}U(x_\sigma)$$

for all $\bar{y} \epsilon \bar{Y}$. Clearly $\bar{\bar{x}}\bar{U}(\bar{y}) = \lim_{\sigma} \bar{y}U(x_\sigma) = \bar{y}(y_0)$ for all $\bar{y}$ of the set $[\bar{y}_\pi] \cup \bar{y}_0$. Consequently

$$\varliminf_{\pi} \bar{\bar{x}}\bar{U}(\bar{y}_\pi) = \varliminf_{\pi} \bar{y}_\pi(y_0) \leq \bar{y}_0(y_0) = \bar{\bar{x}}U(\bar{y}_0) \leq \varlimsup_{\pi} \bar{y}_\pi(y_0) = \varlimsup_{\pi} \bar{\bar{x}}\bar{U}(\bar{y}_\pi).$$

2.3 Corollary. *If $\bar{U}$ is $\aleph_\alpha$-compact, then U is likewise $\aleph_\alpha$-compact.*

By Theorem 2.2, if $\bar{U}$ is $\aleph$-compact, then $\bar{\bar{U}}$ is $\aleph_\alpha$-compact. That is, $\bar{\bar{U}}(\bar{\bar{X}}_1)$ is $\aleph_\alpha$-compact. But X_1 is equivalent to a subset G of $\bar{\bar{X}}_1$. Therefore $U(X_1) \equiv \bar{\bar{U}}(G)$ is $\aleph_\alpha$-compact.

Theorem 2.2 is a generalization of a theorem due to V. Gantmacher (10).

The following lemmas will be found useful in our further work on $\aleph_\alpha$-compact transformations. Lemma 2.4, which is essentially a simplification of a lemma due to Banach (1, pp. 119-121), was first stated by Kakutani (14).

2.4 Lemma. *If Γ is a $\bar{w}(\bar{X})$ closed linear subset of $\bar{X}$ not containing $\bar{x}_0$, then there exists an $x_0 \epsilon X$ such that $\bar{x}_0(x_0) \neq 0$ and $\bar{x}(x_0) = 0$ for all $\bar{x} \epsilon \Gamma$.*

As $\bar{X}$ is a regular Hausdorf space in the $\bar{w}(\bar{X})$ topology, there exists a neighborhood $U_{\bar{x}_0} \equiv [\bar{x} \mid |(\bar{x} - \bar{x}_0)x_i| < \epsilon,\ i = 1, 2, \cdots, n]$ containing no elements of Γ. Consider the linear transformation $T(\bar{x}) = \{\bar{x}(x_1), \bar{x}(x_2), \cdots, \bar{x}(x_n)\}$ which maps $\bar{X}$ into the n-dimensional euclidean space E. $T(\Gamma)$ is a linear subspace of E not containing $T(\bar{x}_0)$. Hence there exists an $(n-1)$-dimensional hyperplane $\{c_i\}$ containing $T(\Gamma)$ but not containing $T(\bar{x}_0)$. That is

$$\bar{x}\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{n} c_i \bar{x}(x_i) \begin{cases} = 0 & \text{for all } \bar{x} \epsilon \Gamma \\ \neq 0 & \text{for } \bar{x} = \bar{x}_0 \end{cases}$$

$x_0 = \sum_{i=1}^{n} c_i x_i$ satisfies the conditions of the lemma.

2.5 Lemma. *If G is $\aleph_\alpha$-compact and if there exists a set $\Gamma \subset \bar{X}$ of power $\leq \aleph_\alpha$ total on G_{LC}, then G is $\aleph$-compact.*

Suppose $[x_\pi \mid \pi \epsilon \text{ directed set}] \subset G$. Applying Lemma 1.8 to the set $[\bar{x}(x_\pi) \mid \bar{x} \epsilon \bar{X}, \pi \epsilon [\pi]]$, we obtain a directed set $[\sigma]$ such that $\lim_{\sigma} \bar{x}(x_{\pi_\sigma})$ exists for all $\bar{x} \epsilon \bar{X}$. The set of all finite subsets ν of Γ is a directed set ($\leq$ means set inclusion) of power $\leq \aleph_\alpha$. For $\nu = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$ there exists an x_{π_σ} such that $|\bar{x}(x_{\pi_\sigma}) - \lim_{\sigma} \bar{x}(x_{\pi_\sigma})| < 1/k$ for all $\bar{x} \epsilon \nu$. Define this x_{π_σ} to be x_ν. Then $\lim_{\nu} \bar{x}(x_\nu) = \lim_{\sigma} \bar{x}(x_{\pi_\sigma})$ for all $\bar{x} \epsilon \Gamma$. As G is $\aleph_\alpha$-compact there exists an x_0 such that $\lim_{\nu} \bar{x}(x_\nu) = \bar{x}(x_0)$ for all $\bar{x} \in \Gamma$. If we repeat this process for $\Gamma' = \Gamma \cup \bar{x}_0$, where $\bar{x}_0$ is an arbitrary element of $\bar{X}$, we obtain an x'_0 such that $\bar{x}(x'_0) = \lim_{\sigma} \bar{x}(x_{\pi_\sigma})$ for all $\bar{x} \in \Gamma'$ and $\bar{x}(x_0) = \bar{x}(x'_0)$ for all $\bar{x}$ belonging to the total set Γ. Hence $x_0 = x'_0$ and as $\bar{x}_0$ was arbitrary, $\bar{x}(x_0) = \lim_{\sigma} \bar{x}(x_{\pi_\sigma})$ for all $\bar{x} \in \bar{X}$. Now, given π_0, there exists a σ such that $\pi_\sigma \geq \pi_0$ for all $\sigma \geq \sigma_0$. It follows that

$$\varliminf_{\pi} \bar{x}(x_\pi) \leq \varliminf_{\sigma} \bar{x}(x_{\pi_\sigma}) = \bar{x}(x_0) = \varlimsup_{\sigma} \bar{x}(x_{\pi_\sigma}) \leq \varlimsup_{\pi} \bar{x}(x_\pi).$$

Lemma 2. 5 is a rather strong result. For instance, if X_1 is $\aleph_a$-compact and there exists a subset of $\bar{X}$ of power $\leq \aleph_a$ total on X, it follows from this lemma, that X is $\aleph_a$-complete for all $\aleph$. This, by a result due to Goldstine (12), implies that X is reflexive (Theorem 1. 1).

2. 6 THEOREM. *If U is $\aleph_a$-compact and if there exists a set of power $\leq \aleph_a$ total on $\bar{U}(X)_C$ then* (1) *U is $\aleph$-compact fo rall $\aleph$, and* (2) *$\bar{U}(\bar{Y})$ is $\aleph_a$-separable.*[2]

By Lemma 2. 5 U is $\aleph$-compact for all $\aleph$. Let $Y_0 = U(X)_C$. U also defines an adjoint transformation $\bar{U}_1$ on $\bar{Y}_0$ to $\bar{X}$. Since 1) any functional on Y_0 can be extended to a functional on Y and conversely any functional on Y defines a functional on Y_0, and 2) $\bar{U}(\bar{y})$ depends only on the behavior of $\bar{y}$ on Y_0, it follows that $\bar{U}(\bar{Y}) \equiv \bar{U}_1(\bar{Y}_0)$. By hypothesis, there exists a subset of $\bar{Y}_0$ of power $\leq \aleph_a$ total on Y_0. Let Γ be the set of all finite linear combinations with rational coefficients of this set. Γ is of power $\leq \aleph_a$ and $\Gamma_C = (\Gamma_L)_C$. $\bar{Y}_0$ is the $\bar{w}(\bar{Y}_0)$ closure of Γ_L, by Lemma 2. 4, and hence of Γ. Therefore given $\bar{y} \in \bar{Y}_0$, there exists a set $[\bar{y}_\pi \mid \pi \in \text{directed set}] \subset \Gamma$ such that $\lim_{\pi} \bar{y}_\pi(y) = \bar{y}(y)$ for all $y \in Y_0$. If we now employ the reasoning used in Theorem 2. 2, we obtain $\lim_{\pi} \bar{\bar{x}}\bar{U}_1(\bar{y}_\pi) = \bar{\bar{x}}\bar{U}_1(\bar{y})$ for all $\bar{\bar{x}} \subset \bar{\bar{X}}$. It follows that $\bar{U}_1(\bar{y}) \in \bar{U}_1(\Gamma)_{LC} = \bar{U}_1(\Gamma)_C$, which concludes the proof.

2. 7 COROLLARY. *If U is $\aleph_a$-compact and if there exists a set of power $\leq \aleph_a$ total on $U(X)_C$, then $\bar{\bar{U}}(\bar{\bar{X}}) \subset U(X)_C$ which is $\aleph_a$-separable.*

Given $\bar{\bar{x}} \in \bar{\bar{X}}$, let π be any finite subset of $\bar{Y}$. Then $[\pi]$ is a directed set ($\leq$ means set inclusion). By Helly's theorem (14), there exists for each π an $x_\pi \in X$ such that both $\bar{U}(\bar{y})x_\pi = \bar{\bar{x}}\bar{U}(\bar{y})$ for $\bar{y} \in \pi$ and $\|x_\pi\| \leq \|\bar{\bar{x}}\| + 1$.

---

[2] A subset of X will be called $\aleph_a$-separable if it contains a strongly dense set of power $\leq \aleph_a$.

As $U(X_1)$ is $\aleph$-compact for all $\aleph$ (Lemma 2.5), there exists a $y_0 \in U(X)_C$ (Lemma 1.3) for which $\bar{\bar{x}}\bar{U}(\bar{y}) = \lim_{\pi} \bar{y}U(x_\pi) = \bar{y}(y_0)$ for all $\bar{y} \in \bar{Y}$. By definition of $\bar{\bar{U}}$, $\bar{\bar{U}}(\bar{\bar{x}}) = y_0$. $U(X)_C$ is $\aleph_\alpha$-separable, by Theorem 2.6, since $\bar{U}$ satisfies the same conditions that U satisfies.

2.8 THEOREM. *If* $\{U_n\}$ *is a sequence of* $\aleph_\alpha$*-compact transformations and* $\lim_n \| U_n - U_0 \| = 0$, *then* U_0 *is* $\aleph_\alpha$*-compact.*

Suppose $[x_\pi \mid \pi \in$ directed set of power $\leq \aleph_\alpha] \subset X_1$. Let $X_0 \equiv [x_\pi]_{LC}$. Then there will exist a subset of $\bar{Y}$ of power $\leq \aleph_\alpha$ total on $U(X_0)_C$ (see the argument used in Theorem 1.9). By Lemma 2.5 we infer that $U_n((X_0)_1)$ is $\aleph$-closed. Let $[\alpha]$ be the class of all pairs $(\bar{y}, n)$ where $\bar{y} \in \bar{Y}$ and n is one of the integers $(0, 1, 2, \cdots)$. We can now apply Lemma 1.8 to the numbers $[a_{\alpha\pi} = \bar{y}U_n(x_\pi)]$. Hence there exists a directed set $[\sigma]$ such that $\lim_{\sigma} \bar{y}U_n(x_{\pi_\sigma})$ exists and

$$\underline{\lim}_{\pi}\, \bar{y}U_n(x_\pi) \leq \lim_{\sigma} \bar{y}U_n(x_{\pi_\sigma}) \leq \overline{\lim}_{\pi}\, \bar{y}U_n(x_\pi)$$

for all $\bar{y} \in \bar{Y}$ and $n = (0, 1, 2, \cdots)$. As $U_n((X_0)_1)$ is $\aleph$-compact for $n \geq 1$, there exists a $y_n \in Y$ such that $\lim_{\sigma} \bar{y}U_n(x_{\pi_\sigma}) = \bar{y}(y_n)$ for all $\bar{y} \in \bar{Y}$ and $n \geq 1$. Now given ε, there exists an N such that for $n, m \geq N$ $\| U_n - U_m \| \leq \varepsilon$. Hence $| \bar{y}(y_n - y_m) | = \lim_{\sigma} | \bar{y}(U_n - U_m)x_{\pi_\sigma} | \leq \| \bar{y} \| \varepsilon$ so that $\| y_n - y_m \| \leq \varepsilon$ for $n, m \geq N$. The sequence y_n will therefore converge to an element y_0 of Y. As $\lim_{n} \bar{y}U_n(x_\pi) = \bar{y}U(x_\pi)$ uniformly in π, the iterated limits theorem gives $\lim_{\sigma} \bar{y}U(x_{\pi_\sigma}) = \bar{y}(y_0)$. By the above inequalities

$$\underline{\lim}_{\pi}\, \bar{y}U_0(x_\pi) \leq \bar{y}(y_0) \leq \overline{\lim}_{\pi}\, \bar{y}U_0(x_\pi)$$

for all $\bar{y} \in \bar{Y}$. U_0 is therefore $\aleph_\alpha$-compact.

3. The convex extension of $\aleph_0$ compact sets. The principal theorem of this section proves that the closed convex extension of an $\aleph_\alpha$-compact set is likewise $\aleph_\alpha$-compact. Making use of this theorem, we obtain a characterization of $\aleph_\alpha$-compact subsets G of X which is independent of the separability of X. The set of numbers $[\bar{x}(x) \mid \bar{x} \in \bar{X}_1, x \in G]$ can be considered as a class of functions having the range $\bar{X}_1$ or as a class of functions having the range G. Our characterization of $\aleph_\alpha$-compact sets exhibits these two function classes as duals of one another.

3.1 THEOREM. *If* G *is an* $\aleph_0$*-compact subset of a separable Banach space* X, *then* $U(\bar{x}) = [\bar{x}(x) \mid x \in G]$ *on* $\bar{X}$ *to* M_G *is* $\aleph_0$*-compact.*

Since G is bounded (by Corollary 1.7), $[\bar{x}(x) \mid x \in G]$ belongs to M_G. Suppose that $\{\bar{x}_n\}$ is a sequence of functionals bounded by one. As X is separable, $\bar{X}_1$ is a compact metric space in its $\bar{w}(\bar{X})$ topology (1, pp. 185-186). Hence there exists a subsequence, which we renumber for convenience, and an $\bar{x}_0 \in \bar{X}$ such that $\lim_n \bar{x}_n(x) = \bar{x}_0(x)$ for all $x \in X$. We will show that $\lim_n \bar{b}\bar{U}(\bar{x}_n) = \bar{b}\bar{U}(\bar{x}_0)$ or that $\lim_n \bar{b}\bar{U}\bar{y}_n = 0$ where $\bar{y}_n = \bar{x}_n - \bar{x}_0$ for all $\bar{b} \in \bar{M}_G$. By a theorem due to Hildebrandt (13) this is equivalent to $\lim_n \int_G \bar{y}_n(x)\,d\beta = 0$ for all finitely additive bounded set functions β on all subsets of G. Suppose this were not the case. Then by a theorem due to Banach (1, p. 219), there exists a sequence $\{x_i\} \subset G$ such that $\lim_n \varliminf_i |\bar{y}_n(x_i)| > \varepsilon > 0$. Since G is $\aleph_0$-compact there exists a subsequence $\{x_{i_j}\}$ which converges weakly as elements to an $x_0 \in X$. Therefore $\varliminf_i |\bar{y}(x_i)| \leq |\bar{y}(x_0)|$. However $\lim_n \bar{y}_n(x_0) = 0$, which is in contradiction to our supposition.

3.2 Corollary. *If G is an $\aleph_0$-compact subset of a Banach space X, then $[\Sigma_G a_x x \mid \Sigma_G \mid a_x \mid \leq 1]$ is $\aleph_0$-compact.*

Suppose $x_n = \sum_{i=1}^{\infty} a_i^n x_i^n$ where $\sum_{i=1}^{\infty} |a_i^n| \leq 1$ and $x_i^n \in G$. We shall show that $\{x_n\}$ is contained in an $\aleph_0$ compact subset of X. Let $X_0 = [x_i^n]_{LC}$ and $G_0 = G \cap X_0$. Then G_0 is $\aleph_0$-compact (by Lemma 1.3) and is contained in the separable space X_0. Define $T(a) = \Sigma_{G_0} a_x x$ where $\Sigma_{G_0} |a_x| < \infty$. As G is bounded (by Corollary 1.7), T is linear on l_{G_0} to X. $\bar{T}(\bar{x}) = [\bar{x}(x) \mid x \in G_0]$ on $\bar{X}$ to M_{G_0} is by Theorem 3.1 $\aleph_0$-compact. By Corollary 2.3, T is likewise $\aleph_0$-compact. Therefore $T[(l_{G_0})_1]$ which contains $\{x_n\}$ is $\aleph_0$-compact.

Krein (15) has demonstrated the slightly weaker result that the convex extension of an $\aleph_0$ compact subset of a separable Banach space is likewise $\aleph_0$-compact. His proof made use of the Fubini Theorem. The following lemma is due to Smulian (25). For the sake of completeness we shall give another proof of this result.

3.3 Lemma. *If G is $\aleph_0$-compact, then G_C is $\aleph_0$-compact.*

If $\{x_n\} \subset G_C$, then there exist y_n in G such that $\| x_n - y_n \| < 1/n$. As G is $\aleph_0$-compact, there exists an x_0 for which

$$\varliminf_n \bar{x}(y_n) \leq \bar{x}(x_0) \leq \varlimsup_n \bar{x}(y_n)$$

for all $\bar{x} \in \bar{X}$. Now if $\| \bar{x} \| \leq 1$ and $n \geq k$, then $\bar{x}(x_n) - 1/k \leq \bar{x}(y_n) \leq \bar{x}(x_n) + 1/k$. Thus

$$\underline{\lim}_n \bar{x}(x_n) - 1/k \leq \underline{\lim}_n \bar{x}(y_n) \leq \bar{x}(x_0) \leq \overline{\lim}_n \bar{x}(y_n) \leq \overline{\lim}_n \bar{x}(x_n) + 1/k$$

for all k. In other words

$$\underline{\lim}_n \bar{x}(x_n) \leq \bar{x}(x_0) \leq \overline{\lim}_n \bar{x}(x_n).$$

Hence by Theorem 1. 6, G_C is $\aleph_0$-compact.

3. 4 Theorem. *If G is $\aleph_\alpha$-compact, then G_C is $\aleph_\alpha$-compact.*

If $[x_\pi \mid \pi \in \text{directed set of power} \leq \aleph_\alpha] \subset G_C$, then there exist elements x_π^n in G such that $\| x_\pi - x_\pi^n \| < 1/n$. As G is $\aleph_\alpha$-compact, there exist x^n in X for which

$$\underline{\lim}_\pi \bar{x}(x_\pi^n) \leq \bar{x}(x^n) \leq \overline{\lim}_\pi \bar{x}(x_\pi^n)$$

for all $\bar{x} \in \bar{X}$. It follows that

$$\underline{\lim}_\pi \bar{x}(x_\pi) - 1/n \leq \bar{x}(x^n) \leq \overline{\lim}_\pi \bar{x}(x_\pi) + 1/n$$

for all $\bar{x} \in \bar{X}_1$. By Lemma 1. 3 x_n belongs to the closed convex extension of G which, according to Corollary 3. 2 and Lemma 3. 3, is $\aleph_0$-compact. Hence we can find an $x_0 \in X$ such that

$$\underline{\lim}_n \bar{x}(x^n) \leq \bar{x}(x_0) \leq \overline{\lim}_n \bar{x}(x^n)$$

for all $\bar{x} \in \bar{X}$. Finally

$$\underline{\lim}_\pi \bar{x}(x_\pi) \leq \underline{\lim}_n \bar{x}(x^n) + 1/n \leq \bar{x}(x_0) \leq \overline{\lim}_n \bar{x}(x^n) - 1/n \leq \overline{\lim}_\pi \bar{x}(x_\pi)$$

for all $\bar{x} \in \bar{X}_1$.

We remark that the same argument shows G $\aleph_\alpha$-closed to imply G_C $\aleph_\alpha$-closed. The only lemmas used in the proof concerned $\aleph_0$-compactness which is equivalent to $\aleph_0$-closedness.

3. 5 Lemma. *If G is $\aleph$-compact, then $[\Sigma_G a_x x \mid \Sigma_G \mid a_x \mid \leq 1]$ is $\aleph$-compact.*

Let $S_n \equiv [\sum_{i=1}^{n} c_i x_i \mid \sum^{n} \mid c_i \mid \leq 1, x_i \in G]$. We first show that S_n is $\aleph$-compact. Suppose $[x_\pi \mid \pi \in \text{directed set}] \subset S_n$. That is $x_\pi = \sum_{i=1}^{n} c_i^\pi x_i^\pi$ where $\sum_{i=1}^{n} \mid c_i^\pi \mid \leq 1$ and $x_i^\pi \in G$. Let $a_{\bar{x}\pi}$ be the point in a $2n$-dimensional euclidean space having the coördinates $(\bar{x}(x_1^\pi), \bar{x}(x_2^\pi), \cdots, \bar{x}(x_n^\pi), c_1^\pi, c_2^\pi, \cdots, c_n^\pi)$. The set $[a_{\bar{x}\pi} \mid \pi \in [\pi]]$ is contained in a bounded and hence bicompact subset of this space. We can therefore apply Lemma 1. 8 and obtain a directed set $[\sigma]$

such that $\lim_{\sigma} \bar{x}(x_i^{\pi\sigma})$ exists for each i and $\bar{x} \in \bar{X}$, and $\lim_{\sigma} c_i^{\pi\sigma} = c_i$ exists for each i. As G is $\aleph$-compact, there exist $x_i \in X$ such that $\lim_{\sigma} \bar{x}(x_i^{\pi\sigma}) = \bar{x}(x_i)$ for all $\bar{x} \in \bar{X}$. Consequently $\lim_{\sigma} \bar{x}(c_i^{\pi\sigma} x_i^{\pi\sigma}) = \bar{x}(c_i x_i)$ and hence

$$\lim_{\sigma} \bar{x}\Big(\sum_{1}^{n} c_i^{\pi\sigma} x_i^{\pi\sigma}\Big) = \bar{x}\Big(\sum_{1}^{n} c_i x_i\Big).$$

Now for a given π_0 there exists a σ_0 such that $\pi\sigma \geq \pi_0$ if $\sigma \geq \sigma_0$. Consequently if $x_0 = \sum_{1}^{n} c_i x_i$, then

$$\varliminf_{\pi} \bar{x}(x_\pi) \leq \lim_{\sigma} \bar{x}\Big(\sum_{1}^{n} c_i^{\pi\sigma} x_i^{\pi\sigma}\Big) = \bar{x}(x_0) \leq \varlimsup_{\pi} \bar{x}(x_\pi).$$

We shall next show that $S = \Sigma S_n$ is $\aleph$-closed for all $\aleph$. Suppose therefore that $[x_\mu \mid \mu \in \text{ordinal numbers} < \Omega] \subset S$. Let $\Pi_n = [\mu \mid x_\mu \in S_n]$. Then $\Sigma \Pi_n = [\mu \mid \mu < \Omega]$ and $\Pi_{n+1} \supset \Pi_n$. As S_n is $\aleph$-compact and Π_n is a directed set, there exists an $x_n \in S_G$ by Lemma 1. 3 such that

$$\varliminf_{\mu \in \Pi_n} \bar{x}(x_\mu) \leq \bar{x}(x_n) \leq \varlimsup_{\mu \in \Pi_n} \bar{x}(x_\mu)$$

for all $\bar{x} \in \bar{X}$. By Lemmas 3. 5 and 3. 3, S_G is $\aleph_0$-compact. Thus there exists an $x_0 \in X$ such that

$$\varliminf_{n} \bar{x}(x_n) \leq \bar{x}(x_0) \leq \varlimsup_{n} \bar{x}(x_n).$$

For a given $\bar{x} \in \bar{X}$ and $\varepsilon > 0$, there exists a μ_0 such that $\mu \geq \mu_0$ implies $\bar{x}(x_\mu) \geq \varliminf_{\mu} \bar{x}(x_\mu) - \varepsilon$. Further there exists an N such that $n \geq N$ implies $\mu_0 \in \Pi_n$ and hence $\varliminf_{\mu \in \Pi_n} \bar{x}(x_\mu) \geq \varliminf_{\mu} \bar{x}(x_\mu) - \varepsilon$. Therefore $\bar{x}(x_0) \geq \varliminf_{\mu} \bar{x}(x_\mu) - \varepsilon$ for all ε. A symmetric argument gives

$$\varliminf_{\mu} \bar{x}(x_\mu) \leq \bar{x}(x_0) \leq \varlimsup_{\mu} \bar{x}(x_\mu)$$

for all $\bar{x} \in \bar{X}$.

S is certainly convex and hence by Theorem 1. 10, S is $\aleph$-compact for all $\aleph$. Finally the set $[\Sigma_G a_x x \mid \Sigma_G \mid a_x \mid \leq 1]$ is $\aleph$-compact by Theorem 3. 4.

3. 6 THEOREM. *If G is $\aleph_a$-compact, then $[\Sigma_G a_x x \mid \Sigma_G \mid a_x \mid \leq 1]$ is $\aleph_a$-compact.*

Suppose $[x_\pi \mid \pi \in \text{directed set of power} \leq \aleph_a] \subset [\Sigma_G a_x x \mid \Sigma_G \mid a_x \mid \leq 1]$. Hence $x_\pi = \sum_{i=1}^{\infty} a_i^\pi x_i^\pi$ where $x_i^\pi \in G$. Let $X_0 \equiv [x_i^\pi]_{LC}$. By a familiar argument, there exists a set of power $\leq \aleph_a$ total on X_0. By Lemma 2. 5, $G_0 \equiv G \cap X_0$

is $\aleph$-compact. Since $[\Sigma_{G_0} a_x x \mid \Sigma_{G_0} \mid a_x \mid \leq 1]$ is by Lemma 3. 5 $\aleph$-compact and contains $[x_\pi]$, there exists an $x_0 \in X$ such that

$$\varliminf_{\pi} \bar{x}(x_\pi) \leq \bar{x}(x_0) \leq \varlimsup_{\pi} \bar{x}(x_\pi)$$

for all $\bar{x} \in \bar{X}$.

Combining this result with Theorem 3. 4, we obtain

3. 7 COROLLARY. *If G is $\aleph_a$-compact, then its closed convex extension is likewise $\aleph_a$-compact.*

We are now in a position to justify one of the introductory remarks, namely, G is $\aleph_a$-compact if and only if the $w(X)$ closure of G is $\aleph_a$-compact*. For if G is $\aleph_a$-compact, then its convex closed extension (which is weakly closed) is $\aleph_a$-compact and hence by Theorem 1. 9 this set is $\aleph_a$ compact*. The weak closure of G being a weakly closed subset of an $\aleph_a$-compact* set is likewise $\aleph_a$-compact*. The converse follows from Theorem 1. 9.

If G is a bounded subset of X and β a finitely additive set function on all subsets of G, we shall define $\int_G x\,d\beta$ to be an element x_0 of X for which $\bar{x}(x_0) = \int_G \bar{x}(x)\,d\beta$ for all $\bar{x} \in \bar{X}$.

3. 8 THEOREM. *The following are equivalent:*

1). *G is $\aleph$-compact.*

2). *$(cv(G))_G$ is $\aleph$-compact.*

3). *$\int_G x\,d\beta$ exists and is unique for all β.*

4). *$\lim_\sigma \bar{x}_\sigma(x) = 0$ for all $x \in X$ implies $\int_G \bar{x}_\sigma(x)\,d\beta \to 0$ for all β; and G is bounded.*

5). *$U(\bar{x}) = [\bar{x}(x) \mid x \in G]$ is $\aleph$-compact on $\bar{X}$ to M_G for all $\aleph$.*

We shall show that 1) → 2) → 3) → 4) → 5) → 1).

That 1) implies 2) is a restatement of Corollary 3. 7. By Lemma 3. 5, $S \equiv [\sum_G a_x x \mid \sum_G \mid a_x \mid \leq 1]$ is $\aleph$-compact. Let π be a subdivision of G into a finite number of subsets $\{\tau_i\}$. Define $x_\pi = \Sigma_\pi x_i \beta(\tau_i)$ where x_i is some element of τ_i. Then $\int_G \bar{x}(x)\,d\beta = \lim_\pi \bar{x}(x_\pi)$. As $[x_\pi] \subset S$, it follows that there exists an $x_0 \in X$ such that

$$\bar{x}(x_0) = \lim \bar{x}(x_\pi) = \int_G \bar{x}(x)\,d\beta$$

for all $\bar{x} \in \bar{X}$. If $\bar{x}(x'_0) = \int_G \bar{x}(x)\,d\beta$ for all $\bar{x} \in \bar{X}$, then $\bar{x}(x_0 - x'_0) = 0$ for all $\bar{x} \in \bar{X}$ which implies $x_0 = x'_0$. Therefore 2) implies 3). That 3) implies 4) is obvious.

If G is bounded, we can define the linear transformation $T(a) = \sum_G a_x x$ where $\sum_G | a_x | < \infty$ on l_G to X. Then $\bar{T}(\bar{x}) = [\bar{x}(x) \mid x \in G]$ on $\bar{X}$ to M_G. We shall show that $\bar{T}$ is $\aleph$-compact as a consequence of 4). Given $[\bar{x}_\pi \mid \pi \in$ directed set$] \subset \bar{X}_1$. Applying Lemma 1.8 to the numbers $[\bar{x}_\pi(x) \mid x \in X, \pi \in [\pi]]$ we obtain a directed set $[\sigma]$ such that $\lim_\sigma \bar{x}_{\pi_\sigma}(x)$ exists, and given π_0 there exists a σ_0 such that $\pi_\sigma \geqq \pi_0$ for all $\sigma \geqq \sigma_0$. Now $\lim_\sigma \bar{x}_{\pi_\sigma}(x)$ defines an element $\bar{x}_0$ of $\bar{X}$, namely, $\bar{x}_0(x) = \lim_\sigma \bar{x}_{\pi_\sigma}(x)$. For every $\bar{b} \in \bar{M}_G$, $\bar{b}\bar{T}(\bar{x}) = \int_G \bar{x}(x) \, d\beta$. By 4) $\lim_\sigma \bar{b}\bar{T}(\bar{x}_{\pi_\sigma}) = \bar{b}\bar{T}(\bar{x}_0)$ and hence $\lim_\pi \bar{b}\bar{T}(\bar{x}_\pi) \leqq \bar{b}\bar{T}(\bar{x}_0) \leqq \overline{\lim_\pi} \, \bar{b}\bar{T}(\bar{x}_\pi)$ which proves 5). By Corollary 2.3, T is likewise $\aleph$-compact for all $\aleph$. Therefore $G \subset T((l_G)_1)$ is $\aleph$-closed. This concludes the proof.

An application of this theorem gives a criterion for the interchange of order of integration for finitely additive set functions. Let $f(s,t)$ on $S \times T$ have the property that $[f(s, .) \mid s \in S]$ is an $\aleph$-compact subset of M_T. Then for any finitely additive set function α on all subsets of S and any finitely additive set function β on all subsets of T,

$$\int_T \int_S f(s,t) \, d\alpha \, d\beta = \int_S \int_T f(s,t) \, d\beta \, d\alpha.$$

If G is an $\aleph_a$-compact subset of X and $H \subset \bar{X}$ is such that for $x \in G$

$$k_1 \| x \| \leqq \sup_{\bar{x} \in H} | \bar{x}(x) | \leqq k_2 \| x \| \qquad \text{where } k_1, k_2 > 0$$

then $V(x) = [\bar{x}(x) \mid \bar{x} \in H]$ maps G isomorphically into a subset $\mathfrak{G}$ of M_H. Clearly $\mathfrak{G}$ is likewise $\aleph_a$-compact (see Lemma 2.1). Further $U(\bar{x}) = [\bar{x}(x) \mid x \in G]$ maps H into a subset $\mathfrak{H}$ of M_G since G is bounded. The following theorem states that $\mathfrak{G} = [\,.\,(x) \mid x \in \mathfrak{G}]$ is $\aleph_a$-compact if and only if $\mathfrak{H} = [\bar{x}(\,.\,) \mid \bar{x} \in H]$ is $\aleph_a$-compact. In this sense the sets $\mathfrak{G}$ and $\mathfrak{H}$ are the duals of one another. To make this duality more striking, let us consider the real valued bounded function $f(s,t)$ on $S \times T$. Then $\mathfrak{G} = [f(\,.\,, t) \mid t \in T]$ is a subset of M_S and $\mathfrak{H} = [f(s, .) \mid s \in S]$ is a subset of M_T. Using these definitions of $\mathfrak{G}$ and $\mathfrak{H}$ we have

3.9 THEOREM. *$\mathfrak{G}$ is $\aleph_a$-compact if and only if $\mathfrak{H}$ is $\aleph_a$-compact.*

By symmetry we need only show that if $\mathfrak{G}$ is $\aleph_a$-compact, then $\mathfrak{H}$ is $\aleph_a$-compact. Now $W(a) = \sum_T a_t f(\,.\,, t)$ where $\sum_T | a_t | < \infty$ is a linear transformation on l_T to M_S since $\mathfrak{G}$ is bounded. As $\mathfrak{G}$ is $\aleph_a$-compact, Theorem 3.6 implies that $W((l_T)_1)$ is $\aleph_a$-compact. Hence W is $\aleph_a$-compact. By Theorem 2.2, $\bar{W}$ on $\bar{M}_S$ to M_T is $\aleph_a$-compact. Since $\bar{x}_s(x) = x(s)$ on M_S to real numbers is an element of $(\bar{M}_S)_1$, it follows that $\mathfrak{H} \equiv [\bar{W}(\bar{x}_s) = f(s, .) \mid s \epsilon S]$ is $\aleph_a$-compact which was to be proved.

3.10 Corollary. *A necessary and sufficient condition that G be $\aleph_a$-compact is that both* 1). *G be bounded, and* 2). *$U(\bar{x})$ on a determining manifold*[3] *Γ in $\bar{X}$ to M_G be $\aleph_a$-compact.*

In the above notation let H be the unit sphere in Γ. Then $\mathfrak{H} = U(H)$ is by Theorem 3.9 $\aleph_a$-compact. U is therefore $\aleph_a$-compact on Γ to M_G. Conversely if U is $\aleph_a$-compact, then $\mathfrak{H}$ is $\aleph_a$-compact. By Theorem 3.9, $\mathfrak{G}$ is then $\aleph_a$-compact and consequently $G = V^{-1}(\mathfrak{G})$ is $\aleph_a$-compact.[4]

We conclude this section with the following theorem:

3.11 Theorem. *If G is $\aleph_a$-compact and there exists a subset of $\bar{X}$ of power $\leq \aleph_a$ total on G_{LC}, then G_{LC} is $\aleph_a$-separable.*

Let $T(a) = \sum_G a_x x$ where $\sum_G |a_x| < \infty$. T is linear limited on l_G to X and by Corollary 3.10 T is $\aleph_a$-compact. Hence by Theorem 2.2, $\bar{T}$ is $\aleph_a$-compact. As $G_{LC} = (T(l_G))_C$, the conclusion follows immediately from Corollary 2.7.

4. $\aleph$-compact sets.

In this section we shall consider certain restrictions on the geometry of the space under which $\aleph_a$-compactness implies $\aleph$-compactness. Lemma 2.5 is an instance of this type of theorem. The principal problem here is still open. This is the problem of determining whether or not $\aleph_0$ compactness always implies $\aleph$-compactness. A subproblem to this is that of determining whether or not every space whose unit sphere is $\aleph_0$ compact is necessarily reflexive.

4.1 *Definition.* A *face* is a convex set of constant norm.

Milman (17) has introduced considerations of the type employed in the following theorem. In view of Theorem 3.11, this theorem is an extension of Lemma 2.5.

4.2 Theorem. *If G is $\aleph_a$-compact and if there exist strongly dense subsets in every face of power $\leq \aleph_a$, then G is $\aleph$-compact.*

By Corollary 3.7, $(cv(G))_C$ is $\aleph_a$-compact. Theorem 1.10 reduces the problem to that of proving a closed convex set G which is $\aleph_a$-compact to be $\aleph$-closed for all $\aleph$. Suppose $[x_\mu \mid \mu \in \text{ordinals} < \Omega] \subset G$. Define $F_\mu = cv[x_\lambda \mid \lambda \geq \mu]$ and $A_\mu = (F_\mu)_C$. We first show that the A_μ have a common point. Since $A_\mu \subset G$ is $\aleph_0$-compact, convex, and weakly closed, it contains an element y_μ such that $d_\mu = \| y_\mu \| = \text{g.l.b.}\ [\| x' \| \mid x' \in A_\mu]$. Now if $\mu_1 > \mu_2$ then $A_{\mu_1} \subset A_{\mu_2}$, and consequently $d_{\mu_1} > d_{\mu_2}$. Thus $\lim_{\mu} d_\mu = d$ exists. We have the two following cases: 1). There exists no μ_0 such that $d_\mu = d$ for all $\mu \geq \mu_0$. In this case we can obtain a spanning sequence μ_n such that $| d_{\mu_n} - d | < 1/n$. Then a subsequence of $\{y_{\mu_n}\}$ converges weakly to an x_0, which is easily shown to lie in $\bigcap A_\mu$. 2). There exists a μ_0 such that $d_\mu = d$ for all $\mu \geq \mu_0$. The set $[y_\mu \mid \mu \geq \mu_0]$ therefore lies in a face. By hypothesis there exists a strongly dense subset in this face of power $\leq \aleph_a$. Let X_0 be the linear closed extension of this face. By a familiar argument, there exists a set of power $\leq \aleph_a$ total on X_0. Therefore $G_0 = G \bigcap X_0$ is by Lemma 2.5 $\aleph$-compact. Hence there exists an x_0 such that

---

[3] A determining manifold Γ in $\bar{X}$ is a closed linear subset of $\bar{X}$ such that l. u. b. $[|\bar{x}(x)| \mid \bar{x} \epsilon \Gamma, \|\bar{x}\| \leq 1] = \|x\|$ for all $x \epsilon X$.

[4] In this case V is an isometric transformation on X to a closed linear subset of M_H on which V^{-1} is defined.

$$\underline{\lim_{\mu}}\, \bar{x}(y_\mu) \leq \bar{x}(x_0) \leq \overline{\lim_{\mu}}\, \bar{x}(y_\mu)$$

for all $\bar{x} \in \bar{X}$. If x_0 were not an element of A_λ, then we could find an $\bar{x} \in \bar{X}$ and a constant c such that $\bar{x}(A_\lambda) \leq c$ and $\bar{x}(x_0) > c$ (8, p. 331). Thus $\bar{x}(y_\mu) \leq c$ for all $\mu \geq \lambda$, contrary to the above inequalities. In both cases we can therefore find an $x_0 \in \bigcap A_\mu$. For a given ε and μ_0 there exist $\mu_1, \mu_2, \cdots, \mu_n \geq \mu_0$ such that $\| \Sigma\, c_i x_{\mu_i} - x_0 \| < \varepsilon$ where $c_i \geq 0$ and $\Sigma\, c_i = 1$. For one of these μ_i's, $\bar{x}(x_\mu) \leq \bar{x}(x_0) + \varepsilon$ while for another $\bar{x}(x_\mu) \geq \bar{x}(x_0) - \varepsilon$. Hence

$$\underline{\lim_{\mu}}\, \bar{x}(x_\mu) \leq \bar{x}(x_0) \leq \overline{\lim_{\mu}}\, \bar{x}(x_\mu)$$

for all $\bar{x} \in \bar{X}$.

A space is said to be *strictly convex* if every face contains but a single element. We obtain as a corollary

4.3 COROLLARY. *If G is $\aleph_0$-compact and X is strictly convex, then G is $\aleph$-compact.*

Clarkson (7) has shown that every separable space is isomorphic to a strictly convex space.[5] By employing the same method, one can easily prove that any space on which there exists a total set of power $\leq \aleph_0$ is isomorphic to a strictly convex space. This, however, gives no more than Lemma 2.5.

Let $\alpha(\tau)$ be a non-negative measure function on the sigma field $\mathfrak{T}$ of subsets of T. We assume that $T = \Sigma \tau_i$ where $\alpha(\tau_i) < \infty$. Define $\mathfrak{M}(\mathfrak{T})$ to be the class of all essentially bounded measurable functions on T. $\mathfrak{M}(\mathfrak{T})$ is a

---

[5] The author has shown that the space M_T (where the power of $T > \aleph_0$) is not isomorphic to a strictly convex space.—Added September 1942.

Banach space under the norm $\|\phi\| = \text{ess. l.u.b.}\,[|\phi(t)| \mid t \in T]$. Let us define

$$\|\phi\|_1 = \left[\frac{\|\phi\|^2}{2} + \sum_{i=1}^{\infty} \frac{\int_{\tau_i} |\phi(t)|^2\, d\alpha}{2^{i+1}\cdot\alpha(\tau_i)}\right]^{1/2}$$

Then $\|\phi\|/2^{1/2} \leq \|\phi\|_1 \leq \|\phi\|$. Now

$$\frac{\|\phi+\psi\|^2}{2} + \sum \frac{\int_{\tau_i} |\phi+\psi|^2\, d\alpha}{2^{i+1}\cdot\alpha(\tau_i)} \leq \frac{(\|\phi\|+\|\psi\|)^2}{2} + \sum \frac{\int_{\tau_i} |\phi+\psi|^2\, d\alpha}{2^{i+1}\cdot\alpha(\tau_i)}$$

$$= 2\,(\|\phi\|_1^2 + \|\psi\|_1^2) - \left\{ \frac{(\|\phi\|-\|\psi\|)^2}{2} + \sum \frac{\int_{\tau_i} |\phi-\psi|^2\, d\alpha}{2^{i+1}\cdot\alpha(\tau_i)} \right\}.$$

Thus if $\|\phi+\psi\|_1 = 2\|\phi\|_1 = 2\|\psi\|_1$, then $\int_{\tau_i} |\phi-\psi|^2\, d\alpha = 0$ for all i. That is $\phi(t) = \psi(t)$ a. e. The renormed $\mathfrak{M}(\mathfrak{T})$ is therefore strictly convex and isomorphic to the original $\mathfrak{M}(\mathfrak{T})$. Making use of Corollary 4. 3 we have

4. 4 THEOREM. *If G is an $\aleph_0$-compact subset of $\mathfrak{M}(\mathfrak{T})$, then G is $\aleph$-compact.*

4. 5 THEOREM. *If G is an $\aleph_0$-compact subset of $\mathfrak{L}(\mathfrak{T})$, then G is $\aleph$-compact.*[6]

Dunford and Pettis (8, p. 376) have shown that $G \subset \mathfrak{L}(\mathfrak{T})$ is $\aleph_0$-compact if and only if

1). G is bounded,

2). given $\varepsilon > 0$ there is a $\delta > 0$ such that $\alpha(\tau) < \delta$ implies $\left|\int_{\tau} \phi\, d\alpha\right| < \varepsilon$ for every $\phi \in G$,

3). for each decomposition $\{\tau_i\}$ of T $\lim_{m\to\infty} \sum_{m}^{\infty} \left|\int_{\tau_i} \phi\, d\alpha\right| = 0$ uniformly over G.

We shall now show that these conditions are sufficient to imply that G is $\aleph$-compact. Suppose $[\phi_\pi \mid \pi \in \text{directed set}] \subset G$. Let $a_{\tau\pi} = \int_{\tau} \phi_\pi\, d\alpha$. Since G is bounded, we can apply Lemma 1. 8 and obtain a directed set $[\sigma]$ for which $\lim_{\sigma} a_{\tau\pi_\sigma} = a_\tau$ exists for all $\tau \in \mathfrak{T}$. a_τ is clearly finitely additive on $\mathfrak{T}$

---

[6] $\mathfrak{L}(\mathfrak{T})$ is defined to be the class of all summable functions on T where $\|\phi\| = \int_{T} |\phi(t)|\, d\alpha$ and $T = \Sigma\tau_i$ $[\alpha(\tau_i) < \infty]$.

and by condition 2) absolutely continuous with respect to α. If $\{\tau_n\}$ is a monotonic ascending sequence of sets $\tau_n \in \mathfrak{T}$ and $\lim \tau_n = \tau$, then by condition 3) $\lim_n a_{\tau_n \pi} = a_{\tau\pi}$ uniformly in π. Hence by the familiar iterated limits theorem $\lim_n a_{\tau_n} = a_\tau$. a_τ is therefore completely additive and by the Radon-Nikodym theorem the indefinite integral of an element ϕ of $\mathfrak{L}(\mathfrak{T})$. If ψ is a bounded step function on $\mathfrak{T}$, then $\lim_\sigma \int_T \psi\phi_{\pi_\sigma}\, d\alpha = \int_T \psi\phi\, d\alpha$. Since the bounded step functions are strongly dense in $\mathfrak{M}(\mathfrak{T}) \equiv \overline{\mathfrak{L}(\mathfrak{T})}$, it follows that $\lim_\sigma \int_T \psi\phi_{\pi_\sigma}\, d\alpha = \int_T \psi\phi\, d\alpha$ for all $\psi \in \mathfrak{M}(\mathfrak{T})$. Finally, for every π_0, there exists a σ_0 such that $\pi_\sigma \geqq \pi_0$ for all $\sigma \geqq \sigma_0$. Consequently

$$\underline{\lim}_\pi \int_T \psi\phi_\pi\, d\alpha \leqq \int_T \psi\phi\, d\alpha \leqq \overline{\lim}_\pi \int_T \psi\phi_\pi\, d\alpha$$

for all $\psi \in \mathfrak{M}(\mathfrak{T})$, which was to be proved.

For purposes of the next theorem, we define c_0 to be the class of all real valued functions $x(t)$ on an abstract class T which converge to zero for any sequence $\{t_i\} \subset T$. With $\|x\| = \text{l. u. b.}\ [|x(t)| \mid t \in T]$, c_0 is a Banach space.

4. 6 Theorem. *If G is an $\aleph_0$-compact subset of c_0, then G is $\aleph$-compact.*

We first show that if G is $\aleph_0$-compact then

1). G is bounded,

2). for all sequences $\{t_i\}$ of distinct elements of T $\lim_n \{\text{g. l. b.}\ [|x(t_i)| \mid i = 1, 2, \cdots, n]\} = 0$ uniformly over G.

1) follows from Corollary 1. 7. Let us suppose that G $\aleph_0$-compact does not imply 2). Then there exists an $\varepsilon > 0$ and a sequence $\{t_i\}$ of distinct elements such that for every n we can find an $x_n \in G$ for which $|x_n(t_i)| \geqq \varepsilon$, $i = 1, 2, \cdots, n$. As G is $\aleph_0$-compact, a subsequence $\{x_{n_i}\}$ converges weakly to some $x_0 \in c_0$. Since $x_{n_i}(t) \to x_0(t)$ for all $t \in T$, it follows that $|x_0(t_i)| \geqq \varepsilon$ for all t_i. This is impossible for $x_0 \in c_0$.

We next show that conditions 1) and 2) are sufficient to make G $\aleph$-compact. Suppose $[x_\pi \mid \pi \in \text{directed set}] \subset G$. By 1), the numbers $x_\pi(t)$ are bounded. Applying Lemma 1. 8, we obtain a directed set $[\sigma]$ such that $\lim_\sigma x_{\pi_\sigma}(t) = a(t)$ exists for all $t \in T$. If $a(t)$ did not belong to c_0, there would exist an $\varepsilon > 0$ and a sequence $\{t_i\}$ of distinct points such that $|a(t_i)| \geqq \varepsilon$ for all i. By 2), there exists an N such that $\text{g. l. b.}\ [|x(t_i)| \mid i = 1, 2, \cdots, N] < \varepsilon/2$, for all $x \in G$. This is contrary to the fact that we can find a σ_0 such that $\sigma \geqq \sigma_0$

implies $| x_{\pi_\sigma}(t_i) - a(t_i) | < \varepsilon/2$ for $i = 1, 2, \cdots, N$. Hence $a(t) \in c_0$. Since $\bar{c}_0 \equiv l_T$, it is easy to show that $\lim_{\sigma} \bar{x}({}_{\pi_\sigma}) = \bar{x}(a)$ for all $\bar{x} \in l_T$. Finally, for every π_0, there exists a σ_0 such that $\pi_\sigma \geq \pi_0$ for all $\sigma \geq \sigma_0$. Consequently

$$\varliminf_{\pi} \bar{x}(x_\pi) \leq \bar{x}(a) \leq \varlimsup_{\pi} \bar{x}(x_\pi)$$

for all $\bar{x} \in l_T$.

4.7 THEOREM. *If G is an $\aleph_0$-compact subset of l_T, then G is $\aleph$-compact.*

To prove this, we show that l_T is isomorphic to a strictly convex space. The result then follows by Corollary 4.3. Define

$$|| a ||_1 = [(\sum_T | a_t |)^2 + \sum_T | a_t |^2]^{1/2}.$$

Then $|| a || \leq || a ||_1 \leq 2^{1/2} \cdot || a ||$. The remainder of the proof is similar to that of Theorem 4.4.

The corresponding theorem for $M_T \equiv l_T$ is, to our knowledge, unproved. Such a theorem would settle the general problem since any Banach space is equivalent to a subspace of an M_T space. We shall conclude this section with a characterization of $\aleph_0$-compact subsets of M_T in the hope that this characterization may be found useful in attacking the general problem.

4.8 THEOREM. *$G \subset M_T$ is $\aleph_0$-compact if and only if G is bounded and for all sequences $\{t_i\}$ there exists a subsequence $\{t_{i_j}\}$ and an $\bar{x} \in \bar{M}_T$ such that for no $\varepsilon > 0$ and no subsequence $\{t_{i_{j_k}}\}$ is*

$$\bigcap_{k=1}^{n} [x \mid | x(t_{i_{j_k}}) - \bar{x}(x) | > \varepsilon] \neq 0$$

for all n.

We can apply Theorem 3.9 to $f(t, x) \equiv x(t)$ on $T \times G$. The following statements are equivalent: 1). $G \equiv \mathfrak{G} \equiv [f(\,.\,, x) \mid x \in G]$ is $\aleph_0$-closed; 2). $\mathfrak{H} \equiv [f(t,\,.\,) \mid t \in T] \subset M_G$ is $\aleph_0$-compact; 3). Given any sequence $\{t_i\}$ there exists a subsequence $\{t_{i_j}\}$ and an element $y(x)$ of M_G such that $\lim_j \varliminf_n | x_n(t_{i_j}) - y(x_n) | = 0$ for all sequences $\{x_n\} \subset G$. The last equivalence follows from a theorem due to Banach (1, p. 219). Now $y(x)$ can be extended over M_T to be the functional $\bar{x}(x) = \operatorname{Lim}_j x(t_{i_j})$ where Lim stands for the Banach generalized limit function (1, p. 34). 3) can be restated as: given any sequence $\{t_i\}$ there exists a subsequence $\{t_{i_j}\}$ and an $\bar{x} \in \bar{M}_T$ such that for no $\varepsilon > 0$ and no subsequence $\{t_{i_{j_k}}\}$ of $\{t_{i_j}\}$ is $\varliminf_n | x_n(t_{i_{j_k}}) - \bar{x}(x_n) | > \varepsilon$ for some sequence $\{x_n\} \subset G$. The required condition clearly implies this. Suppose the converse were not true. Then there would exist an $\varepsilon > 0$ and a subsequence $\{t_{i_{j_k}}\}$ of $\{t_{i_j}\}$ such that $\bigcap_{k=1}^{n} [x \mid |x(t_{i_{j_k}}) - \bar{x}(x) | > \varepsilon] \neq 0$ for all n. Choose

x_n out of $\bigcap_{k=1}^{n}[x \mid |x(t_{i_{j_k}}) - \bar{x}(x)| > \varepsilon]$. Then $\underline{\lim}_n |x_n(t_{i_{j_k}}) - \bar{x}(x_n)| \geq \varepsilon$ in contradiction to our assumption.

5. Integral representations of absolutely continuous set functions. The principal theorems of this section deal with an extension of the Radon-Nikodym theorem from real valued set functions to certain abstract valued functions. Gelfand (11), Clarkson (7), Pettis (19), and Dunford and Pettis (8) have obtained many results on this problem. By assuming that the set function lies in an "essentially" $\aleph_0$-compact subset, we have been able to extend these results to sigma fields which are not separable. As an application, we obtain the general form of the weakly completely continuous transformation on $\mathfrak{L}(\mathfrak{T})$ to an arbitrary Banach space and prove that it is separable valued.

We shall consider an abstract class T of elements t possessing a sigma field $\mathfrak{T}$ of subsets τ of T. $\alpha(\tau)$ will be a single-valued, non-negative, completely additive measure function of $\mathfrak{T}$. We shall assume that there exists a denumerable decomposition $\{\tau_i\}$ of T such that $\alpha(\tau_i) < \infty$. It will be convenient to designate by $|\tau|$ the value $\alpha(\tau)$. We shall define the class of additive X valued set functions

$$V^\infty(X) \equiv [x(\tau) \mid \|x(\tau)\| \leq M \cdot |\tau|, \tau \epsilon \mathfrak{T}, |\tau| < \infty].$$

Finally we shall define $B^\infty(X)$ to be the class of weakly measurable point functions $y(t)$ on T to X whose values are contained in a *separable* $\aleph$-compact subspace of X. With the norm $\|x(.)\| = \text{l. u. b.}\, [(\|x(\tau)\|/|\tau|) \mid \tau \epsilon \mathfrak{T}, 0 < |\tau| < \infty]$ for $V^\infty(X)$ and $\|y(.)\| = \text{ess. l. u. b.}\, [\|y(t)\| \mid t \epsilon T]$ for $B^\infty(X)$, both of these classes become Banach spaces.

Integration with respect to α will be realized by means of the Bochner integral (2). Since $y(t) \epsilon B^\infty(X)$ is contained in a separable subspace of X, $y(t)$ is measurable (20, theorem 1.1) and hence integrable on all sets $\tau \epsilon \mathfrak{T}$ of finite measure.

5.1 THEOREM. *If $x(.) \epsilon V^\infty(X)$ and $S \equiv [(x(\tau)/|\tau| \mid 0 < |\tau| < \infty]$ is $\aleph_0$-compact, then there exists a $y(.) \epsilon B^\infty(X)$ such that $x(\tau) = \int_\tau y(t)\, d\alpha$ for all τ of finite measure.*

If R is the space of real numbers, then by the Radon-Nikodym theorem there exists for each $f(.) \epsilon V^\infty(R)$ a $g(.) \epsilon B^\infty(R)$ such that $f(\tau) = \int_\tau g(t)\, d\alpha$ for all τ of finite measures. $V(f(.)) = g(.)$ defines an equivalence between $V^\infty(R)$ and $B^\infty(R)$. Hence by Theorem 4.4, $\aleph_0$ compact subsets of $V^\infty(R)$ are $\aleph$-compact. Define $W(\bar{x}) = [(\bar{x}(x(\tau))/|\tau|) \mid \tau \epsilon \mathfrak{T}, 0 < |\tau| < \infty]$ on $\bar{X}$ to $V^\infty(R)$. Since $S \equiv [(x(\tau)/|\tau|) \mid \tau \epsilon \mathfrak{T}, 0 < |\tau|$

$< \infty]$ is $\aleph_0$-compact, it follows by Corollary 3. 10 that W is $\aleph_0$-compact. As $W(\bar{X}_1)$ is an $\aleph_0$-compact subset of $V^\infty(R)$, it must be $\aleph$-compact. Hence by this same corollary S is $\aleph$-compact.

Let π be a finite number of disjoint sets τ of $\mathfrak{T}$ for which $0 < |\tau| < \infty$. $\pi_1 \geqq \pi_2$ will mean that $\Sigma\tau^1 \supset \Sigma\tau^2$ and that every set $\tau^1 \epsilon \pi_1$, is either a subset of some $\tau^2 \epsilon \pi_2$ or that τ^1 is disjoint from every $\tau^2 \epsilon \pi_2$. For each π, we define $y_\pi(t) = x(\tau_i)/|\tau_i|$ for $t \epsilon \tau_i$ and $y_\pi(t) = 0$ elsewhere. Then if $V(\bar{x}(x(.))) = g_{\bar{x}}(t)$, $\lim_\pi |g_{\bar{x}}(t) - \bar{x}(y_\pi(t))| = 0$ for *all* $t \epsilon T$ and uniformly for any set of finite measure. As $[y_\pi(t)] \subset S$ is $\aleph$-compact, there exists a function $y(t)$ such that $\bar{x}(y(t)) = \lim_\pi \bar{x}(y_\pi(t)) = g_{\bar{x}}(t)$ for all $\bar{x} \epsilon \bar{X}$. $y(t)$ is clearly weakly measurable. By Lemma 1. 3, $y(t)$ belongs to $(cv(S))_C$ which is $\aleph$-compact by Theorem 3. 6. Further, it is clear that $\bar{x}(x(\tau)) = \int_\tau g_{\bar{x}}(t)\, d\alpha = \int_\tau \bar{x}(y(t))\, d\alpha$ for all τ of finite measure.

We shall now show that $y(t)$ is separable valued. Actually we shall prove a stronger result, namely if $|\tau_0|\tau < \infty$, then $[x(\tau) \mid \tau \subset \tau_0]$ is compact valued. This is equivalent to the statement that $[\bar{x}(x(\tau)) \mid \tau \subset \tau_0, \bar{x} \epsilon \bar{X}_1]$ is a compact subset of $M_{(\tau \subset \tau_0)}$ (21, theorem 3. 1). As $(cv(S))_C$ is $\aleph$-compact, it follows from Corollary 3. 10 that given any sequence $\{\bar{x}_n\} \subset \bar{X}_1$ there exists a subsequence $\{\bar{x}_{n'}\}$ and an $\bar{x}_0 \epsilon \bar{X}$ such that $\lim_n \bar{x}_{n'}(x) = \bar{x}_0(x)$ for all $x \epsilon (cv(S))_C$. $y(t)$ is bounded. Consequently $\bar{x}_{n'}(x(\tau)) = \int_\tau \bar{x}_{n'}(y(t))\, d\alpha$ converges to $\bar{x}_0(x(\tau))$ uniformly for all $\tau \subset \tau_0$. Now $T = \Sigma\tau_i$ where $|\tau_i| < \infty$. Consequently $[x(\tau) \mid \tau \epsilon \mathfrak{T}, |\tau| < \infty]$ is separable valued. But S and hence $(cv(S))_C$ is contained in the closed linear extension of this set which is clearly a separable subspace of X. Therefore $y(t)$ is separable valued. Thus $y(.) \epsilon B^\infty(X)$. That $x(\tau) = \int_\tau y(t)\, d\alpha$ follows from $\bar{x}(x(\tau)) = \int_\tau \bar{x}(y(t))\, d\alpha$ for all $\bar{x} \epsilon \bar{X}$.

5. 2 THEOREM. *The general form of the weakly completely continuous transformation U on $\mathfrak{L}(\mathfrak{T})$ to X is*

$$U(\phi) = \int \phi(t) y(t)\, d\alpha \tag{1}$$

where $y(.) \epsilon B^\infty(X)$ and $\|y(.)\| = \|U\|$. U is separable valued and $\aleph$-compact for all $\aleph$.

Let χ_τ be the characteristic function of the set $\tau \epsilon \mathfrak{T}$. Define $x(\tau) = U(\chi_\tau)$ for all $\tau \epsilon \mathfrak{T}$ such that $|\tau| < \infty$. It is clear that $x(.)$ belongs to $V^\infty(X)$ and that $[(x(\tau)/|\tau|) \mid \tau \epsilon \mathfrak{T}, 0 < |\tau| < \infty]$ is $\aleph_0$-compact. Applying Theorem

5.1, we obtain $y(.) \epsilon B^{\infty}(X)$ where $x(\tau) = \int_{\tau} y(t) d\alpha$ for all τ of finite measure. Now (1) is satisfied by all step functions in $\mathfrak{L}(\mathfrak{T})$. Since

$$2) \qquad || \int \phi(t) y(t) d\alpha || \leq || \phi || \cdot || y(.) ||$$

and since all functions in $\mathfrak{L}(\mathfrak{T})$ can be approximated by step functions, (1) is valid in the general case. By (2) we have $|| U || \leq || y(.) ||$. As $(|| x(\tau) ||/| \tau | \leq || U(\chi_{\tau}/| \tau |) || \leq || U ||$, it follows from the way in which $y(t)$ was found that $|| y(t) || \leq$ l. u. b. $[(|| x(\tau) ||/| \tau |) | 0 < \tau < \infty] \leq || U ||$. Hence $|| U || = || y(.) ||$. Further $\int \phi(t) y(t) d\alpha \, \epsilon [y(t)]_{LC}$. Therefore U is separable valued. Conversely, it is easily seen that (1) defines a linear limited transformation. Suppose $|| \phi || \leq 1$. Then $\int \phi(t) y(t) d\alpha$ is approximated by elements of the type $\Sigma \phi(t_i) | \tau_i | y(t_i)$ where $t_i \epsilon \tau_i$ and $\{\tau_i\}$ is a decomposition of T. But these are contained in the set $[\sum_T a_t y(t) \mid \sum_T | a_t | \leq 1]$ which by Theorem 3.6 is $\aleph$-compact. Therefore $U((\mathfrak{L})_1)$ lies in the closure of this set which by Theorem 3.4 is also $\aleph$-compact. This concludes the proof.

An alternative proof for Theorems 5.1 and 5.2 will now be presented. This argument is not as direct, depending upon a long chain of results. Suppose $T = \Sigma \tau_i$ where τ_i are disjoint sets of $\mathfrak{T}$ of finite measure. Set

$$C_n \equiv [\phi(t) \mid | \phi(t) | \leq n \text{ for } t \,\epsilon \sum_{i=1}^{n} \tau_i,\ \phi(t) = 0 \text{ elsewhere}].$$

As shown in Theorem 4.5, C_n is $\aleph_0$-compact. Suppose that U is $\aleph_0$-compact on $\mathfrak{L}(\mathfrak{T})$ to X. It can be shown that $U(C_n)$ is conditionally compact (21, theorem 5.5) and hence contained in a separable closed linear subspace X_n of X. Now ΣC_n is dense in $\mathfrak{L}(\mathfrak{T})$. Therefore $U(\mathfrak{L}) \subset (\Sigma X_n)_C$ which is a separable subspace of X. U is therefore separable valued. The general form of the weakly completely continuous *separable* transformation has been shown to be that given by Theorem 5.2 (21, theorem 5.4). From this one can easily obtain Theorem 5.1.

We remark that Theorem 5.2 can not be extended to the general space of Lebesgue integrable functions where T is not the sum of a denumerable number of sets of finite measure. For suppose $\mathfrak{L} = l_T$ where $\bar{\bar{T}} \geq \aleph_1$. Let $X \equiv l_T{}^2 \equiv [x_t \mid \sum_T x_t{}^2 < \infty]$. Define $U(a) = x$ where $a_t = x_t$ for all $t \,\epsilon\, T$. Now

$$|| U(a) || = [\Sigma a_t{}^2]^{1/2} \leq [(\Sigma | a_t |)^2]^{1/2} = || a ||.$$

U is therefore bounded and certainly not separable valued. Finally since $l_T{}^2$ is reflexive, U is $\aleph$-compact.

5.3 LEMMA. *If* $x(\tau)$ *is a completely additive* α-*absolutely continuous set function on* $\mathfrak{T}$ *to* X *where* $\alpha(T) < \infty$, *then for each* n *there exists a decomposition* $(\tau_0, \tau_1, \tau_2, \cdots)$ *of* T *into disjoint sets of* $\mathfrak{T}$ *such that*

$$||\, x(\tau) \,|| \leq n \,|\, \tau \,| \qquad \textit{for all } \tau \subset \tau_0$$

and

$$||\, x(\tau) \,|| \geq n \,|\, \tau \,| \qquad \textit{for all } \tau \subset \tau_i \quad i = 1, 2, \cdots.$$

Given $\bar{x} \in \bar{X}$, it follows from a theorem due to Hahn (22, p. 32) that there exists a set $\tau_{\bar{x}} \in \mathfrak{T}$ such that $\bar{x}(x(\tau)) \geq n \,|\tau| \cdot ||\, \bar{x} \,||$ for all $\tau \subset \tau_{\bar{x}}$ and $\bar{x}(x(\tau)) \leq n \cdot |\, \tau \,| \cdot ||\, \bar{x} \,||$ for all $\tau \subset T - \tau_{\bar{x}}$. Let $O \equiv [\tau \mid \tau \subset \tau_{\bar{x}}$ for some $\bar{x}, |\, \tau \,| > 0]$. We define a transfinite sequence $(\tau_1, \tau_2, \cdots; \tau_\omega, \cdots, \tau_\lambda, \cdots)$ as follows: τ_1 is an arbitrary element of O. Suppose τ_λ to have been defined for all $\lambda < \mu$. If there exists τ such that $\tau \cdot \sum_{\lambda < \mu} \tau_\lambda = 0$ and $\tau \in O$, then set $\tau \equiv \tau_\mu$. As α is bounded, $\alpha(\tau)$ cannot differ from zero on a nondenumerable number of disjoint sets. The sequence therefore contains only a denumerable set of elements. Let us put this set into one-to-one correspondence with the natural numbers. Define $\tau_0 = T - \sum_{i=1}^{\infty} \tau_i$. Then if $\tau \subset \tau_0$, it follows that $|\, \tau \cdot \tau_{\bar{x}} \,| = 0$ for all $\bar{x} \in \bar{X}$. Hence $\bar{x}(x(\tau)) \leq n \,|\, \tau \,|\, ||\, \bar{x} \,||$ for all $\bar{x} \in \bar{X}$, so that $||\, x(\tau) \,|| \leq n \,|\, \tau \,|$. On the other hand τ_i is a subset of some $\tau_{\bar{x}_i}$. Thus if $\tau \subset \tau_i$ then $\bar{x}_i(x(\tau)) \geq n \,|\, \tau \,|\, ||\, \bar{x}_i \,||$ and therefore $||\, x(\tau) \,|| \geq n \,|\, \tau \,|$.

Lemma 5.3 can obviously be extended to the case of $T = \Sigma \tau_i$ where τ_i is of finite measure. In the following work we will consider such T.

We define $x(\tau)$ to be of *p-bounded variation* for $p \geq 1$ if for all finite sets $\pi = (\tau_1, \tau_2, \cdots, \tau_n)$ of disjoint sets of finite measure different from zero

$$V_p(x(.)) = \underset{\pi}{\text{l. u. b.}} [\sum_{\pi} (||\, x(\tau_i) \,||^p / |\, \tau_i \,|^{p-1})] < \infty.$$

5.4 LEMMA. *If* $x(\tau)$ *is a completely additive* α *absolutely continuous set function on* $\mathfrak{T}$ *to* X *and of p-bounded variation* $(p \geq 1)$, *then there exist sets* $\{\tau_n\}$ *such that* $\tau_n \supset \tau_{n+1}$, $\lim_n |\, \tau_n \,| = 0$, *and* $||\, x(\tau) \,|| \leq n \,|\, \tau \,|$ *for all* $\tau \subset T - \tau_n$.

Applying Lemma 5.3 to n, we obtain the decomposition $(\sigma^0, \sigma^1, \sigma^2, \cdots)$ of T into disjoint sets such that $||\, x(\tau) \,|| \leq n \,|\, \tau \,|$ for all $\tau \subset \sigma^0$, and $||\, x(\tau) \,|| \geq n \,|\, \tau \,|$ for all $\tau \subset \sigma^i$ $(i = 1, 2, \cdots)$. As $x(\tau)$ is of p-bounded variation, there exists an M such that

$$M \geq \sum_{i=1}^{k} (||\, x(\sigma^i) \,||^p / |\, \sigma^i \,|^{p-1}) \geq n^p \cdot |\, \sum_{i=1}^{k} \sigma^i \,|.$$

Consequently if we define $\sigma_n = \sum_{i=1}^{\infty} \sigma^i$, then $\alpha(\sigma_n) \leq M/n^p$. Let $\sigma =$

$\sigma_n \cdot (T - \sigma_{n-1})$. Then $(n-1) \cdot |\sigma \cdot \sigma^i| \geq \|x(\sigma \cdot \sigma^i)\| \geq n \cdot |\sigma \cdot \sigma^i|$. Thus $|\sigma \cdot \sigma^i| = 0$ and hence $|\sigma| = 0$. Define $\tau_n = \sum_n^\infty \sigma_i$. Then $|\tau_n - \sigma_n| = 0$. The $\{\tau_n\}$ clearly satisfy the conditions of the theorem.

5.5 THEOREM. *If $x(\tau)$ is a completely additive α-absolutely continuous set function on T to X of p-bounded variation such that*

$$[(x(\tau)/|\tau|) \mid (\|x(\tau)\|)/|\tau| \leq n]$$

is $\aleph_0$-compact for all n, then there exists a measurable function $y(t)$ such that

$$x(\tau) = \int_\tau y(t)\, d\alpha \begin{cases} \text{for all } \tau \text{ if } p = 1 \\ \text{for all } \tau \text{ of finite measure if } p > 1 \end{cases}$$

and

$$V_p(x(\,.\,)) = \underset{\pi}{\text{l. u. b.}} [\sum_\pi (\|x(\tau_i)\|^p)/|\tau_i|^{p-1}] = \int_T \|y(t)\|^p\, d\alpha.$$

We can apply Lemma 5.4 and obtain sets τ_n such that $\tau_n \supset \tau_{n+1}$, $\lim_n |\tau_n| = 0$, and $\|x(\tau)\| \leq n\,|\tau|$ for all $\tau \subset T - \tau_n$. By Theorem 5.1 there exists for each n a $y_n(\,.\,) \subset B^\infty(X)$ such that $y_n(t) = 0$ for all $t \in \tau_n$ and $x_n(\tau) \equiv x(\tau - \tau \cdot \tau_n) = \int_\tau y_n(t)\, d\alpha$ for all τ of finite measure. Since the $y_n(t)$ are separable valued $y_n(t) = y_{n+1}(t)$ a. e. on $T - \tau_n$. We define $y(t) = y_n(t)$ for $t \in \tau_{n-1} - \tau_n$ where $T = \tau_0$. Then as $|\tau_n| \to 0$, $y(t)$ is measurable and $y(t) = y_n(t)$ a. e. on $T - \tau_n$.

Now if $z(t)$ is measurable and $\int_T \|z(t)\|^p d\alpha < \infty$, one shows by step function approximations that $V_p(z(\tau)) = \int_T \|z(t)\|^p d\alpha$ where $z(\tau)$ $\int_\tau z(t)\, d\alpha$. Consequently $\int_T \|y_n(t)\|^p d\alpha = V_p(x_n(\,.\,)) \leq V_p(x(\,.\,))$. Hence $\int_T \|y(t)\|^p d\alpha = V_p(x(\,.\,))$. Since $\|y_n(t) - y(t)\|^p \leq \|y(t)\|^p$, it follows that $\lim_n \int_T \|y_n(t) - y(t)\|^p d\alpha = 0$. Therefore for all τ of finite measure $x(\tau) = \lim_n x_n(\tau) = \lim_n \int_\tau y_n(t)\, d\alpha = \int_\tau y(t)\, d\alpha$. If $p = 1$, this holds for all τ. For given a τ there exists a monotonic ascending sequence of sets τ_n of finite measure such that $\lim_n \tau_n = \tau$. $x(\tau) = \lim_n x(\tau_n) = \lim_n \int_{\tau_n} y(t)\, d\alpha = \int_\tau y(t)\, d\alpha$ when $p = 1$.

5.6 COROLLARY. *If $x(\tau)$ is a completely additive α-absolutely continuous* *set function on* $\mathfrak{T}$ *to a reflexive Banach space X of p-bounded variation, then there exists a measurable function* $y(t)$ *such that*

$$x(\tau) = \int_\tau y(t)d\alpha \begin{cases} \textit{for all } \tau \textit{ if } p = 1 \\ \textit{for all } \tau \textit{ of finite measure if } p > 1 \end{cases}$$

and

$$V_p(x(.)) = \int_T ||\, y(t)\,||^p d\alpha.$$

We shall conclude this section with an application of Corollary 5. 6 to the spaces $L^p(X)$ introduced by Bochner and Taylor (3). For this purpose we do not insist upon the usual decomposition of T into a denumerable sum of sets of finite measure. $L^p(X)$ is defined to be the class of all measurable functions $x(t)$ on T to X for which $||\, x(.)\,|| = (\int_T ||\, x(t)\,||^p dt)^{1/p}$ exists.

Let U be a linear functional on $L^p(X)$ where $p > 1$. Define $\chi(\tau, x) = x$ for all $t \epsilon \tau$ and $\chi(\tau, x) = 0$ elsewhere. If τ is of finite measure then $\chi(\tau, x) \epsilon L^p(X)$ and $U(\chi(\tau, x))$ exists. Following the reasoning of Bochner and Taylor (3, theorem 3. 2), one can show that $U(\chi(\tau, x))$ is a linear functional $\bar{x}(\tau)$ on X, that $\bar{x}(\tau)$ is additive on sets of finite measure, that $||\, U\,||_q = V_q(\bar{x}(\tau))$ $(1/p + 1/q = 1)$, and that $U(x(.)) = \int d\bar{x}(\tau) \cdot x(t)$. Since $V_q(\bar{x}(\tau)) < \infty$, $\bar{x}(\tau)$ is completely additive and α-absolutely continuous on all subsets of a set of finite measure to $\bar{X}$. Further one can easily prove $V_q(\bar{x}(\tau)) < \infty$ implies the existence of a set τ_0 which has a denumerable decomposition into disjoint sets τ_i of finite measure and for which $\tau \cdot \tau_0 = 0$ gives $x(\tau) = 0$.

If we assume X to be reflexive, then $\bar{X}$ is likewise reflexive and we can apply Corollary 5. 6 to the sets $\{\tau_i\}$. On each of these there will exist a function $\bar{x}_i(t)$ such that $\bar{x}(\tau \cdot \tau_i) = \int_\tau \bar{x}_i(t)dt$ and $V_q(\bar{x}(\tau \cdot \tau_i)) = \int_{\tau_i} ||\, \bar{x}_i(t)\,||^q dt$. Define $\bar{x}(t) = \bar{x}_i(t)$ for $t \epsilon \tau_i$ $(i = 1, 2, \cdots)$ and $\bar{x}(t) = 0$ elsewhere. Then $\bar{x}(t)$ is measurable, $\bar{x}(\tau) = \int_\tau \bar{x}(t)d\alpha$ for all sets of finite measure, and $||\, U\,||^q = V_q(\bar{x}(.)) = \int_T ||\, \bar{x}(t)\,||^q d\alpha$. In other words $\overline{L^p(X)} = L^q(\bar{X})$.

5. 7 THEOREM. *If X is reflexive, then* $\overline{L^p(X)} = L^q(\bar{X})$ *and hence* $L^p(X)$ *is reflexive.*

HARVARD UNIVERSITY.

REFERENCES

1. S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

2. S. Bochner, " Integration von Funktionen deren Werte die Elemente eines Vektorraumes sind," *Fundamenta Mathematicae*, vol. 20 (1933), pp. 262-276.

3. S. Bochner and A. E. Taylor, " Linear functionals on certain spaces of abstractly-valued functions," *Annals of Mathematics*, 2, vol. 39 (1938), pp. 913-944.

4. N. Bourbaki, " Sur les espaces de Banach," *Comptes Rendus*, 206 (1938), pp. 1701-1704.

5. D. G. Bourgin, " Some properties of Banach spaces," *American Journal of Mathematics*, vol. 64 (1942), pp. 597-612.

6. Eduard Čech, " On bicompact spaces," *Annals of Mathematics*, vol. 38 (1939), pp. 823-846.

7. J. A. Clarkson, " Uniformly convex spaces," *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 396-414.

8. Nelson Dunford and B. J. Pettis, " Linear operations on summable functions," *Transactions of the American Mathematical Society*, vol. 47 (1940), pp. 323-392.

9. V. Gantmacher and V. Smulian, " Sur les espaces linéaires dont la sphère unitaire est faiblement compacte," *Comptes Rendus U. R. S. S.*, vol. 17 (1937), pp. 91-94.

10. V. Gantmacher, " Über schwache totalstetige Operatoren," *Recueil Mathématique*, vol. 7 (1940), pp. 301-308.

11. I. Gelfand, "Abstrakte Funktionen und lineare Operatoren," *Recueil Mathématique*, vol. 4 (1938), pp. 235-284.

12. H. Goldstine, " Weakly complete Banach spaces," *Duke Mathematical Journal*, vol. 4 (1938), pp. 125-131.

13. T. H. Hildebrandt, " On bounded linear functional operations," *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 868-875.

14. S. Kakutani, " Weak topology and regularity of Banach spaces," *Proceedings of the National Academy of Sciences of Tokyo*, vol. 25 (1939), pp. 169-173.

15. M. Krein, " Sur quelques questions de la géometrie des ensembles convexes situé dans un espace linéaire normé et complet," *Comptes Rendus U. R. S. S.*, vol. 14 (1937), pp. 5-7.

16. S. Mazur, " Über konvex Mengen in linearen normierten Räumen," *Studia Mathematica*, vol. 4 (1933), pp. 70-84.

17. D. Milman, " On some criteria for the regularity of spaces of the type (*B*)," *Comptes Rendus U. R. S. S.*, vol. 20 (1938), pp. 243-246.

18. E. H. Moore and H. L. Smith, " A general theory of limits," *American Journal of Mathematics*, vol. 44 (1922), pp. 102-121.

19. B. J. Pettis, " Differentiation in Banach spaces," *Duke Mathematical Journal*, vol. 5 (1939), pp. 254-269.

20. B. J. Pettis, " On integration in vector spaces," *Transactions of the American Mathematical Society*, vol. 44 (1938), pp. 277-304.

21. R. S. Phillips, " On linear transformations," *Transactions of the American Mathematical Society*, vol. 48 (1940), pp. 516-541.

22. S. Saks, *Theory of the integral*, Warsaw, 1937.

23. J. Sirvint, " Schwache Kompaktheit in den Banachschen Räumen," *Comptes Rendus U. R. S. S.*, vol. 28 (1940), pp. 199-202.

24. V. Smulian, " Über lineare topologische Räume," *Recueil Mathématique*, vol. 49 (1940), pp. 425-444.

25. V. Smulian, " Sur les ensembles faiblement compact dans les espaces linéaires normés," *Communications de la Société Mathématique de Kharkhoff*, 4, vol. 14 (1937), pp. 239-242.

CONTINUA OF FINITE LINEAR MEASURE I.*

By Samuel Eilenberg and O. G. Harrold, Jr.

1. The α-dimensional measures. For a metric space X and numbers $\alpha \geqq 0$, $\epsilon > 0$ define[1] $L_\epsilon^\alpha(X) = \text{g.l.b.} \sum_{i=1}^{\infty} [\delta(X_i)]^\alpha$ for all decompositions $X = X_1 + X_2 + \cdots$, $\delta(X_i) < \epsilon$. If no such decomposition exists set $L_\epsilon^\alpha(X) = +\infty$. Since L_ϵ^α is a non-decreasing function of ϵ we may define

$$L^\alpha(X) = \lim L_\epsilon^\alpha(X), \epsilon \to 0.$$

This number (which may be $+\infty$) is called the *α-dimensional measure* of X. The number $L^\alpha(A)$ is also defined for all subsets $A \subset X$ and is an outer Carathéodory measure [10, p. 53].[2]

It is agreed that $[\delta(A)]^0 = 0, 1$ according as A is vacuous or not, and hence $L^0(X)$ is the number of points in X if it is a finite set and $+\infty$ otherwise.

The value of $L^\alpha(X)$ depends strictly on the choice of the distance function ρ in X. For the sake of clarity we shall sometimes write $L^\alpha(X, \rho)$ instead of $L^\alpha(X)$.

2. Szpilrajn's theorem. The problem. The connection between α-dimensional measures and the Menger-Urysohn dimension of X was studied by Szpilrajn [11, 7] who, among other results, has proved that a separable metric space X has a dimension $\leqq n$ if and only if there is a metrization ρ of X such that $L^{n+1}(X, \rho) = 0$. This implies that if $\dim X = n$, $L^n(X, \rho) > 0$ for every metrization ρ of X.

If X_0 is a compact metric space consisting of a countably infinite set of points and X_n is the cartesian product $I^n \times X_0$, where I^n is the n-dimensional cube, it is easily seen that $\dim X_n = n$ but $L^n(X_n, \rho) = +\infty$ for every metrization ρ. This suggests the following

Problem. *To characterize a separable metric space X of dimension n for which there is a metrization ρ such that $L^n(X, \rho) < +\infty$.*

---

* Received July 31, 1941.

[1] For $A \subset X$ we define $\delta(A) = \text{l.u.b.}\ \rho(x, y)$, $x, y \in A$.

[2] The numbers in brackets refer to the bibliography at the end of the paper.

3. The integral theorem. A necessary condition which the space must satisfy can be easily obtained from the following[3]

THEOREM 1. *Given a metric space X and a real valued function $f(x)$ on X such that $|f(x_1) - f(x_2)| \leqq \rho(x_1, x_2)$ for all $x_1, x_2 \epsilon X$, then*

$$\int_{-\infty}^{+\infty} L^{\alpha}[f^{-1}(t)]dt \leqq L^{\alpha+1}(X), \qquad \alpha \geqq 0,$$

where the integral is to be understood as an upper integral in the sense of Lebesgue.[4]

Since all the quantities involved are non-negative it follows that if $L^{\alpha+1}(X) < +\infty$ (or $=0$) then $L^{\alpha}[f^{-1}(t)] < +\infty$ (or $=0$) for almost all t. In particular if $L^1(X) < +\infty$, then $f^{-1}(t)$ is a finite set for almost all t.

If x_0 is a fixed point of X and $f(x) = \rho(x, x_0)$, we get the following

COROLLARY. *If $L^1(X, \rho) < \infty$ and $x_0 \epsilon X$ then the set*

$$S(t) = \{x \mid \rho(x_0, x) = t\}$$

is finite for almost all t.

Using the function $f(x) = \rho(x_0, x)$ we also easily deduce from Theorem 1 that if $L^{n+1}(X) = 0$ then $\dim X \leqq n$, which is one half of the theorem of Szpilrajn mentioned above.

4. Rectifiable continua. In this paper attention is restricted to the case $n = 1$ and X is a continuum (i. e. a metric, compact and connected space containing at least two points). For this case the problem will be completely solved (see Section **6**). This case, the authors believe, is of particular importance on account of its connection with the problem of rectification of continuous curves. A continuum X is called rectifiable if there is a continuous mapping, $f(I) = X$, of the closed unit interval I, $0 \leqq t \leqq 1$, such that the path-length $S(f)$ is finite.[5] Since the inequality $L^1(X) \leqq S(f)$ is readily established from the definitions, it follows that, if X is rectifiable, the linear measure of X is finite. The converse of this is a consequence of the following

---

[3] This is a generalization of a theorem of Eilenberg [2].

[4] We understand the upper Lebesgue integral of a function $f \geqq 0$ to be the g. l. b. of Lebesgue integrals of all measurable functions $g \geqq f$.

[5] For a definition of $S(f)$ and simple properties of this functional see Saks [10, p. 121].

THEOREM 2. *If* $L^1(X) < \infty$ *there is a continuous mapping* $f(I) = X$ *such that* $S(f) \leqq 2 \cdot L^1(X) - \delta(X)$.

This is a generalization of a theorem of Ważewski [12] (see also [8] and [9]).

5. Continua of finite degree. The point p of the continuum X is said to be of *finite degree* provided that to each $\epsilon > 0$ there is an uncountable family of neighborhoods $\{U_\alpha\}$ of p such that (a) $\delta(U_\alpha) < \epsilon$, (b) for arbitrary U_α, U_β in $\{U_\alpha\}$ either $\bar{U}_\alpha \subset U_\beta$ or $\bar{U}_\beta \subset U_\alpha$ and (c) each $\bar{U}_\alpha - U_\alpha = F(U_\alpha)$ is a finite set. If each point of X is of finite degree, the continuum X is of *finite degree.* A continuum of finite degree is already regular in the Menger-Urysohn sense.

The point p of the continuum X is called a *local separating point* provided that some pair of points, of the component C containing p of an arbitrary neighborhood R of p, are separated by p in R. The development of these concepts is due to G. T. Whyburn who has shown that a continuum X is of finite degree if and only if every subcontinuum of X contains uncountably many local separating points of X [13]. This theorem of Whyburn implies that if X is of finite degree at all points except perhaps those of a compact 0-dimensional set it is of finite degree.

6. The main theorem. Denote by I the unit interval $0 \leqq t \leqq 1$. If $f(X) = Y$, denote by $\Phi(f)$ the set of all points in Y such that $f^{-1}(y)$ is a finite set. We come now to the principal

THEOREM 3. *For every continuum* X *the following conditions are equivalent:* [6]

(A) X *can be imbedded in the Hilbert cube* Q_ω *so as to have a finite linear measure.*

(B) X *has a homeomorphic image of finite linear measure.*

(C) X *is of finite degree.*

(D) *Given any two disjoint, closed subsets* X_0, X_1 *in* X *there is a continuous mapping* $f(X) = I$ *such that*

(d$_1$) $f(x) = i$, *for* $x \in X_i$, $i = 0, 1$;
(d$_2$) $\Phi(f)$ *is uncountable.*

---

[6] For the equivalence of (A), (B) and (D) see [3]. For the equivalence of (B) and (C), originally established by other methods, see abstract 45-9-321 of the *Bulletin of the American Mathematical Society,* vol. 45 (1939), p. 668.

(E) *Given any two closed, disjoint subsets X_0, X_1 in X there is a continuous mapping $f(X) = I$ such that*

(e$_1$) $f(x) = i$, *for* $x \in X_i$, $i = 0, 1$;

(e$_2$) *for every irrational* $t \in I$, $t \in \Phi(f)$.

(F) *Every subcontinuum of X contains uncountably many local separating points of X.*

(G) *X is locally connected and for every pair of closed, disjoint subsets X_0, X_1 in X there is a finite collection of disjoint, perfect sets $N_1, N_2, \cdots N_k$ such that any continuum in X intersecting both X_0 and X_1 contains some N_i.*

(H) *Given a sequence $X_0, X_1, X_2, \cdots$ of subcontinua such that* $\lim X_i = X_0$ *there is an integer n for which $\prod_{n}^{\infty} X_i$ is uncountable.*

The problem, as to whether or not the 3-dimensional cube, Q^3, can be substituted for the Hilbert cube in (A) is open.

In applications the condition (F) proves most useful in deciding whether or not a given continuum has a homeomorph of finite length.

It is evident that (A) → (B). The implication (B) → (C) is a consequence of the Corollary to Theorem 1. The theorem of Whyburn mentioned in Section **5** states the equivalence of (C) and (F). The equivalence of (C), (G) and (H) has been proved elsewhere [5]. Hence only the implications

$$(C) \rightarrow (D) \rightarrow (E) \rightarrow (A)$$

need to be proved. These proofs as well as the proofs of Theorem 1 and Theorem 2 are given in the following sections.

7. Proof of Theorem 1.

Given a subset A of X define

$$d(A, t) = 1 \text{ if } t \in \overline{f(A)}; \qquad d(A, t) = 0 \text{ if } t \text{ non} \in \overline{f(A)}.$$

Let now

$$X = A_1{}^n + A_2{}^n + \cdots \qquad \delta(A_i{}^n) < 1/n$$

be a sequence of decompositions such that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \{\delta(A_i{}^n)\}^{a+1} = L^{a+1}(X). \tag{7.1}$$

Define

$$d(t) = \liminf_{n\to\infty} \sum_{i=1}^{\infty} \{\delta(A_i{}^n)\}^a d(A_i{}^n, t). \tag{7.2}$$

Since $d(A_i^n, t)$ is a measurable function of t,

(7. 3) $$d(t) \text{ is measurable.}$$

In view of the definition of L^α we have

$$L^\alpha[f^{-1}(t)] \leqq \liminf_{n\to\infty} \sum_{i=1}^{\infty} \{\delta[A_i^n \cdot f^{-1}(t)]\}^\alpha$$

But since

$$\{\delta[A_i^n \cdot f^{-1}(t)]\}^\alpha = \{\delta[A_i^n \cdot f^{-1}(t)]\}^\alpha \, d(A_i^n, t) \leqq \{\delta(A_i^n)\}^\alpha \, d(A_i^n, t)$$

it follows that

(7. 4) $$L^\alpha[f^{-1}(t)] \leqq d(t).$$

From (7. 2) and the lemma of Fatou [10, p. 29] it follows that

$$\int_{-\infty}^{\infty} d(t)\,dt \leqq \liminf_{n\to\infty} \sum_{i=1}^{\infty} \{\delta(A_i^n)\}^\alpha \int_{-\infty}^{\infty} d(A_i^n, t)\,dt.$$

But since

$$\int_{-\infty}^{\infty} d(A_i^n, t)\,dt = \operatorname{meas} \overline{f(A_i^n)} \leqq \delta[f(A_i^n)] \leqq \delta(A_i^n)$$

it follows that

$$\int_{-\infty}^{\infty} d(t)\,dt \leqq \liminf_{n\to\infty} \sum_{i=1}^{\infty} \{\delta(A_i^n)\}^{\alpha+1}.$$

This implies

(7. 5) $$\int_{-\infty}^{\infty} d(t)\,dt \leqq L^{\alpha+1}(X)$$

because of (7. 1). The theorem is a consequence of (7. 3), (7. 4) and (7. 5).

Notice that the case $\alpha = 0$ needs no special proof provided we understand that $\{\delta(A)\}^0 = 0$ if $A = 0$ and that $\{\delta(A)\}^0 = 1$ if $A \neq 0$.

8. Proof of Theorem 2.

We shall first establish the more general

THEOREM 2a. *Let X be a continuum such that $L^1(X) < \infty$ and let $A \subset X$ be an arc with endpoints a_0, a_1. There is a mapping $f(I) = X$ such that $f(0) = a_0$, $f(1) = a_1$ and $S(f) \leqq 2L^1(X) - L^1(A)$.*

Proof. In case X is a tree (i. e. a connected graph containing no closed curve) the theorem can be established by an easy induction with respect to the number of ramification points of X.

Since $L^1(X) < \infty$, X is a locally connected continuum and therefore there is a sequence of trees $\{T_i\}$ such that

$$A \subset T_i \subset X; \qquad \operatorname{Lim} T_i = X.$$

Hence we obtain a sequence of mappings $f_i(I) = T_i$ such that $f_i(0) = a_0$, $f_i(1) = a_1$ and

$$S(f_i) \leqq 2L^1(T_i) - L^1(A) \leqq 2L^1(X) - L^1(A).$$

Since the functions $\{f_i\}$ have a uniformly bounded variation there is a subsequence $\{f_{i_k}\}$ converging uniformly to a mapping $f(I) \subset X$ which clearly will satisfy all the requirements of Theorem 2a.

To prove Theorem 2 we notice that since $L(X) < \infty$, X is arcwise connected and we can chose the arc A so that $\delta(A) = \delta(X)$. Since $\delta(A) \leqq L^1(A)$ we have $2L^1(X) - L^1(A) \leqq 2L^1(X) - \delta(X)$.

9. Proof that (C) → (D). Let X_0 and X_1 be two disjoint closed subsets of X. Identifying all the points belonging to X_0 one obtains a continuum X^* which is a continuous image of X. If X is of finite degree then X^* is of finite degree at every point except perhaps the point x_0 which is the image of X_0. As we have remarked in Section **5**, X^* must therefore be of finite degree also at x_0. This implies the existence of an uncountable family $\{U_\alpha\}$ of open subsets of X such that: $X_0 \subset U_\alpha$, $U_\alpha \cdot X_1 = 0$, $\bar{U}_\alpha - U_\alpha$ is finite, either $U_{\alpha_1} \subset U_{\alpha_2}$ or $U_{\alpha_2} \subset U_{\alpha_1}$.

By a theorem of Kuratowski[7] a continuous mapping $f(X) = I$ can be constructed such that $f(x) = 0$ for $x \in X_0$, $f(x) = 1$ for $x \in X_1$ and that for all indices α, except for a countable number of them, the equality $\bar{U}_\alpha - U_\alpha = f^{-1}(t)$ holds for some t. Consequently $\Phi(f)$ is uncountable and (D) is established.

10. Proof that (D) → (E). We shall use the following

LEMMA 1. *Given a perfect set* $N \subset I$ *there is a continuous mapping* $g(I) = I$ *such that* $g(0) = 0$, $g(1) = 1$, *and for every irrational* $t \in I$ *the set* $g^{-1}(t)$ *is a single point of* N.

We omit the proof.

Let now f be a mapping given by (D). Since $\Phi(f)$ is uncountable and $\Phi(f)$ is a Borel set (in fact, a $G_{\delta\sigma}$) there is a perfect set $N \subset \Phi(f)$. Let $g(I) = I$ be the mapping given by Lemma 1, and define $f_1 = gf$. For every irrational t, we have $g^{-1}(t) \in N \subset \Phi(f)$ hence $t \in \Phi(f_1)$. This proves (E).

11. Regular decompositions. A decomposition

$$X = F_1 + F_2 + \cdots + F_k$$

---

[7] C. Kuratowski, "Sur les familles monotones d'ensembles fermés et leurs applications à la théorie des espaces connexes," *Fundamenta Mathematicae*, vol. 30 (1938), pp. 17-33.

will be called regular if (a) F_i is a continuum for $i = 1, 2, \cdots, k$ and (b) $F_i \cdot F_j$ is finite for $i \neq j$.

LEMMA 2. *A continuum X is regular in the Menger-Urysohn sense if and only if for every $\epsilon > 0$ there is a regular decomposition of mesh*[8] ϵ.

Proof. It is obvious that if the condition of Lemma 2 is satisfied then X is regular. Assume now that X is regular. There is then a decomposition

$$X = U_1 + U_2 + \cdots + U_m$$

where U_i is open, $\delta(U_i) < \epsilon/2$ and $B(U_i) = \bar{U}_i - U_i$ is finite for $i = 1, 2, \cdots, m$. Consider the finite set

$$B = \sum_{i=1}^{m} B(U_i) = (x_1, x_2, \cdots, x_r)$$

It is clear that every component of $X - B$ has diameter $< \epsilon/2$ and only a finite number of components of $X - B$ have a boundary consisting of more than one point. Hence we have a decomposition

$$X - B = D_1 + D_2 + \cdots + D_r + D_{r+1} + \cdots + D_k$$

where for $i \leqq r$, D_i is the sum of all components of $X - B$ with x_i as boundary and $D_{r+1}, \cdots, D_k$ are the remaining components. Taking $F_i = \bar{D}_i$ we obtain a regular decomposition of mesh ϵ.

12. The condition (I). In terms of regular decompositions we can formulate the following condition for a continuum X.

(I) *Given any two disjoint, closed subsets X_0, X_1 in X there is a continuous mapping $g(X) \subset I$ such that*

(i_1) *$g(x) = 0$ for $x \epsilon X_0$, $g(x) > 0$ for $x \epsilon X_1$;*

(i_2) *for every regular decomposition $\{F_i\}$ of X, $\sum_i \delta[g(F_i)] \leqq 1$.*

LEMMA 3. (E) → (I).

Proof. Let f be the function given by (E). Let $k(t)$ denote the number of points of $f^{-1}(t)$ if $f^{-1}(t)$ is finite and let $k(t) = \infty$ otherwise. It follows from (e_2) that $k(t) < \infty$ for every irrational $t \epsilon I$.

We define the mapping $g(X) \subset I$ as follows

$$g(x) = \int_0^{f(x)} \frac{dt}{k(t)}.$$

---

[8] A decomposition will be said to be of mesh ϵ provided the diameter of each element is $< \epsilon$.

If $x \in X_0$ then $f(x) = 0$ and $g(x) = 0$. If $x \in X_1$ then $f(x) = 1$ and

$$g(x) = \int_0^1 \frac{dt}{k(t)};$$

therefore $g(x) > 0$ since $k(t) < \infty$ on a set of positive measure. Condition (i_1) is thus fulfilled.

Now let a regular decomposition $\{F_i\}$ be given. Since F_i is a continuum, $f(F_i)$ is an interval or a point:

$$f(F_i) = [a_i, b_i] \text{ where } a_i \leqq b_i.$$

It follows that

$$\delta[g(F_i)] = \int_0^{b_i} \frac{dt}{k(t)} - \int_0^{a_i} \frac{dt}{k(t)} = \int_{a_i}^{b_i} \frac{dt}{k(t)}$$

and therefore

$$\sum_i \delta[g(F_i)] = \sum_i \int_{a_i}^{b_i} \frac{dt}{k(t)} \tag{12.1}$$

Let $0 = c_0 < c_1 \cdots < c_r = 1$ be the sequence obtained by arranging the points $\{a_i\}$, $\{b_i\}$ in the natural order. Each of the intervals $[a_i, b_i]$ will be the sum of a number of consecutive intervals $[c_j, c_{j+1}]$ and therefore from (12.1) we obtain

$$\sum_i \delta[g(F_i)] = \sum_{j=0}^{r-1} k_j \int_{c_j}^{c_{j+1}} \frac{dt}{k(t)} \tag{12.2}$$

where k_j is the number of indices i for which $[c_j, c_{j+1}] \subset [a_i, b_i]$. It follows that $[c_j, c_{j+1}] \subset f(F_i)$ for k_j different indices i. Since $F_i \cdot F_j$ is finite for $i \neq j$ we deduce that $f^{-1}(t)$ contains at least k_j points for every $t \in [c_j, c_{j+1}]$ except perhaps a finite set. Hence $k_j \leqq k(t)$ for almost every $t \in [c_j, c_{j+1}]$ and therefore

$$k_j \int_{c_j}^{c_{j+1}} \frac{dt}{k(t)} \leqq c_{j+1} - c_j.$$

Using (12.2) we obtain (i_2).

13. Proof that (E) → (A). Let $U_1, U_2, \cdots$ be a sequence of open sets forming a base for X. Let $\{(X_0^n, X_1^n)\}$ be the sequence of all pairs (X_0^n, X_1^n) such that $X_0^n = \bar{U}$, $X_1^n = X - U_j$ and $X_0^n \cdot X_1^n = 0$.

Since (E) implies (I), for every pair (X_0^n, X_1^n) we may consider a function g_n satisfying (I). Given $x \in X$ the sequence

$$h(x) = \{g_1(x), 2^{-2}g_2(x), \cdots, n^{-2}g_n(x), \cdots\}$$

is a point of the Hilbert cube Q_ω. Clearily the mapping $h(X) \subset Q_\omega$ is continuous.

If $x_0, x_1 \epsilon X$ and $x_0 \neq x_1$ then for some index n we have $x_0 \epsilon X_0{}^n$, $x_1 \epsilon X_1{}^n$ and because of (i_1), $g_n(x_0) < g_n(x_1)$ and consequently $h(x_0) \neq h(x_1)$. Hence h is a homeomorphism.

Since for $x_0, x_1 \epsilon X$

$$\rho[h(x_0), h(x_1)] = \{\sum_{n=1}^{\infty} n^{-4} \mid g_n(x_0) - g_n(x_1) \mid {}^2\}^{\frac{1}{2}}$$

it follows that

$$\delta[h(A)] \leqq \{\sum_{n=1}^{\infty} n^{-4}(\delta[g_n(A)])^2\}^{\frac{1}{2}} \leqq \sum_{n=1}^{\infty} n^{-2}\, \delta[g_n(A)]$$

for every subset A of X.

Given a regular decomposition $\{F_i\}$ of X we have

$$\sum_i \delta[h(F_i)] \leqq \sum_i \sum_{n=1}^{\infty} n^{-2}\delta[g_n(F_i)] \leqq \sum_{n=1}^{\infty} n^{-2} \sum_i \delta[g_n(F_i)]$$

and since $\sum_i \delta[g_n(F_i)] \leqq 1$ because of (i_2), we obtain

$$(13.1) \qquad \sum_i \delta[h(F_i)] \leqq \sum_{n=1}^{\infty} n^{-2}.$$

Now, since X satisfies (E), it is clear that X is regular and therefore in view of Lemma 2 there are regular decompositions of X of arbitrarily small mesh. Hence (13.1) implies that

$$L^1[h(X)] \leqq \sum_{n=1}^{\infty} n^{-2} < \infty.$$

This concludes the proof of Theorem 3. Notice that in proving (A) we have used (I) and the fact that X is regular. Since (E) implies (I) we therefore obtain

THEOREM 3a. *For a continuum X regular in the sense of Menger-Urysohn conditions* (A) — (H) *and* (I) *are equivalent.*

UNIVERSITY OF MICHIGAN.
LOUISIANA STATE UNIVERSITY.

BIBLIOGRAPHY

1. Damköhler, W., " Struktur der geschlossenen rectifizierbaren Kurven," *Journal für die reine und angewandte Mathematik*, vol. 177 (1937), pp. 37-54.
2. Eilenberg, S., " On Φ measures," *Annales de la Société Polonaise de Mathématique*, vol. 17 (1938), pp. 252-253.

3. ———, "On continua of finite length," *Annales de la Société Polonaise de Mathématique,* vol. 17 (1938), pp. 253-254.
4. Favard, J., "Sur la structure des continus rectifiables," *Journal de Mathématiques pures et appliquées,* vol. 15 (1936), pp. 321-332.
5. Harrold, O. G., "Continua of finite degree and certain product sets," *Bulletin of the American Mathematical Society,* vol. 46 (1940), pp. 951-953.
6. Hausdorff, F., "Dimension und äusseres Mass," *Mathematische Annalen,* vol. 79 (1919), pp. 157-179.
7. Hurewicz, W. and Wallman, H., *Dimension theory, Princeton Mathematical Series,* vol. 4 (Princeton), 1941.
8. Kolmogoroff, A., "Beiträge zur Masstheorie," *Mathematische Annalen,* vol. 107 (1933), pp. 351-366.
9. Milgram, A. N., "On shortest paths through a set," *Reports of a mathematical Colloquium,* second series 2 (1940), pp. 39-44.
10. Saks, S., *Theory of the integral, Monografje Matematyczne,* vol. 7, Warszawa (1937).
11. Szpilrajn, E., "La dimension et la mesure," *Fundamenta Mathematicae,* vol. 28 (1937), pp. 81-89.
12. Ważewski, T., "Kontinua prostowalne w związku z funkcjami i odwzorowaniami absolutnie ciągłemi," *Annales de la Société Polonaise de Mathématique,* supplement, vol. 3 (1937), pp. 9-49.
13. Whyburn, G. T., "Concerning continua of finite degree and local separating points," *American Journal of Mathematics,* vol. 57 (1935), pp. 11-18.

A FUNCTION-THEORETIC IDENTITY.*

By D. C. SPENCER.

1. Let $n(r, w)$ denote the number of roots of $f = w$ in $|z| < r$. We have then the following rather general result:

THEOREM 1. *Suppose that* $g(R)$, *defined for* $R \geq 0$, *is absolutely continuous*[1] *and that*

$$G(R) = \int g(R)\, d(\ln R). \tag{1.1}$$

Then for any f *regular in* $|z| \leq r$ *we have*

$$\begin{aligned} r\frac{d}{dr}\int_{-\pi}^{\pi} G(|f(re^{i\phi})|)\, d\phi \\ = \int_0^r \int_{-\pi}^{\pi} g'(|f|)\cdot|f|^{-1}\cdot|f'|^2 \rho\, d\rho\, d\phi + 2\pi g(0)\cdot n(r, 0) \end{aligned} \tag{1.2}$$

(except in cases when $g(0) \neq 0$ and $f = 0$ on $|z| = r$).

Since the derivative appears on the left-hand side of (1.2), the additive constant implicit in the definition of $G(R)$ does not matter.

For functions meromorphic in $|z| \leq r$, the identity becomes

$$\begin{aligned} r\frac{d}{dr}\int_{-\pi}^{\pi} G(|f(re^{i\phi})|)\, d\phi \\ = \int_0^r \int_{-\pi}^{\pi} g'(|f|)\cdot|f|^{-1}\cdot|f'|^2 \rho\, d\rho\, d\phi + 2\pi g(0) n(r, 0) - 2\pi g(\infty) n(r, \infty). \end{aligned}$$

This can be seen almost at once from the argument given in Section **5**, the only modification (which is rather trivial) being in (5.3) of Theorem 6. For convenience we suppose throughout that f is regular[2] in $|z| < 1$, and that $0 < |z| = r < 1$.

Taking $g(R) = 1$, so that $g'(R) = 0$, $G(R) = \ln R$, we obtain

$$r\frac{d}{dr}\int_{-\pi}^{\pi} \ln|f|\, d\phi = 2\pi n(r, 0), \tag{1.3}$$

---

* Received August 24, 1941.

[1] Absolute continuity on the set of values of $|f|$ is of course sufficient.

[2] There is no loss of generality whatever in considering regular functions; the method given here is applicable with trivial modification to meromorphic functions.

which is Jensen's formula. Next if $\lambda > 0$ [3], the choice $g(R) = \lambda R^\lambda$ gives

$$(1.4) \qquad r\frac{d}{dr}\int_{-\pi}^{\pi} |f|^\lambda d\phi = \lambda^2 \int_0^r \int_{-\pi}^{\pi} |f|^{\lambda-2}\cdot|f'|^2 \rho d\rho d\phi,$$

the identity of Hardy-Stein. Thirdly, if $f \neq 0$, we may take $g(R) = \ln R$, obtaining

$$(1.5) \qquad r\frac{d}{dr}\int_{-\pi}^{\pi} \ln^2 |f|\, d\phi = 2\int_0^r \int_{-\pi}^{\pi} \left|\frac{d}{dz}\ln f\right|^2 \rho d\rho d\phi.$$

This formula may also be found by dividing both sides of (1.4) by λ^2 and letting $\lambda \to 0$.

Taking $g(R) = R^2/(1+R^2)$, $G(R) = \ln[(1+|a|^2)(1+R^2)]^{\frac{1}{2}}$, we have (since $g(0) = 0$)

$$(1.6) \quad r\frac{d}{dr}\int_{-\pi}^{-\pi} \ln[(1+|a|^2)(1+|f|^2)]^{\frac{1}{2}} d\phi = 2\int_0^r \int_{-\pi}^{\pi} \frac{|f'|^2}{(1+|f|^2)^2}\rho d\rho d\phi$$

Replacing f by $f - a$ in (1.3), subtracting the resulting identity from (1.6) and then integrating, we obtain, in the notation of Ahlfors,[4]

$$(1.7) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\frac{1}{[f, a]}\, d\phi + N(r, a) = \int_0^r \frac{S(\rho)}{\rho}\, d\rho.$$

This is Ahlfors' formulation of Nevanlinna's first main theorem; similar identities were found earlier by Shimizu (3) and Ahlfors (1). Both authors use methods quite different from that given here.

Finally suppose that $\lambda > 0$, and take $g(R) = \mathrm{Min}\,(\lambda R^\lambda, \lambda)$. Then if $f(0) = 0$ we obtain, after integrating, the following curious identity,

$$(1.8) \quad \lambda\int_{-\pi}^{\pi} \ln^+ |f|\, d\phi = \lambda^2 \int_0^r \left\{ \int_0^\rho \int_{\substack{-\pi \\ |f|\leqq 1}}^{\pi} |f|^{\lambda-2}|f'|^2 \rho_1 d\rho_1 d\phi \right\}\frac{d\rho}{\rho} - \int_{\substack{-\pi \\ |f|\leqq 1}}^{\pi} |f|^\lambda d$$

where $\ln^+ |f| = \mathrm{Max}\,(\ln|f|, 0)$, and where the integration on the right-hand side is extended only over values z for which $|f| \leq 1$. Integration by ϕ alone refers (as above) to the circumference $|z| = r$.

2. We write

$$(2.1) \qquad A(r) = A_g(r) = \int_0^r \int_{-\pi}^{\pi} g'(|f|)\cdot|f|^{-1}\cdot|f'|^2 \rho d\rho d\phi.$$

$A(r)$, an area-functional, may be given the following geometrical interpretation. We suppose first that $g(R)$ is non-negative and non-decreasing. Let $W(r)$ denote the map of $|z| < r$ by f, $w = Re^{i\Phi}$ a point of $W(r)$, and let $V(r) = V_g(r)$ be the aggregate of points $v = [2g(R)]^{\frac{1}{2}} \cdot e^{i\Phi}$ for which $Re^{i\Phi} \subset W(r)$. Then $V(r)$ is a uniform distortion of $W(r)$ in the radial direction, and $A(r)$ is the area of $V(r)$, multiply covered points being counted multiply. In general, since $g(R)$ is absolutely continuous, we may write

---

[3] If $f \neq 0$ in $|z| \leqq r$, (1.4) is true for all real λ.

[4] Ahlfors (2). $S(r)$ is the area, divided by π, of the stereographic projection on the Riemann sphere (of diameter unity) of the transform of $|z| < r$ by f; and $[a, b]$ is the straight-line distance between the projections of a and b on this sphere.

$$(2.2) \qquad g(R) = g_1(R) - g_2(R),$$

where g_1 and g_2 are each non-negative, non-decreasing, and absolutely continuous; and

$$(2.3) \qquad A(r) = A_1(r) - A_2(r),$$

where A_1 and A_2 are the areas of V_{g_1} and V_{g_2} respectively.

3. We define

$$(3.1) \qquad p(r, R) = \frac{1}{2\pi} \int_{-\pi}^{\pi} n(r, Re^{i\Phi}) d\Phi,$$

$$(3.2) \qquad n(w) = n(1, w) = \lim_{r \to 1} n(r, w),$$

$$(3.3) \qquad p(R) = p(1, R) = \lim_{r \to 1} p(r, R) = \frac{1}{2\pi} \int_{-\pi}^{\pi} n(Re^{i\Phi}) d\Phi,$$

the limits (though perhaps infinite) existing in virtue of the fact that $n(r, Re^{i\Phi})$, and so $p(r, R)$, are non-decreasing functions of r, $0 < r < 1$.

If

$$(3.4) \qquad \int_0^{R_1} p(R) d[\psi(R)] \leq \boldsymbol{p} \int_0^{R_1} d[\psi(R)]$$

for all $R_1 > 0$, where $\boldsymbol{p}$ is a positive (though not necessarily integral) number, we say that f is mean $\boldsymbol{p}$-valent (ψ). Here we suppose throughout that ψ is an absolutely continuous, non-decreasing function, and that $\psi(0) = 0$. Then (by absolute continuity) (3.4) becomes

$$(3.5) \qquad \int_0^{R_1} p(R) d[\psi(R)] = \int_0^{R_1} p(R) \cdot \psi'(R) dR \leq \boldsymbol{p}\psi(R_1) \qquad (R_1 > 0).$$

In particular, $\boldsymbol{p}$-valent functions are mean $\boldsymbol{p}$-valent (ψ). The special case of (3.4) when $\psi = R^2$ is ordinary mean $\boldsymbol{p}$-valency, and this has been discussed elsewhere.[5] Roughly, (3.4) is the stronger the more rapidly ψ increases with R.

We note first that mean $\boldsymbol{p}$-valency (ψ) implies that $n(0) \leq \boldsymbol{p}$. For there is a circle of radius $\delta = \delta(f) > 0$ every point of which is covered exactly $n(0)$-times, and if $n(0) > \mathbf{p}$ we should have

---

[5] See Spencer (4), (5), and (7).

$$(3.6) \quad \int_0^\delta p(R)d[\psi(R)] \geqq n(0) \cdot \int_0^\delta d[\psi(R)] > \mathbf{p} \int_0^\delta d[\psi(R)],$$

in contradiction with (3.4). If $\mathbf{p} = n(0)$, we say that f is mean $n(0)$-valent (ψ). Since f is regular in $|z| < 1$, $p(\infty) = 0$, and

$$(3.7) \quad \int_0^\infty d[-p(R)] = p(0) - p(\infty) = p(0) = n(0).$$

Mean $n(0)$-valency may therefore be written in the form

$$(3.8) \quad \int_0^{R_1} p(R)d[\psi(R)] \leqq \psi(R_1) \cdot \int_0^\infty d[-p(R)] \qquad (R_1 > 0).$$

Secondly, since $p(r, R) \leqq p(R)$, we see that

$$(3.9) \quad \int_0^{R_1} p(r, R)d[\psi(R)] \leqq \int_0^{R_1} p(R)d[\psi(R)] \qquad (0 < r < 1, R_1 > 0).$$

Therefore a mean $\mathbf{p}$-valent f is *a fortiori* mean $\mathbf{p}$-valent inside $|z| < r < 1$.

The following theorem is useful when combined with Theorem 1.

THEOREM 2. *Suppose that f is mean $\mathbf{p}$-valent (ψ). Then*

$$(3.10) \quad \int_0^\infty \phi\{\psi(R)\}d[-p(R)] \geqq \mathbf{p}\phi\left\{\frac{1}{\mathbf{p}}\int_0^\infty \psi(R)d[-p(R)]\right\}$$

if ϕ is convex (that is, if $\phi'' \geqq 0$) and if $\phi(0) \leqq 0$; and

$$(3.11) \quad \int_0^\infty \phi\{\psi(R)\}d[-p(R)] \leqq \mathbf{p}\phi\left\{\frac{1}{\mathbf{p}}\int_0^\infty \psi(R)d[-p(R)]\right\}$$

if ϕ is concave (that is, if $\phi'' \leqq 0$) and if $\phi(0) \geqq 0$.

When f is mean $n(0)$-valent (ψ), we may replace $\mathbf{p}$ by $n(0) = \int_0^\infty d[-p(R)]$ in (3.10) and (3.11), and these inequalities assume a more elegant form.

The restrictions that $\phi(0)$ be non-positive and non-negative respectively in (3.10) and (3.11) arise from the fact that a mean $\mathbf{p}$-valent function f is *a fortiori* mean $\mathbf{p}_1$-valent if $\mathbf{p}_1 > \mathbf{p}$, and may therefore be dropped when f is mean $n(0)$-valent and $\mathbf{p}$ is replaced by $n(0)$. To see the necessity of these conditions we have only to take $\phi(R) = R + \delta$, δ positive in (3.10), negative in (3.11), to obtain contradictions of both if $\mathbf{p}$ is large enough.

Choosing now $\psi(R) = R^2$, $\phi(R) = g([R]^{\frac{1}{2}})$, so that mean $\mathbf{p}$-valency (ψ)

becomes ordinary mean $\mathbf{p}$-valency, and $g(R) = \phi\{\psi(R)\}$, we have, on integrating by parts,

$$(3.12)\qquad \int_0^\infty \phi\{\psi(R)\}\, d[-p(r,R)] = \int_0^\infty p(r,R)\, d[\phi\{\psi(R)\}] + \phi\{\psi(0)\}\cdot n(r,0)$$

since $p(\infty) = 0$;

$$= \int_0^\infty p(r,R)\, d[g(R)] + g(0)\cdot n(r,0)$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} n(r, Re^{i\Phi})\cdot \frac{g'(R)}{R}\, R\, dR\, d\Phi + g(0)\cdot n(r,0)$$

$$= \frac{1}{2\pi}\int_0^r \int_{-\pi}^{\pi} g'(|f|)\cdot |f|^{-1}\cdot |f'|^2 \rho\, d\rho\, d\phi + g(0)\cdot n(r,0)$$

$$= r\frac{d}{dr}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} G(|f|)\, d\phi$$

by Theorem 1. Finally

$$(3.13)\qquad \int_0^\infty \psi(R)\, d[-p(r,R)] = \int_0^\infty R^2 d[-p(r,R)] = \int_0^\infty p(r,R)\, d(R^2),$$

and so, by Theorem 2, we obtain

THEOREM 3. *Suppose that f is mean $\mathbf{p}$-valent. Then*

$$(3.14)\qquad r\frac{d}{dr}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} G(|f|)\, d\phi \geqq \mathbf{p}g\left\{\left[\frac{1}{\mathbf{p}}\int_0^\infty p(r,R)\, d(R^2)\right]^{\frac{1}{2}}\right\}$$

if $g([R]^{\frac{1}{2}})$ is convex and $g(0) \leq 0$;

$$(3.15)\qquad r\frac{d}{dr}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} G(|f|)\, d\phi \leqq \mathbf{p}g\left\{\left[\frac{1}{\mathbf{p}}\int_0^\infty p(r,R)\, d(R^2)\right]^{\frac{1}{2}}\right\}$$

if $g([R]^{\frac{1}{2}})$ is concave and $g(0) \geq 0$.

If in addition $f(0) = 0$, $g(0) = 0$, and $G(0) = 0$, we obtain by integrating (3.14) and (3.15) from 0 to r, $0 < r < 1$, the following result:

THEOREM 4. *Suppose that f is mean $\mathbf{p}$-valent, and that $f(0) = 0$, $g(0) = 0$, and $G(0) = 0$. Then*

$$(3.16)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} G(|f|)\, d\phi \geqq \mathbf{p}\int_0^r g\left\{\left[\frac{1}{\mathbf{p}}\int_0^\infty p(\rho,R)\, d(R^2)\right]^{\frac{1}{2}}\right\}\frac{d\rho}{\rho}$$

if $g([R]^{\frac{1}{2}}$ is convex;

$$(3.17)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} G(|f|)\, d\phi \leqq \mathbf{p}\int_0^r g\left\{\left[\frac{1}{\mathbf{p}}\int_0^\infty p(\rho,R)\, d(R^2)\right]^{\frac{1}{2}}\right\}\frac{d\rho}{\rho}$$

if $g([R]^{\frac{1}{2}})$ is concave.

Taking in particular $g=\lambda R^{\lambda}$, $G(R)=R^{\lambda}$, we have

$$(3.18)\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^{\lambda}d\phi \leq \boldsymbol{p}^{1-(\lambda/2)}\cdot\lambda\cdot\int_{0}^{r}\left\{\int_{0}^{\infty}p(\rho,R)\,d(R^{2})\right\}^{\lambda/2}\frac{d\rho}{\rho}$$

$$=\boldsymbol{p}^{1-(\lambda/2)}\cdot\lambda\cdot\int_{0}^{r}\left\{\frac{1}{\pi}\int_{0}^{\rho}\int_{-\pi}^{\pi}|f'|^{2}\rho_{1}d\rho_{1}d\phi\right\}^{\lambda/2}\frac{d\rho}{\rho}$$

$$=\boldsymbol{p}^{1-(\lambda/2)}\cdot\lambda\cdot\int_{0}^{r}\left\{\sum_{\nu=1}^{\infty}\nu\,|a_{\nu}|^{2}\rho^{2\nu}\right\}^{\lambda/2}\frac{d\rho}{\rho}.$$

if $0<\lambda\leq 2$, where

$$(3.19)\qquad f=\sum_{\nu=1}^{\infty}a_{\nu}z^{\nu};$$

and the reverse inequality is true if $\lambda>2$. Writing

$$(3.20)\qquad M(\rho,f)=\operatorname*{Max}_{|z|=\rho}|f|,$$

mean $\boldsymbol{p}$-valency implies that

$$(3.21)\quad \int_{0}^{\infty}p(\rho,R)\,d(R^{2})=\int_{0}^{M(\rho,f)}p(\rho,R)\,d(R^{2})\leq\int_{0}^{M(\rho,f)}p(R)\,d(R^{2})\leq \boldsymbol{p}M^{2}(\rho,f),$$

and substitution of (3. 21) into (3. 18) yields the weaker result (proved in my paper (4))

$$(3.22)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^{\lambda}d\phi\leq\boldsymbol{p}\lambda\int_{0}^{r}M^{\lambda}(\rho,f)\frac{d\rho}{\rho}\qquad(0<\lambda\leq 2).$$

The significance of these inequalities becomes clearer when they are expressed in terms of integrals taken over the boundary of $W(r)$, the transform of $|z|<r$ by f. For simplicity we consider (3. 18). Taking $g=\lambda R^{\lambda}$, $G=R^{\lambda}$, in formula (5. 1) below, we have

$$(3.23)\qquad r\frac{d}{dr}\int_{-\pi}^{\pi}|f|^{\lambda}d\phi=\lambda\int_{|z|=r}R^{\lambda}d\Phi,$$

where $f(re^{i\phi})=Re^{i\Phi}$. If now

$$(3.24)\qquad d\Phi\geq 0$$

for $0<r<1$, we have for $0<\lambda\leq 2$, by Hölder's inequality applied to the right-hand side of (3. 23),

$$(3.25)\quad r\frac{d}{dr}\int_{-\pi}^{\pi}|f|^{\lambda}d\phi\leqq\lambda\Big\{\int_{|z|=r}d\Phi\Big\}^{1-(\lambda/2)}\cdot\Big\{\int_{|z|=r}R^{2}d\Phi\Big\}^{\lambda/2}$$

$$=\lambda\{2\pi n(r,0)\}^{1-(\lambda/2)}\cdot\Big\{\int_{|z|=r}R^{2}d\Phi\Big\}^{\lambda/2}$$

by the argument principle, if $f\neq 0$ on $|z|=r$. But, by formula (5. 3) below with $g=R^{2}$,

$$(3.26) \qquad \int_{|z|=r} R^2 d\Phi = 2\pi \int_0^\infty p(r,R)\,d(R^2).$$

Substituting from (3.26) into (3.25), and integrating from 0 to r, we obtain (3.18) (with $\boldsymbol{p}$ replaced by $n(r,0)$). Similarly if $\lambda > 2$ we have the reverse inequality.

But (3.24) is much stronger than mean $\boldsymbol{p}$-valency, and is not satisfied even by *schlicht* functions. We see, however, that integrals taken over the boundaries of mean $\boldsymbol{p}$-valent domains behave like integrals in which $d\Phi \geqq 0$.

4. The final result of the paper lies in a somewhat different direction, and is an application of the Ahlfors-Shimizu identity (1.7) in which we obtain a localization of the following well-known theorem of Nevanlinna:

(N) *If*

$$(4.1) \qquad T(r) = O(1) \qquad (0 < r < 1),$$

where $T(r)$ *is Nevanlinna's characteristic function, then for almost all* θ, f *tends to a limit as* z *tends non-tangentially to* $e^{i\theta}$.

We suppose that $0 < \alpha < \pi/2$, denote by $\Gamma_\alpha(\theta)$ the domain defined by

$$|z| < 1, \qquad |e^{i\theta} - z| < 5/4, \qquad |\arg(e^{i\theta} - z) - \theta| < \alpha,$$

and write

$$(4.2) \qquad S_\alpha(\theta) = \frac{1}{\pi} \iint_{\Gamma_\alpha(\theta)} \frac{|f'|^2}{(1+|f|^2)^2}\, r\,dr\,d\phi.$$

THEOREM 5. *Suppose that* $f(z)$ *is regular for* $|z| < 1$, *and that to each* θ *of a set* $\mathcal{E}$, $0 < m\mathcal{E} \leqq 2\pi$,[6] *there is a positive number* $\alpha = \alpha(\theta)$ *for which* $S_\alpha(\theta)$ *is finite. Then for almost all* θ *of* $\mathcal{E}$ *and for every* β, $0 < \beta < \pi/2$, *the function* $f(z)$ *tends to a finite limit as* $z \to e^{i\theta}$ *in* $\Gamma_\beta(\theta)$.

We note that

$$(4.3) \qquad S_\alpha(\theta) < \iint_{\Gamma_\alpha(\theta)} |f'|^2 r\,dr\,d\theta = A_\alpha(\theta),$$

say; hence if $A_\alpha(\theta)$ is finite in $\mathcal{E}$ the result of Theorem 4 is true *a fortiori*. Theorem 5 with the hypothesis replaced by the stronger condition that $A_\alpha(\theta)$ is finite in $\mathcal{E}$ was suggested to the author by Professor A. Zygmund.

---

[6] $m\mathcal{E}$ denotes the linear measure of $\mathcal{E}$.

Let $\bar{\Gamma}_a(\theta)$ be the subset of $\Gamma_a(\theta)$ in which $|f| \leq 1$, and write

$$\bar{A}_a(\theta) = \iint\limits_{\bar{\Gamma}_a(\theta)} |f'|^2 \, r dr d\theta. \tag{4.4}$$

Using the identity (1.8) in place of (1.7) and noting that

$$\int_{-\pi}^{\pi} ln^+ |f| \, d\phi = O(1) \tag{4.5}$$

is equivalent to (4.1), a trivial modification of the proof given in Section 7 shows that the statement of Theorem 5 is true with $S_a(\theta)$ replaced by $\bar{A}_a(\theta)$.

5. We turn now to the proofs of Theorems 1, 2, and 5, and begin with Theorem 1, the proof of which is striking in its simplicity.

Theorem 1 is equivalent to the following theorem, in which we write $f = Re^{i\Phi}$:

THEOREM 6.

$$\int\limits_{|z|=r} g(R)\, d\Phi = r \frac{d}{dr} \int_{-\pi}^{\pi} G(|f(re^{i\phi})|)\, d\phi; \tag{5.1}$$

$$\int_0^r \int_{-\pi}^{\pi} g'(|f|) \cdot |f|^{-1} \cdot |f'|^2 \rho d\rho d\phi = 2\pi \int_0^{\infty} p(r, R)\, d\{g(R)\}; \tag{5.2}$$

$$\int\limits_{|z|=r} g(R)\, d\Phi = 2\pi \int_0^{\infty} p(r, R)\, d\{g(R)\} + 2\pi g(0) \cdot n(r, 0). \tag{5.3}$$

We have given elsewhere[7] a proof of the special case of Theorem 6 when $g = \lambda R^{\lambda}$, and the idea here is similar. The main difference is that we must now consider the possibility that $g(0) \neq 0$.

We suppose that $f \neq 0$ on $|z| = r$, and we recall again that, by the absolute continuity of g,

$$\int_0^{\infty} p(r, R)\, d\{g(R)\} = \int_0^{\infty} p(r, R) \cdot g'(R) \cdot dR. \tag{5.4}$$

We begin with (5.1). Since $f \neq 0$ on $|z| = r$, $\ln f = \ln R + i\Phi$ is regular on this circumference, and we have

$$\frac{\partial(\ln R)}{\partial r} = \frac{\partial \Phi}{r \partial \phi}; \qquad \frac{\partial \Phi}{\partial \phi} = \frac{r}{R} \frac{\partial R}{\partial r}. \tag{5.5}$$

Since, differentiating both sides of (1.1), $g(R) = RG'(R)$, we have by (5.5)

---

[7] Spencer (6).

$$(5.6)\qquad \int g(R)\,d\Phi = \int g(R)\frac{\partial\Phi}{\partial\phi}\,d\phi$$
$$= \int g(R)\,\frac{r}{R}\,\frac{\partial R}{\partial r}\,d\phi = r\frac{d}{dr}\int_{-\pi}^{\pi} G(R)\,d\phi,$$

and this is (5.1).

Next,

$$(5.7)\qquad \int_0^r\int_{-\pi}^{\pi} g'(|f|)\cdot|f|^{-1}\cdot|f'|^2\rho\,d\rho\,d\phi$$
$$= \int_0^\infty\int_{-\pi}^{\pi} n(r, Re^{i\Phi})\cdot\frac{g'(R)}{R}\cdot R\,dR\,d\Phi$$
$$= 2\pi\int_0^\infty p(r,R)\cdot g'(R)\cdot dR = 2\pi\int_0^\infty p(r,R)\,d\{g(R)\},$$

which is (5.2).

Finally, since $g(R)$ is absolutely continuous,

$$(5.8)\qquad g(R) = g_1(R) - g_2(R),$$

where g_1 and g_2 are non-negative, non-decreasing, absolutely continuous functions, and

$$(5.9)\qquad \int g(R)\,d\Phi = \int g_1(R)\,d\Phi - \int g_2(R)\,d\Phi.$$

We write

$$(5.10)\qquad \tfrac{1}{2}\underline{R}^2 = g_1(R),$$

so that

$$(5.11)\qquad \int g_1(R)\,d\Phi = \tfrac{1}{2}\int \underline{R}^2\,d\Phi.$$

As above, let $W(r)$ denote the transform of $|z| < r$ by f, and let $B(r)$ be its boundary ($B(r)$ is then the transform of $|z| = r$ by f). Also let $V(r)$ be the set of points $v = [2g_1(R)]^{\frac{1}{2}}\cdot e^{i\Phi}$ for which $Re^{i\Phi}\subset W(r)$, $\mathfrak{B}(r)$ its *external* boundary—that is to say, $\mathfrak{B}(r)$ is the curve composed of points $v = [2g_1(R)]^{\frac{1}{2}}\cdot e^{i\Phi}$, $Re^{i\Phi}\subset B(r)$. Then (See Fig. 1) $V(r)$ and $\mathfrak{B}(r)$ both lie in the annulus $[2g_1(0)]^{\frac{1}{2}}\leq \underline{R}\leq [2g_1(\infty)]^{\frac{1}{2}}$, and $\mathfrak{B}(r)$ bounds a domain D which, if $g_1(0)\neq 0$, is composed of two parts: (i) $V(r)$; (ii) the circle C, $0\leq \underline{R} < [2g_1(0)]^{\frac{1}{2}}$, every point of which is covered exactly $n(r,0)$-times since

$$(5.12)\qquad \frac{1}{2\pi}\int_{\mathfrak{B}(r)} d\Phi = \frac{1}{2\pi}\int_{|z|=r} d\Phi = n(r,0)$$

by the argument principle.

We write $A(G)$ for the area, multiply-covered points counted multiply, of a domain G. Since D is closed by the curve $w = \underline{R}e^{i\Phi}$, the right-hand side of (5.11) is the integral over this domain of a sectorial element, and therefore

$$(5.13)\qquad \int g'(R)\,d\Phi = \tfrac{1}{2}\int \underline{R}^2 d\Phi = A(D) = A(V(r)) + A(C).$$

Next, $n\{r, g_1^{-1}(\frac{1}{2}\underline{R}^2)\cdot e^{i\Phi}\}$ sheets of $V(r)$ lie over the point $\underline{R}e^{i\Phi}$, and hence

$$(5.14)\qquad A(V(r)) = \int_{-\pi}^{\pi}\int_{[2g_1(0)]^{\frac{1}{2}}}^{[2g_1(\infty)]^{\frac{1}{2}}} n\{r, g_1^{-1}(\tfrac{1}{2}\underline{R}^2)\cdot e^{i\Phi}\}\underline{R}\,d\underline{R}\,d\Phi = 2\pi\int_0^\infty p(r,R)\,d[g_1(R)$$

Finally

$$(5.15)\qquad A(C) = n(r,0)\cdot\pi\{[2g_1(0)]^{\frac{1}{2}}\}^2 = 2\pi g_1(0)\cdot n(r,0).$$

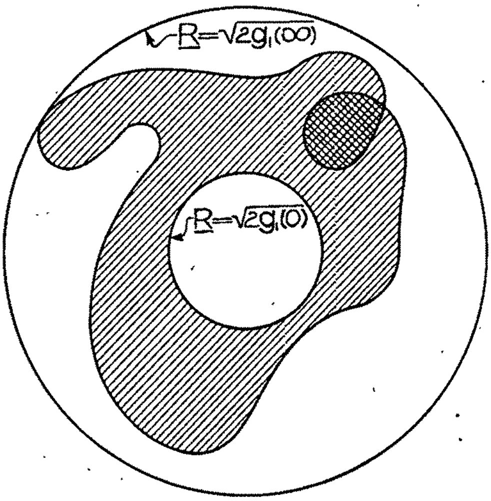

$\underline{R}=\sqrt{2g_1(\infty)}$
$\underline{R}=\sqrt{2g_1(0)}$

Fig. 1.

Substituting (5.14) and (5.15) into (5.13), we obtain (5.3) with g replaced by g_1. But the same formula is true for g_2, and by subtraction (using (5.8)) we obtain (5.3).

This completes the proof of Theorem 6, and so of Theorem 1.

6. We take next Theorem 2, and we may suppose that

$$(6.1)\qquad \int_0^\infty \psi(R)\,d[-p(R)] = \boldsymbol{p}\cdot\psi(R_0) < \infty.$$

For, having proved the theorem in this case, the general result follows by a trivial passage to the limit. We define

$$(6.2)\qquad p_0(R) = \begin{cases} \boldsymbol{p}, & \text{if } 0 \leq R < R_0,\\ 0, & \text{otherwise,}\end{cases}$$

and then

$$\int_0^{R_1} p(R)\,d[\psi(R)] \leq \operatorname{Min}\,\{\boldsymbol{p}\psi(R_1), \boldsymbol{p}\psi(R_0)\} = \int_0^{R_1} p_0(R)\,d[\psi(R)] \tag{6.3}$$

by the hypothesis of mean $\boldsymbol{p}$-valency and (6.1). Next, integrating twice by parts,[8]

$$\begin{aligned}\int_0^{\infty} \phi\{\psi(R)\}\,d[-p(R)] &= [-p(R)\cdot\phi\{\psi(R)\}]_0^{\infty} + \int_0^{\infty} p(R)\,d[\phi\{\psi(R)\}] \\ &= n(0)\cdot\phi(0) + \int_0^{\infty} p(R)\phi'\{\psi(R)\}\cdot\psi'(R)\,dR \\ &= n(0)\phi(0) + [\phi'\{\psi(R)\}\cdot\int_0^{R} p(R)\,d[\psi(R)]]_0^{\infty} \\ &\quad -\int_0^{\infty}\{\int_0^{R_1} p(R)\,d[\psi(R)]\}\cdot\phi''\{\psi(R_1)\}\cdot\psi'(R_1)\,dR_1\end{aligned} \tag{6.4}$$

since $\psi(0)=0$, $p(0)=n(0)$, and $p(\infty)=0$;

$$\begin{aligned}&= n(0)\phi(0) + \phi'\{\psi(\infty)\}\cdot\boldsymbol{p}\psi(R_0) \\ &\quad -\int_0^{\infty}\{\int_0^{R_1} p(R)\,d[\psi(R)]\}\cdot\phi''\{\psi(R_1)\}\cdot\psi'(R_1)\,dR_1\end{aligned}$$

by (6.1);

$$\begin{aligned}&\geq \boldsymbol{p}\phi(0) + \phi'\{\psi(\infty)\}\cdot\boldsymbol{p}\psi(R_0) \\ &\quad -\int_0^{\infty}\{\int_0^{R_1} p_0(R)\,d[\psi(R)]\}\cdot\phi''\{\psi(R_1)\}\cdot\psi'(R_1)\,dR_1\end{aligned} \tag{6.5}$$

if ϕ is convex and $\phi(0) \leq 0$, since $\psi' \geq 0$ (by hypothesis) and $n(0) \leq \boldsymbol{p}$;

$$= \int_0^{\infty} \phi\{\psi(R)\}\,d[-p_0(R)] = \boldsymbol{p}\phi\{\psi(R_0)\} = \boldsymbol{p}\phi\{\frac{1}{\boldsymbol{p}}\int_0^{\infty}\psi(R)\,d[-p(R)]\}$$

by (6.1) and (6.2). If ϕ is concave and $\phi(0) \geq 0$, it is plain that we have the opposite inequality.

7. It remains to prove Theorem 5. Let $\mathcal{E}_n$ be the subset of $\mathcal{E}$ for which $\alpha > 1/n$. We may suppose that $\mathcal{E}_n$ is measurable; then since $m\mathcal{E}_n \to m\mathcal{E}$ as $n \to \infty$, it is enough to prove the theorem for $\mathcal{E}_n$. In other words we may suppose that, for some fixed $\alpha > 0$, $S_\alpha(\theta)$ is finite in $\mathcal{E}$.

Next, given $\epsilon > 0$, we can find a perfect subset $\mathcal{E}_1$ of $\mathcal{E}$ such that $m(\mathcal{E} - \mathcal{E}_1) < \epsilon$, and

$$S_\alpha(\theta) < K_1(\epsilon) \tag{7.1}$$

for θ of $\mathcal{E}$. That is to say, we may suppose that (7.1) is satisfied in $\mathcal{E}$, and that $\mathcal{E}$ is perfect.

---

[8] If $\phi'\{\psi(\infty)\} = \infty$, we suppose that $p(R) = 0$ for $M \leq R < \infty$, repeat the same argument with $\int_0^{\infty}$ replaced by $\int_0^{M}$ throughout, and then let $M \to \infty$.

We write

$$D^* = \sum_{\theta \subset \mathcal{E}} \Gamma_\alpha(\theta), \tag{7.2}$$

and Δ for the set complementary to D^* with respect to $|z| < 1$. Then Δ consists of an enumerable set of "triangular" domains Δ_ν. Let $P_1(\nu)$, $P_2(\nu)$ be the two boundary points $e^{i\theta_1}$, $e^{i\theta_2}$ of Δ_ν for which $\theta_1 \subset \mathcal{E}$, $\theta_2 \subset \mathcal{E}$, and let C_{r_0} be the circle $|z| < r_0 < 1$. We shall require the few simple facts about D^* which are summed up in the following lemma:

LEMMA 1. (i) *The distance* d *of the boundary of* D^* *from* $z = 0$ *satisfies*

$$d \geq \sin\alpha > 0. \tag{7.3}$$

(ii) *To any real number* θ *which does not belong to* $\mathcal{E}$, *there corresponds a* $\nu = \nu(\theta)$ *such that*

$$\Gamma_\alpha(\theta) = \sum_{k=1}^{4} \Gamma_k, \tag{7.4}$$

where $\Gamma_1 \subset \Delta_\nu$, $\Gamma_2 \subset C_{r_0}$, $\Gamma_3 \subset \Gamma_\alpha\{P_1(\nu)\}$ *and* $\Gamma_4 \subset \Gamma_\alpha\{P_2(\nu)\}$ (r_0 *depending only on* α).

In fact, let $P_0(\nu)$ be the point on the boundary of Δ_ν which is nearest the origin (P_0 is the "vertex" of Δ_ν), and (See Fig. 2) let ϕ be the angle P_0OP_1.

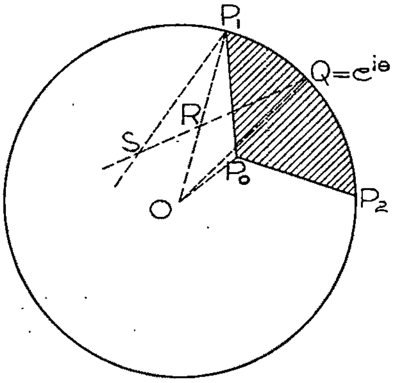


Fig. 2.

Then we have

$$OP_0 = \frac{\sin\alpha}{\sin(\alpha + \phi)} \geq \sin\alpha > 0. \tag{7.5}$$

As for the second part, since θ is not a point of $\mathcal{E}$, there exists a $\nu = \nu(\theta)$ such that $\theta_1(\nu) < \theta < \theta_2(\nu)$, where $P_1(\nu) = e^{i\theta_1}$, $P_2(\nu) = e^{i\theta_2}$. We construct angles OQR and RP_1S each equal to α, and then triangles OQR and RSP_1 are similar. We have, writing $\psi =$ angle $P_1SR =$ angle ROQ,

$$\frac{P_1S}{\sin(\alpha+\psi)}=\frac{P_1R}{\sin\psi}=\frac{1-OR}{\sin\psi}, \tag{7.6}$$

and

$$OR=\frac{\sin\alpha}{\sin(\alpha+\psi)}. \tag{7.7}$$

Substituting from (7.7) into (7.6) and simplifying, we obtain

$$P_1S=\cos\alpha-\sin\alpha\cdot\tan\psi/2, \tag{7.8}$$

and it follows that, if $\psi<\psi_0(\alpha)$, then

$$P_1S\geq K_2(\alpha)>0. \tag{7.9}$$

Hence if $\psi<\psi_0(\alpha)$, part (ii) of Lemma 1 is true. But if $\psi\geq\psi_0(\alpha)$, (ii) is obviously true, and this completes the proof of the lemma.

By (i) of Lemma 1, $z=0$ is an interior point of D^*, and there exists a function $h(z_1)$ which maps the unit circle $|z_1|<1$ on D^* with $h(0)=0$. We write $F(z_1)=f\{h(z_1)\}$, $z_1=\rho e^{i\phi}$. It is, then, sufficient to prove that

$$\int_0^1 S_F(r)\,dr=\int_0^1\left\{\frac{1}{\pi}\int_0^r\int_{-\pi}^{\pi}\frac{|F'|^2}{(1+|F|^2)^2}\rho d\rho d\phi\right\}dr<\infty. \tag{7.10}$$

For then $T_F(r)=O(1)$ by (1.7), and so, by (N), for almost all ϕ, F tends to a limit as $z_1\to e^{i\phi}$ in any $\Gamma_\gamma(\phi)$ for which $\gamma<\pi/2$. Also $h(z_1)$, a bounded function, tends to a limit for almost all ϕ as $z_1\to e^{i\phi}$ in $\Gamma_\gamma(\phi), \gamma<\pi/2$. Since the boundary of D^* is rectifiable, it has a tangent almost everywhere. Hence almost everywhere the angle between the right-hand and left-hand tangents is π, and it follows at once that $\lim f(z)$ exists in every $\Gamma_\beta(\theta), \beta<\pi/2$, almost everywhere in $\mathcal{E}$.

Let $t(r)$ be the transform of $|z|<r$ by h, and let $D^*(r)=D^*\cap C_r$ (C_r being the circle with centre at $z=0$ and radius r). Then since $|h|\leq 1$, $h(0)=0$,

$$t(r)\subset D^*(r)$$

by Schwarz's lemma, and so

$$\begin{aligned} S_F(r)&=\frac{1}{\pi}\int_0^r\int_{-\pi}^{\pi}\frac{|F'|^2}{(1+|F|^2)^2}\rho d\rho d\phi \\ &=\frac{1}{\pi}\iint_{t(r)}\frac{|f'|^2}{(1+|f|^2)^2}\rho d\rho d\theta\leq\iint_{D^*(r)}\frac{|f'|^2}{(1+|f|^2)^2}\rho d\rho d\theta. \end{aligned} \tag{7.11}$$

Next, writing

$$\psi(\rho, \theta) = \begin{cases} \dfrac{|f'|^2}{(1+|f|^2)^2}, & \text{if } (\rho, \theta) \subset D^* \\ 0, & \text{otherwise,} \end{cases}$$

we have, for suitable $K_3 = K_3(\alpha)$,

$$(7.12) \quad \int_0^1 \int_{\theta - K_3 \cdot (1-\rho)}^{\theta + K_3 \cdot (1-\rho)} \psi(\rho, \phi) \rho d\rho d\phi \leq \iint\limits_{\Gamma\alpha(\theta)} \psi(\rho, \phi) \rho d\rho d\phi = \sum_{k=1}^{4} \iint\limits_{\Gamma_k} \leq 2K_1 + K_4($$

if $\theta \not\subset \mathcal{E}$, by (ii) of Lemma 1 and (7.1); and (7.12) is true *a fortiori* if $\theta \subset \mathcal{E}$. Hence by (7.11) and (7.12)[9]

$$(7.13) \quad \int_0^1 S_F(r) dr \leq \int_0^1 \left\{ \iint\limits_{D^*(r)} \frac{|f'|^2}{(1+|f|^2)^2} \rho d\rho d\phi \right\} dr$$

$$= \int_0^1 \int_{-\pi}^{\pi} (1-\rho) \cdot \psi(\rho, \phi) \rho d\rho d\phi$$

$$= \frac{1}{2K_3} \int_{-\pi}^{\pi} d\theta \left\{ \int_0^1 \int_{\theta - K_3 \cdot (1-\rho)}^{\theta + K_3 \cdot (1-\rho)} \psi(\rho, \phi) \rho d\rho d\phi \right\} \leq \frac{\pi}{K_3} \{2K_1 + K_4\} < \infty.$$

This proves (7.10), and completes the proof of Theorem 5.

Stanford University, California.

REFERENCES

1. Ahlfors, L., "Beitrage zur Theorie der meromorphen Funktionen," 7. *Congr. Math. Scand.*, Oslo (1929).
2. Ahlfors, L., "Über eine Methode in der Theorie der meromorphen Funktionen," *Soc. sci. fenn. Comment. Phys.-math. 8*, Nr. 10 (1935).
3. Shimizu, T., "On the theory of meromorphic functions," *Jap. Jour. Math.*, vol. 6 (1929), pp. 119-171.
4. Spencer, D. C., "On finitely mean valent functions," *Proc. London Math. Soc.*, vol. 47 (1941), pp. 201-211.
5. Spencer, D. C., "On finitely mean valent functions II," *Trans. Amer. Math. Soc.*, vol. 48 (1940), pp. 418-435.
6. Spencer, D. C., "Note on some function-theoretic indentities," *Jour. London Math. Soc.*, vol. 15 (1940), pp. 84-86.
7. Spencer, D. C., "On mean one-valent functions," *Annals of Math.*, vol. 42 (1941), pp. 614-633.

---

[9] These final steps in the proof, as well as the use of the domain D^*, were suggested by some remarks of Professor J. E. Littlewood in a letter to Professor Zygmund.

ON FUNCTIONS OF CLASS 1 HAVING THE PROPERTY OF DARBOUX.*

By ISAIAH MAXIMOFF.

1. *Definition* 1. A point x_0 of a perfect set P is called a point of second species of P if it is a both-sided limit point of P. Let E be a linear B-measurable set.

Definition 2. We shall say that a point x_0 of E is a point of second species of E in the segment $[a, b]$ if there exists a perfect set $P(x_0)$ having the following properties:

(i) $P(x_0)$ contains x_0 and is contained in $[a, b]E$;

(ii) if $a < x_0 < b$, then x_0 is a point of second species of $P(x_0)$.

Let $f(x)$ be a finite function of class 1 on $[a, b]$.

Definition 3. We shall say that a perfect set $P(x_0)$ is a *perfect road* for $f(x)$ in $[a, b]$ at the point x_0 if

(i) x_0 is a point of second species of $P(x_0)$ in $[a, b]$;

(ii) $f(x)$ is continuous at the point x_0 relative to $P(x_0)$.

The author has established the following theorem:

If a finite function $f(x)$ of Baire's classification has the property of Darboux in $[a, b]$, then there exists for every point x_0 of $[a, b]$ a perfect road $Q(x_0)$ for $f(x)$ in $[a, b]$.[1]

Definition 4. A finite function $f(x)$ will be called *partially continuous* at the point x_0 of the segment $[a, b]$ if there exists at x_0 a perfect road $P(x_0)$ for $f(x)$ in $[a, b]$.

The author has also proved the following theorem:

A necessary and sufficient condition that a finite function $f(x)$ of class 1 *have the property of Darboux in $[a, b]$ is that $f(x)$ be partially continuous at each point of the segment.*[2] This theorem will be called the *first theorem on*

---

* Received May 3, 1939; Revised August 14, 1940.

[1] "Sur les fonctions ayant la propriété de Darboux," *Prace Mat.-Fiz.*, vol. 43 (1935), pp. 241-242, 258-259.

[2] *Loc. cit.*[2], p. 260.

functions of class 1 having the property of Darboux. There is an analogy between this theorem and a well-known theorem of Baire on functions of class 1: *La condition nécessaire et suffisante pour qu'une fonction quelconque finie ou infinie soit limite de fonctions continues, est qu'elle soit ponctuellement discontinue sur tout ensemble parfait.*[3]

Furthermore there is a well-known theorem of Lebesgue on functions of class 1:

La condition nécessaire et suffisante pour que f, supposée discontinue, soit de classe 1 *sur P, est que les ensembles de points de P, $E(f > A)$, $E(f < A)$, soient des sommes d'ensembles fermés quelque soit A.*[4]

2. We thus have two theorems on functions of class 1, a) Baire's theorem and b) Lebesgue's theorem. The author has obtained the analogue of Baire's theorem for functions of class 1 having the property of Darboux; it is of interest to obtain the second theorem on these functions analogous to Lebesgue's theorem. In order to formulate and to prove this theorem, we consider any two rational numbers w, v $(w < v)$ and the set E_v^w of all points of the segment $[a, b]$ such that $w < f(x) < v$.

LEMMA 1. *If $f(x)$ is a finite function of class* 1 *having the property of Darboux in $[a, b]$ and if x_0 is a point of the set E_v^w, then there exists a non-dense perfect set $P(x_0)$ such that*

(i) *$P(x_0)$ is contained in the set E_v^w;*
(ii) *x_0 is a point of second species of $P(x_0)$ in $[a, b]$;*
(iii) *$f(x)$ is continuous at the point x_0 relative to $P(x_0)$.*

It is obvious that $P(x_0)$ is a perfect non-dense road for $f(x)$ in $[a, b]$ at the point x_0.

Proof. Since $f(x)$ is a finite function of class 1 having the property of Darboux in $[a, b]$ we can find a perfect road $Q(x_0)$ for $f(x)$ in $[a, b]$ at the point x_0. Let x_0 be any point of E_v^w. This means that $w < f(x_0) < v$. The function $f(x)$ is continuous at the point x_0 relative to $Q(x_0)$ and so we can find a segment $s = [x_0 - \epsilon, x_0 + \epsilon][a, b]$, where $\epsilon > 0$, such that $sQ(x_0) \subset E_v^w$. Let $R(x_0)$ denote the largest perfect set contained in $sQ(x_0)$. It is readily seen that $R(x_0)$ is a perfect road at the point x_0 for $f(x)$ in $[a, b]$ contained in E_v^w. Evidently, there exists a non-dense perfect set $P(x_0)$ such that

---

[3] R. Baire, *Leçons sur les fonctions discontinues* (1930), p. 124.

[4] H. Lebesgue, "Une propriété caractéristique des fonctions de classe *un*," *Bull. de la Soc. Math. de France*, vol. 32 (1904), pp. 229-242.

(i) $P(x_0)$ is contained in $R(x_0)$;

(ii) x_0 is a point of second species of $P(x_0)$ in $[a, b]$.

This set $P(x_0)$ satisfies all the conditions of our lemma. $P(x_0)$ will be called a perfect non-dense road for $f(x)$ at the point x_0.

If η is any positive number, let $E[f(x_0), \eta]$ denote the set of all points x of $[a, b]$ such that $|f(x) - f(x_0)| < \eta$.

LEMMA 2. *A necessary and sufficient condition that the function $f(x)$ be partially continuous at the point x_0 of the segment $[a, b]$ is that x_0 be a point of second species of $E[f(x_0), \eta]$ in $[a, b]$ for every positive η.*

Proof. x_0 is a point of second species of $E_n = E[f(x_0), 1/n]$ in $[a, b]$ and so there exists a perfect set P_n such that

(i) x_0 is a point of second species of P_n in $[a, b]$;

(ii) P_n is contained in E_n.

Now we form the sets

$$H'_n = (x_0 - 1/n, x_0)P_n, \qquad H''_n = (x_0, x + 1/n)P_n$$

and the sets G'_n, G''_n in such a manner that

(i) if the set $H'_n - H'_{n+1}\{H''_n - H''_{n+1}\}$ is uncountable, then $G'_n\{G''_n\}$ is a perfect set contained in $H'_n - H'_{n+1}\{H''_n - H''_{n+1}\}$;

(ii) if the set $H_n' - H_{n+1}'\{H''_n - H''_{n+1}\}$ is at most countable, then $G'_n\{G''_n\}$ is null.

If $a < x_0 < b$, then the set $x_0 + \sum_{n=1}^{n=\infty} (G'_n + G''_n)$ is a perfect road for $f(x)$ in $[a, b]$ at the point x_0. If $x_0 = a\{x_0 = b\}$, then the set $x_0 + \sum_{n=1}^{n=\infty} G''_n$ $\{x_0 + \sum_{n=1}^{n=\infty} G'_n\}$ is the perfect road for $f(x)$ in $[a, b]$ at the point $x_0 = a\{x_0 = b\}$. From this we conclude at once that $f(x)$ is partially continuous at each point x_0 of $[a, b]$.

We next assume that $f(x)$ is partially continuous at a point x_0 of $[a, b]$. This means that there exists a perfect road $P(x_0)$ for $f(x)$ in $[a, b]$ at the point x_0. Let η be any positive number. We form the set $P_n(x_0) = P(x_0) \cdot E[f(x_0), \eta]$. The function $f(x)$ is continuous at the point x_0 over $P(x_0)$ so that we can find a segment $s = [x_0 - \epsilon, x_0 + \epsilon][a, b]$, where $\epsilon > 0$, such that $sP(x_0) = s \cdot P_n(x_0)$. Since x_0 is a point of second species of $P(x_0)$ in $[a, b]$, x_0 is a point of second species of $sP_\eta(x_0)$ in $[a, b]$. But $sP_\eta(x_0)$ is contained in $E[f(x_0), \eta]$, so that x_0 is a point of second species of $E[f(x_0), \eta]$ in $[a, b]$.

3. Let (y) $y_1, y_2, y_3, \cdots$ be the sequence of all rational numbers and consider any system of perfect sets

$$(H) \qquad H^{y_r}_{y_\rho}[r+\rho+s] \qquad \begin{pmatrix} s=0,1,2,3,\cdots \\ r=1,2,3,4,\cdots \\ \rho=1,2,3,4,\cdots \end{pmatrix}$$

where $y_r < y_\rho$. We shall say that this system H possesses

a) the property $(E = lH)$ if

$$(1) \qquad E^{y_r}_{y_\rho} = \lim_{s=\infty} H^{y_r}_{y_\rho}[r+\rho+s];$$

b) the property $(H \subset E)$ if

$$(2) \qquad H^{y_r}_{y_\rho}[r+\rho+s] \subset E^{y_r}_{y_\rho};$$

c) the property $(H \underset{s}{\subset} H)$ if

$$(3) \qquad H^{y_r}_{y_\rho}[r+\rho+s] \subset H^{y_r}_{y_\rho}[r+\rho+s+1];$$

d) the property $(H \underset{2}{\subset} H)$ if every point of $H^{y_r}_{y_\rho}[M+s]$ where M is the greater of the integers $r+\rho$, $t+\tau$, is a point of second species whenever $y_t < y_r < y_\rho < y_\tau$.

We now state the second theorem on functions of class 1 having the property of Darboux.

THEOREM 1. *A necessary and sufficient condition that a finite function* $f(x)$ *of class* 1 *have the property of Darboux is that there exists a system of perfect sets*

$$(P) \qquad P^{y_r}_{y_\rho}[r+\rho+s] \qquad \begin{pmatrix} s=0,1,2,3,\cdots \\ r=1,2,3,4,\cdots \\ \rho=1,2,3,4,\cdots \end{pmatrix}$$

having the properties:

$$\text{a) } (E = lP), \quad \text{b) } (P \subset E), \quad \text{c) } (P \underset{s}{\subset} P), \quad \text{d) } (P \underset{2}{\subset} P).$$

Proof. Let $f(x)$ be a finite function of class 1, then $E^{y_r}_{y_\rho}$ is the sum of perfect sets Q_n $(n = 1, 2, 3, 4, \cdots)$ and of a countable set of points x_n $(n = 1, 2, 3, 4, \cdots)$. On the other hand, $f(x)$ possesses the property of Darboux in $[a, b]$ so that we can find by Lemma 1 a perfect road $P(x_n)$ at the point x_n for $f(x)$ contained in $E^{y_r}_{y_\rho}$. It is clear that $E^{y_r}_{y_\rho}$ is the sum of perfect sets Q_m and $P(x_n)$ $\{m = 1, 2, 3, 4, \cdots; n = 1, 2, 3, 4, \cdots\}$.

We may thus write

$$(4) \qquad E^{y_r}_{y_\rho} = \sum_{k=1}^{\infty} A^{y_r}_{y_\rho}[r+\rho+k]$$

where $A^{y_r}_{y_\rho}[r+\rho+k]$ are the perfect sets. Hence, we obtain

$$(5) \qquad E^{y_r}_{y_\rho} = \lim_{s=\infty} H^{y_r}_{y_\rho}[r+\rho+s]$$

where $H^{y_r}_{y_\rho}[r+\rho+s]$ are perfect sets defined by the relations

$$(6) \qquad \begin{cases} H^{r+s}_{y_r} = A^1{}_{y_r} + A^2{}_{y_r} + \cdots + A^{r+s}_{y_r} \\ H^{y_r}_{r+s} = A_1{}^{y_r} + A_2{}^{y_r} + \cdots + A^{y_r}_{r+s} \,. \end{cases}$$

It is obvious that the system H of these perfect sets $H^{y_r}_{y_\rho}[r+\rho+s]$ possesses the properties $(E = lH)$ and $(H \subset E)$.

Now we consider any system of perfect sets

$$(H) \qquad H^{y_r}_{y_\rho}[r+\rho+s] \qquad \begin{pmatrix} s = 0, 1, 2, 3, 4, \cdots \\ r = 1, 2, 3, 4, 5, \cdots \\ \rho = 1, 2, 3, 4, 5, \cdots \end{pmatrix}$$

which posseses the properties $(E = lH)$ and $(K \subset E)$.

Let P be an operation which transforms the system H of sets

$$(H) \qquad H^{y_r}_{y_\rho}[r+\rho+s] \qquad \begin{pmatrix} s = 0, 1, 2, 3, \cdots \\ r = 1, 2, 3, 4, \cdots \\ \rho = 1, 2, 3, 4, \cdots \end{pmatrix}$$

into a system PH of perfect sets

$$(PH) \qquad PH^{y_r}_{y_\rho}[r+\rho+s].$$

Here the system PH is the result of the operation P on the system H. We denote by P_2P_1H the system of perfects sets which is the result of an operation P_2 on the system P_1H. It is obvious that the system P_2P_1H may be considered as the result of the operation P_2P_1 on the system H.

We shall now find an operation P which transforms the system H into the system P of perfect sets

$$(P) \qquad P^{y_r}_{y_\rho}[r+\rho+s] \qquad \begin{pmatrix} s = 0, 1, 2, 3, \cdots \\ r = 1, 2, 3, 4, \cdots \\ \rho = 1, 2, 3, 4, \cdots \end{pmatrix}$$

where

$$(7) \qquad P^{y_r}_{y_\rho}[r+\rho+s] = PH^{y_r}_{y_\rho}[r+\rho+s]$$

satisfying all the conditions of our theorem. Formula (7) may be written in the form $P = PH$.

We define the operation P by the formula

$$(8) \qquad P = \cdots \mathfrak{P}_n \cdots \mathfrak{P}_5 \cdot \mathfrak{P}_4 \cdot \mathfrak{P}_3 \cdot \mathfrak{P}_2,$$

where $\mathfrak{P}_n$ is an operation such that if the system H possesses properties $(E = lH)$ and $(H \subset E)$, then the system $\mathfrak{P}_n H$ of perfect sets

$$(\mathfrak{P}_n H) \qquad \mathfrak{P}_n H^{y_r}_{y_\rho}\,[r+\rho+s] \qquad \begin{pmatrix} s = 0, 1, 2, 3, \cdots \\ r = 1, 2, 3, 4, \cdots \\ \rho = 1, 2, 3, 4, \cdots \end{pmatrix}$$

posseses

a₁) the property $(E = l\mathfrak{P}_n H)$, i. e. $E^{y_r}_{y_\rho} = \lim\limits_{s=\infty} \mathfrak{P}_n H^{y_r}_{y_\rho}\,[r+\rho+s]$;

b₁) the property $(H \subset \mathfrak{P}_n H \subset E)$, i. e.

$$H^{y_r}_{y_\rho}\,[r+\rho+s] \subset \mathfrak{P}_n H^{y_r}_{y_\rho}\,[r+\rho+s] \subset E^{y_r}_{y_\rho};$$

c₁) the property $(\mathfrak{P}_n H \underset{s}{\subset} \mathfrak{P}_n H)_k$ for all integers $k \leqq n$, i. e.

$$\mathfrak{P}_n H^{y_r}_{y_\rho}\,[r+\rho+s] \subset \mathfrak{P}_n H^{y_r}_{y_\rho}\,[r+\rho+s+1]$$

whenever $r+\rho+s+1 \leqq 2k \leqq 2n$;

d₁) the property $(\mathfrak{P}_n H \underset{2}{\subset} \mathfrak{P}_n H)_k$ for all integers $k \leqq n$, i. e. every point of the set $\mathfrak{P}_n H^{y_r}_{y_\rho}\,[2k]$ is a point of second species of the set $\mathfrak{P}_n H^{y_t}_{y_\tau}\,[2k]$ whenever $k \leqq n$,

$$y_t < y_r < y_\rho < y_\tau, \quad t \leqq k, \quad r \leqq k, \quad \rho \leqq k, \quad \tau \leqq k;$$

e₁) if the system of sets

$$H^{y_r}_{y_\rho}\,[r+\rho+s] \qquad \begin{pmatrix} s = 0, 1, 2, 3, \cdots \\ r = 1, 2, 3, 4, \cdots \\ \rho = 1, 2, 3, 4, \cdots \end{pmatrix}$$

possesses properties $(E = lH)$, $(H \subset E)$, $H \underset{s}{\subset} H)_k$ and $(H \underset{2}{\subset} H)_k$ for $k \leqq n$, then for all $k \leqq n$ we have $(\mathfrak{P}_n H = H)_k$, i. e. $\mathfrak{P}_n H^{y_r}_{y_\rho}\,[2k] = H^{y_r}_{y_\rho}\,[2k]$ whenever $k \leqq n$, $r \leqq k$, $\rho \leqq k$.

We define the operation $\mathfrak{P}_n$ by the formula

$$(9) \qquad \mathfrak{P}_n = R_n S_n T_n$$

as the combination of certain operations T_n, S_n. R_n. We define the operation T_n by the formulas

$$(10) \qquad \begin{cases} T_n H^{y_r}_{y_\rho}\,[r+\rho+s] = H^{y_r}_{y_\rho}\,[r+\rho] + H^{y_r}_{y_\rho}\,[r+\rho+1] + \cdots \\ \qquad + H^{y_r}_{y_\rho}\,[r+\rho+s] \text{ for all } r+\rho+s \leqq 2n; \end{cases}$$

$$(11) \qquad \begin{cases} T_n H^{y_r}_{y_\rho}\,[r+\rho+s] = H^{y_r}_{y_\rho}\,[r+\rho+s] \\ \qquad \text{for all } r+\rho+s > 2n. \end{cases}$$

It is easily seen tha tthe operation T_n possesses the properties

a$_2$) $(E = lT_nH)$; b$_2$) $(H \subset T_nH \subset E)$; c$_2$) $(T_nH \underset{s}{\subset} T_nH)_k$ for all $k \leqq n$; d$_2$) if for all $k \leqq n$, $(H \underset{s}{\subset} H)_k$, then $(T_nH = H)_k$ for all $k \leqq n$, i. e.,

$$T_nH^{y_r}_{y_\rho}[r+\rho+s] = H^{y_r}_{y_\rho}[r+\rho+s] \text{ for all } r+\rho+s \leqq 2k \leqq 2n.$$

We denote the set $T_nH^{y_r}_{y_\rho}[r+\rho+s]$ by $T^{y_r}_{y_\rho}[r+\rho+s]$, i. e. $TH = T$.

We now define the operation S_n by the formulas

$$(12) \qquad \begin{cases} S_nT^{y_t}_{y_\rho}[2n] = \sum\limits_{r\rho} T^{y_r}_{y_\rho}[2n] \text{ whenever} \\ \qquad r \leqq n,\ t \leqq n,\ \rho \leqq n,\ \tau \leqq n;\ \ y_t \leqq y_r \leqq y_\rho \leqq y_\tau; \end{cases}$$

$$(13) \qquad \begin{cases} S_nT^{y_t}_{y_\rho}[t+\tau+s] = T^{y_t}_{y_\rho}[t+\tau+s] \\ \qquad \text{in all other cases.} \end{cases}$$

Here $\sum\limits_{r\rho}$ extends over all integers r, ρ for which $y_t \leqq y_r \leqq y_\rho \leqq y_\tau$, $r \leqq n$, $t \leqq n$, $\tau \leqq n$. We readily verify that the operation S_n posseses the properties

a$_3$) $(E = lS_nT)$; b$_3$) $(T \subset S_nT \subset E)$; c$_3$) if $(T \underset{s}{\subset} T)_k$ for all $k \leqq n$, then $(S_nT \subset S_nT)_k$ for all $k \leqq n$; d$_3$) if for all $k \leqq n$ we have $(T \underset{y}{\subset} T)_k$, i. e.

$$T^{y_r}_{y_\rho}[2k] \subset T^{y_t}_{y_\tau}[2k] \text{ whenever}$$

$$y_t < y_r < y_\rho < y_\tau, \quad t \leqq k, \quad r \leqq k, \quad \rho \leqq k, \quad \tau \leqq k \leqq n,$$

then for all $k \leqq n$ we have $(S_nT = T_k)$, i. e.

$$S_nT^{y_r}_{y_\rho}[2k] = T^{y_r}_{y_\rho}[2k].$$

We define the operation R_n in the following manner. We first denote the operation S_nT_n by V_n and define the system N of sets

$$(\mathrm{N}) \qquad N^{y_r}_{y_\rho}[r+\rho+s] \qquad \begin{pmatrix} s = 0, 1, 2, 3, \cdots \\ r = 1, 2, 3, 4, \cdots \\ \rho = 1, 2, 3, 4, \cdots \end{pmatrix}$$

by the formula

$$(14) \qquad N^{y_r}_{y_\rho}[r+\rho+s] = V_nH^{y_r}_{y_\rho}[r+\rho+s].$$

Evidently, V_n posseses the properties

a$_4$) $(E = lV_nH)$; b$_4$) $(H \subset V_nH \subset E)$; c$_4$) $(V_nH \underset{s}{\subset} V_nH)_k$ for all $k \leqq n$; d$_4$) $(V_nH \underset{y}{\subset} V_nH)_k$ for all $k \leqq n$, because $V_nH = S_nT$, and therefore the properties a$_4$), b$_4$), c$_4$), d$_4$) coincide respectively with the properties a$_3$), b$_3$), c$_3$), d$_3$).

Now we suppose that $y_t < y_r < y_\rho < y_\tau$, $t \leqq n$, $r \leqq n$, $\rho \leqq n$, $\tau \leqq n$ and that $X^{y_r}_{y_\rho}[2n]$, $X^{y_t}_{y_\tau}[2n]$ are any perfect sets such that

$$X^{y_r}_{y_\rho}[2n] \subset X^{y_t}_{y_\tau}[2n] \subset E^{y_t}_{y_\tau}.$$

The set of all points $x^{rt}_{\rho\tau}$ which belong to the set $X^{y_r}_{y_\rho}[2n]$ and which are not points of second species of the set $X^{y_t}_{y_\tau}[2n]$ in $[a, b]$ is countable. It is clear that $x^{rt}_{\rho\tau}$ is the extremity of a contiguous interval (a, b) of the set $X^{y_t}_{y_\tau}[2n]$. Let $P(x^{rt}_{\rho\tau})$ be a non-dense perfect road at the point $x^{rt}_{\rho\tau}$ contained in $X^{y_t}_{y_\tau}[2n]$. We now adjoin to the set $X^{y_t}_{y_\tau}[2n]$ the set $[a, b]X^{y_t}_{y_\tau}[2n]$ or the set $[a, b]$. $P(x^{rt}_{\rho\tau})$ according as the point $x^{rt}_{\rho\tau}$ is or is not a point of second species of the set $X^{y_t}_{y_\tau}[2n]$. Denote by $A(x^{rt}_{\rho\tau})$ the result of this adjunction and consider the composed operation $A^{rt}_{\rho\tau}$ which consists of the $A(x^{rt}_{\rho\tau})$ for all points $x^{rt}_{\rho\tau}$.

Let $y_{a_1}, y_{a_2}, \cdots y_{a_n}$ be an increasing sequence of the numbers $y_1, y_2, \cdots y_n$. We shall define the operation R_n by the formula

$$R_n = A^{a_{n-2}a_{n-3}\cdot}_{a_{n-1}a_n} S_n \cdots A^{a_4\cdot a_3\cdot}_{a_5\cdot a_6} S_n \cdot A^{a_3a_3\cdot}_{a_4a_5} S_n \cdot A^{a_2a_1\cdot}_{a_3a_4} S_n.$$

Evidently, the operation R_n possesses the properties

a$_5$) $(E = lR_nN)$; b$_5$) $(N \subset R_nN \subset E)$; c$_5$) $(R_n \underset{s}{\subset} R_nN)_k$ for all $k \leqq n$, because in virtue of c$_4$) we have $(V_nH \underset{s}{\subset} V_nH)_k$ or $(N \underset{s}{\subset} N)_k$ for all $k \leqq n$ and the operation R_n changes only the sets $N^{y_r}_{y_\rho}[r + \rho + s]$ of order n, $r + \rho + s = 2n$, in such manner that

$$N^{y_r}_{y_\rho}[2n] \subset R_nN^{y_r}_{y_\rho}[2n];$$

d$_5$) $(R_nN \underset{2}{\subset} R_nN)_k$ for all $k \leqq n$; e$_5$) since $(E = lN)$, $(N \subset E)$, $(N \underset{s}{\subset} N)_n$ and $(N \underset{2}{\subset} N)_k$ for all $k \leqq n$, then $(R_nN = N)_k$ for all $k \leqq n$. It is easily shown that the operation $\mathfrak{P}_n = R_nS_nT_n$ possesses all the properties a$_1$), b$_1$), c$_1$), d$_1$), e$_1$). For this purpose it is sufficient in a$_5$), b$_5$), c$_5$), d$_5$), e$_5$) to substitute R_nN instead of $\mathfrak{P}_nH$ because $R_nN = \mathfrak{P}_nH$.

It is obvious that the operation

$$P = \cdots \mathfrak{P}_n \cdots \mathfrak{P}_5 \cdot \mathfrak{P}_4 \cdot \mathfrak{P}_3 \cdot \mathfrak{P}_2$$

furnishes a system of perfect sets

$$(P) \qquad P^{y_r}_{y_\rho}[r + \rho + s] \quad \begin{pmatrix} s = 0, 1, 2, 3, \cdots \\ r = 1, 2, 3, 4, \cdots \\ \rho = 1, 2, 3, 4, \cdots \end{pmatrix}.$$

where

$$P_{y_\rho}^{y_r}\,[r+\rho+s] = PH_{y_\rho}^{y_r}\,[r+\rho+s]$$

which satisfies all the conditions of our theorem. Besides, the operation P possesses the following property: if $(E = lH)$, $(H \subset E)$, $(H \underset{s}{\subset} H)_k$ and $(H \underset{2}{\subset} H)_k$ for all $k \leqq n$, then $PH_{y_\rho}^{y_r}\,[r+\rho+s] = H_{y_\rho}^{y_r}\,[r+\rho+s]$. Thus, the conditions of our theorem are necessary. To prove that these conditions are sufficient, let x_0 be any point of the segment $[a, b]$ and let $y_0 = f(x_0)$. Then, if η is a sufficiently small positive number we have integers t, τ, r, ρ such that

$$y_0 - \eta < y_t < y_r < y_0 < y_\rho < y_\tau < y_0 + \eta.$$

The point x_0 belongs to the set $E_{y_\rho}^{y_r} \cdot E_{y_\tau}^{y_t}$. Consequently, x_0 belongs to the set

$$P_{y_\rho}^{y_r}\,[2n] \cdot P_{y_\tau}^{rt}\,[2n]$$

for sufficiently large n if $r \leqq n$, $\rho \leqq n$, $t \leqq n$, $\tau \leqq n$. Therefore x_0 is a point of second species of $P_{y_\tau}^{y_t}\,[2n]$. But if $E[f(x_0), \eta]$ is the set of all points x of $[a, b]$ satisfying the condition $|f(x) - f(x)| < \eta$, we have $P_{y_\tau}^{rt}\,[2n] \subset E[f(x_0), \eta]$ so that x_0 is a point of second species of $E[f(x_0), \eta]$ in $[a, b]$. Consequently, in virtue of Lemma 2, the function $f(x)$ is partially continuous at each point x_0 of $[a, b]$. But $f(x)$ is a finite function of class 1, therefore from the first theorem on functions of class 1 having the property of Darboux it follows that $f(x)$ possesses the property of Darboux.

Our theorem is thus proved.

4. We now turn to the functional equation

$$g(x) = f[\phi(x)]$$

where $g(x)$ and $f(y)$ are given functions, and $\phi(x)$ is sought.[5] We consider the case where $\phi(x)$ possesses the property of Darboux in the segment $[a, b]$. Consider the sequence of all rational numbers (y) $y_1, y_2, y_3, \cdots$. Let $E_{y_\rho}^{y_r}$ be the set of all points (x, y) such that

$$g(x) = f(y), \qquad a \leqq x \leqq b, \qquad y_r < y < y_\rho.$$

Denote by PE the orthogonal projection of a set E on the x axis. Suppose now that M is the greater of the integers $\rho + r$, $\tau + t$ and assume $y_t < y_r < y_\rho < y_\tau$.

---

[5] S. Braun, "Sur l'equation fonctionnelle $g(x) = f[\phi(x)]$," *Fundamenta Mathematicae,* vol. 28 (1937), pp. 294-302; W. Sierpinski, "Sur les fonctions dépendantes," *Fundamenta Mathematicae,* vol. 28 (1936), pp. 66-70.

THEOREM 2. *A necessary condition that there exist a finite function* $\phi(x)$ *of class* 1 *having the property of Darboux and satisfying the equation* $g(x) = f[\phi(x)]$ *is that there exists a system P of perfect sets* (*P*) $P_{y_\rho}^{y_r}[r + \rho + s]$ *such that*

(i)
$$\lim_{s=\infty} P_{y_\rho}^{y_r}[r + \rho + s] \subset PE_{y_\rho}^{y_r};$$
$$P_{y_\rho}^{y_r}[r + \rho s] \subset P_{y_\rho}^{y_r}[r + \rho + s + 1] \subset PE_{y_\rho}^{y_r};$$

(ii) *every point of* $P_{y_\rho}^{y_r}[M + S]$ *is a point of second species of the set* $P_{y_\tau}^{rt}[M + S]$;

(iii)
$$\lim_{s=\infty} \sum_{r\rho} P_{y_\rho}^{y_r}[r + \rho + s] = [a, b]$$

Proof. We assume that a finite function of class 1 having the property of Darboux satisfies the functional equation $g(x) = f[\phi(x)]$. It is clear that every point $[x, \phi(x)]$ belongs to the set $E_{y_\rho}^{y_r}$. Since $\phi(x)$ possesses the property of Darboux, then there exists a system P of perfect sets

$$(P) \qquad P_{y_\rho}^{y_r}[r + \rho + s], \qquad y_r < y_\rho,$$

satisfying the second condition of Theorem 1. Besides, it is easily seen that

$$\lim_{s=\infty} P_{y_\rho}^{y_r}[r + \rho + s] \subset PE_{y_\rho}^{y_r},$$

$$P_{y_\rho}^{y_r}[r + \rho + s] \subset P_{y_\rho}^{y_r}[r + \rho + s + 1] \subset PE_{y_\rho}^{y_r},$$

$$\lim_{s=\infty} \sum_{r\rho} P_{y_\rho}^{y_r}[r + \rho + s] = [a, b].$$

Thus, our theorem is proved.

TCHEBOKSARY, U. S. S. R.

THE CALCULUS OF NON-CONTRADICTION.*

By Glenn Edward Dexter.

In this paper an especially simple calculus of propositions, requiring only one assumption, is presented. The principal symbols used are the letters $p, q, r, \cdots$ and a horizontal line, "——." The line forms a *parenthesis*—thus $\overline{p\quad q}$ is a parenthesis in which p and q occur; $\bar{p}$ is a parenthesis in which only p occurs; $\overline{p\quad q\quad \bar{r}}$ is a parenthesis in which p, q and $\bar{r}$ occur. It should be noted that r does NOT occur in $\overline{p\quad q\quad \bar{r}}$. The accompanying table correlates the new notation with ordinary language and with the notation of *Principia Mathematica*.[1] We see that "not (not p)" becomes $\bar{\bar{p}}$; "if p and q, then r" becomes $\overline{\overline{\overline{p\quad q}}\quad \bar{r}}$; and so on.

TABLE

not p	$\sim p$	$\bar{p}$
if p, then q	$p \supset q$	$\overline{p\quad \bar{q}}$
p or q	$p \vee q$	$\overline{\bar{p}\quad \bar{q}}$
p or q or r	$p \vee q \vee r$	$\overline{\bar{p}\quad \bar{q}\quad \bar{r}}$
p and q	$p \cdot q$	$\overline{\overline{p\quad q}}$
p and q and r	$p \cdot q \cdot r$	$\overline{\overline{p\quad q\quad r}}$

The word *proposition* will denote, indifferently, a letter or a parenthesis. To say that we *have* a certain proposition will mean that that proposition can be written without being covered by a line. Thus if we write

$$p \qquad \bar{q} \qquad \overline{\bar{r}\quad \bar{\bar{s}}}$$

---

* Received August 29, 1941; Revised February 16, 1942.

[1] A. N. Whitehead and B. Russell, *Principia Mathematica*, vol. I, 1910. Cambridge University Press.

we evidently have three propositions—p, $\bar q$ and $\overline{\bar r \quad \bar s}$. To say that we have a *contradiction* will mean that we have a parenthesis, P, and have also every proposition which occurs in P. Thus we have a contradiction in

$$p \quad \overline{p \quad q \quad r} \quad q \quad r,$$

or in

$$\overline{\overline{\bar q \quad \bar r}} \quad \overline{p \quad \overline{\overline{\bar q \quad \bar r}}} \quad p.$$

The working of the calculus will depend upon one assumption, as follows: *If P is a parenthesis such that if we had P we would have a contradiction, then we have every proposition which occurs in P.*

A calculation may or may not start from an hypothesis. In either case, the object is to prove that we have one or more propositions designated *conclusions.* (Where there is no hypothesis, as in Example 8, the conclusions will be universally valid propositions.) This can be done, in accordance with the assumption, by showing that there is a parenthesis, P, in which the conclusions occur, and such that if we had P we would have a contradiction. This parenthesis will be called the *trial parenthesis.* Sometimes, as in Example 2, it will necessarily contain propositions other than the conclusions.

Example 1. Given $\sim(\sim p)$; to prove that we have p.

$$\underset{(1)}{\bar{\bar p}} \quad \underset{(2)}{\bar p}$$

We have (1) by hypothesis. Let the trial parenthesis be (2). In (1) and (2) we would have a contradiction.

Example 2. Given $p \cdot q \cdot r$; to prove that we have p.

$$\underset{(1)}{\overline{\overline{p \quad q \quad r}}} \qquad \underset{(2)}{\overline{p \quad q \quad r}}$$

We have (1) by hypothesis. Let the trial parenthesis be (2). In (1) and (2) we would have a contradiction.

Example 3. Given p and $p \supset q$; to prove that we have q.

$$\begin{array}{ccc} p & \overline{p\ \bar{q}} & \bar{q} \\ (1) & (2) & (3) \end{array}$$

We have (1) and (2) by hypothesis. Let the trial parenthesis be (3). In (1), (2) and (3) we would have a contradiction.

Example 4. Given p; to prove that we have $\sim(\sim p)$.

$$\begin{array}{ccc} p & \bar{\bar{\bar{p}}} & \bar{p} \\ (1) & (2) & (3) \end{array}$$

We have (1) by hypothesis. Let the trial parenthesis be (2). But if we had (2) we would have (3), by Example 1. In (1) and (3) we would have a contradiction.

In the proof of Example 4, the transition from (2) to (3) constitutes a *step.* A step, in this calculus, can be thought of as a subsidiary calculation, using as hypothesis (if an hypothesis is needed) one or more propositions already established in the principal calculation. In subsequent proofs, steps will be indicated by the formula "$\cdot\cdot\cdot \rightarrow \cdot\cdot\cdot$," where the two groups of dots represent, respectively, the hypothesis and conclusions of the subsidiary calculation.

Example 5. Given p and q; to prove that we have $p \cdot q$.

$$\begin{array}{cccc} p & q & \overline{\overline{\overline{p\ q}}} & \overline{p\ q} \\ (1) & (2) & (3) & (4) \end{array}$$

We have (1) and (2) by hypothesis. Let the trial parenthesis be (3). (3) → (4). In (1), (2) and (4) we would have a contradiction.

Example 6. Given p; to prove that we have $p \vee q$.

$$\begin{array}{ccc} p & \overline{\overline{\bar{p}\ \bar{q}}} & \bar{p} \\ (1) & (2) & (3) \end{array}$$

We have (1) by hypothesis. Let the trial parenthesis be (2). (2) → (3). In (1) and (3) we would have a contradiction.

Example 7. Given $(p \cdot q) \vee p$; to prove that we have p.

$$\underset{(1)}{\overline{\overline{\overline{\overline{p \quad q}}} \quad \bar{p}}} \qquad \underset{(2)}{\bar{p}} \qquad \underset{(3)}{\overline{\overline{\overline{p \quad q}}}}$$

We have (1) by hypothesis. Let the trial parenthesis be (2). (2) → (3). In (1), (2) and (3) we would have a contradiction.

In Example 7, the step (2) → (3) is a subsidiary calculation which in turn contains subsidiary calculations.

Example 8. To prove that we have $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$.

$$\underset{(1)}{\overline{\overline{\overline{q \quad \bar{r}} \quad \overline{\overline{\overline{\bar{p} \quad \bar{q}} \quad \overline{\overline{\bar{p} \quad \bar{r}}}}}}}} \qquad \underset{(2)}{\overline{q \quad \bar{r}}} \qquad \underset{(3)}{\overline{\overline{\overline{\bar{p} \quad \bar{q}} \quad \overline{\overline{\bar{p} \quad \bar{r}}}}}} \qquad \underset{(4)}{\overline{\bar{p} \quad \bar{q}}} \qquad \underset{(5)}{\overline{\overline{\bar{p} \quad \bar{r}}}}$$

$$\underset{(6)}{\bar{p}} \quad \underset{(7)}{\bar{r}} \quad \underset{(8)}{\bar{q}}$$

Let the trial parenthesis be (1). (1) → (2) (3). (3) → (4) (5). (5) → (6) (7). (2) (7) → (8). In (4), (6) and (8) we would have a contradiction.

It will now be demonstrated that if we have $\overline{p_1 \cdots p_m \quad \overline{q_1 \cdots q_n}}$ and $\overline{q_1 \cdots q_n \quad \overline{p_1 \cdots p_m}}$ $(m = 1, 2, \cdots; n = 1, 2, \cdots)$, then $q_1, \cdots, q_n$ collectively can be substituted for $p_1, \cdots, p_m$ collectively in any formula where $p_1, \cdots, p_m$ are not covered or are all covered by the same lines.

To take first the simpler case, assume that $p_1, \cdots, p_m$ are not covered. We have $q_1, \cdots, q_n$ by calculation as follows:

$$\underset{(1)}{p_1 \cdots p_m} \qquad \underset{(2)}{\overline{p_1 \cdots p_m \quad \overline{q_1 \cdots q_n}}} \qquad \underset{(3)}{\overline{q_1 \cdots q_n}}$$

We have (1) and (2) by hypothesis. Let the trial parenthesis be (3). In (1),

(2) and (3) we would have a contradiction. Hence $q_1, \cdots, q_n$ can be written without being covered, and can be substituted for $p_1, \cdots, p_m$.

But if $p_1, \cdots, p_m$ are covered by j lines ($j = 1, 2, \cdots$) then we find that $p_1, \cdots, p_m$ occur in P_1, P_1 occurs in $P_2, \cdots, P_{j-1}$ occurs in P_j; where $P_1, \cdots, P_j$ are parentheses, and P_j is not covered. Now consider Q_j, obtained from P_j by substituting $q_1, \cdots, q_n$ for $p_1, \cdots, p_m$. We find that $q_1, \cdots, q_n$ occur in Q_1, Q_1 occurs in $Q_2, \cdots, Q_{j-1}$ occurs in Q_j; where $Q_1, \cdots, Q_j$ are parentheses. We also find that P_1 can be represented by $\overline{p_1 \cdots p_m \quad s_1 \cdots s_k}$, and Q_1 by $\overline{q_1 \cdots q_n \quad s_1 \cdots s_k}$ ($k = 0, 1, 2, \cdots$), where $s_1, \cdots, s_k$ are the propositions other than $p_1, \cdots, p_m$ which occur in P_1. (In our notation here we take advantage of the circumstance that propositions within a parenthesis can be written in any order without affecting the calculus.)

We have $\overline{P_1 \quad \overline{Q_1}}$ by calculation as follows:

$$\underset{(1)}{\overline{q_1 \cdots q_n \quad \overline{p_1 \cdots p_m}}} \qquad \underset{(2)}{\overline{\overline{\overline{p_1 \cdots p_m \quad s_1 \cdots s_k} \quad \overline{\overline{q_1 \cdots q_n \quad s_1 \cdots s_k}}}}}$$

$$\underset{(3)}{\overline{p_1 \cdots p_m \quad s_1 \cdots s_k}} \qquad \underset{(4)}{\overline{\overline{q_1 \cdots q_n \quad s_1 \cdots s_k}}}$$

$$\underset{(5)}{q_1 \cdots q_n} \qquad \underset{(6)}{s_1 \cdots s_k} \qquad \underset{(7)}{p_1 \cdots p_m}$$

We have (1) by hypothesis. Let the trial parenthesis be (2). (2) → (3) (4). (4) → (5) (6). (1) (5) → (7). In (3), (6) and (7) we would have a contradiction. Similarly we have $\overline{Q_1 \quad \overline{P_1}}$.

Again, let i be any integer $2, 3, \cdots, j$. If we have $\overline{P_{i-1} \quad \overline{Q_{i-1}}}$ and $\overline{Q_{i-1} \quad \overline{P_{i-1}}}$, then we have $\overline{P_i \quad \overline{Q_i}}$ and $\overline{Q_i \quad \overline{P_i}}$ by a calculation similar to that in the foregoing paragraph. (Let P_{i-1} take the place of $p_1, \cdots, p_m$; Q_{i-1} the place of $q_1, \cdots, q_n$; P_i the place of P_1; Q_i the place of Q_1.) But we have $\overline{P_1 \quad \overline{Q_1}}$ and $\overline{Q_1 \quad \overline{P_1}}$; therefore by induction we have $\overline{P_j \quad \overline{Q_j}}$ and $\overline{Q_j \quad \overline{P_j}}$.

Since we have P_j and $\overline{P_j \,.\, \overline{Q_j}}$, we have Q_j by Example 3. Hence Q_j can be written without being covered; in effect, $q_1, \cdots, q_n$ can be substituted for $p_1, \cdots, p_m$ in P_j.

Thus in all cases, $q_1, \cdots, q_n$ can be substituted for $p_1, \cdots, p_m$. Similarly $p_1, \cdots, p_m$ can be substituted for $q_1, \cdots, q_n$.

The principle demonstrated above can be used to shorten a great many calculations. We have $\overline{\overline{\overline{p_1 \cdots p_m}} \quad \overline{p_1 \cdots p_m}}$ and $\overline{p_1 \cdots p_m \,.\, \overline{\overline{\overline{p_1 \cdots p_m}}}}$ by calculation; therefore $p_1, \cdots, p_m$ collectively can be substituted for $\overline{\overline{p_1 \cdots p_m}}$; in effect, *two lines of the same length, one above the other, cancel each other.* Thus in Example 8 we might go from (1) to (2), (4), (6) and (7) simply by cancelling pairs of lines in (1).

Another rule for cancellation can be stated as follows: *If we have a proposition, p, and have also a parenthesis in which p occurs, then p may be cancelled in the parenthesis* (*unless p is the* ONLY *proposition in the parenthesis—in that case we have a contradiction, and the calculation is already complete*). Thus in Example 8, by referring to (7) we might cancel the $\bar{r}$ in (2), making (2) equivalent to (8).

We see that the entire proof of Example 8 could be accomplished by cancellations in the trial parenthesis. The same may be said of any theorem in the first three chapters of *Principia Mathematica.*

It remains to be demonstrated that the Calculus of Non-contradiction is not weaker than the calculus of propositions in *Principia Mathematica.* Examples 9, 10, 11 and 12, below, and Example 8, above, correspond to the five axioms of *Principia*—propositions *1. 2, *1. 3, *1. 4, *1. 5 and *1. 6 respectively. In Example 3 we have the principle of inference. In Examples 13 to 20 we have $D \supset D'$ and $D' \supset D$, where D is the definiendum and D' the definiens, for each of the following definitions in *Principia*: *1. 01, *2. 33, *3. 01, *4. 34. It is evident, then, that we can prove any theorem which can be proved by the method of *Principia Mathematica,* for if necessary we can imitate that method.

The proofs of Examples 9 to 20 can be accomplished, in each instance, by cancellations in the trial parenthesis, using only the first rule for cancellation.

Example 9. To prove that we have $(p \vee p) \supset p$.

Let the trial parenthesis be $\overline{\overline{\overline{\bar{p} \quad \bar{p}} \quad \bar{p}}}$.

Example 10. To prove that we have $q \supset (p \vee q)$.

Let the trial parenthesis be $\overline{\overline{q \quad \overline{\overline{\bar{p} \quad \bar{q}}}}}$.

Example 11. To prove that we have $(p \vee q) \supset (q \vee p)$.

Let the trial parenthesis be $\overline{\overline{\overline{\bar{p} \quad \bar{q}} \quad \overline{\overline{\bar{q} \quad \bar{p}}}}}$.

Example 12. To prove that we have $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$.

Let the trial parenthesis be $\overline{\overline{\overline{\bar{p} \quad \overline{\overline{\bar{q} \quad \bar{r}}}} \quad \overline{\overline{\bar{q} \quad \overline{\overline{\bar{p} \quad \bar{r}}}}}}}$.

Example 13. To prove that we have $(p \supset q) \supset (\sim p \vee q)$.

Let the trial parenthesis be $\overline{\overline{\overline{p \quad \bar{q}} \quad \overline{\overline{\bar{\bar{p}} \quad \bar{q}}}}}$.

Example 14. To prove that we have $(\sim p \vee q) \supset (p \supset q)$.

Let the trial parenthesis be $\overline{\overline{\overline{\bar{\bar{p}} \quad \bar{q}} \quad \overline{\overline{p \quad \bar{q}}}}}$.

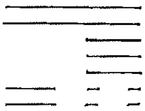
Example 15. To prove that we have $(p \vee q \vee r) \supset [p \vee (q \vee r)]$.

Let the trial parenthesis be $\overline{\overline{\overline{\bar{p} \quad \bar{q} \quad \bar{r}} \quad \overline{\overline{\bar{p} \quad \overline{\overline{\bar{q} \quad \bar{r}}}}}}}$.

Example 16. To prove that we have $[p \vee (q \vee r)] \supset (p \vee q \vee r)$.

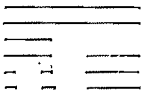
Let the trial parenthesis be $\overline{\overline{\overline{\bar{p} \quad \overline{\overline{\bar{q} \quad \bar{r}}}} \quad \overline{\overline{\bar{p} \quad \bar{q} \quad \bar{r}}}}}$.

Example 17. To prove that we have $(p \cdot q) \supset [\sim(\sim p \vee \sim q)]$.



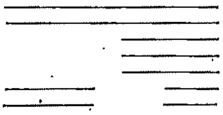
Let the trial parenthesis be $p\ \ q\ \ p\ \ q$.

Example 18. To prove that we have $[\sim(\sim p \vee \sim q)] \supset (p \cdot q)$.



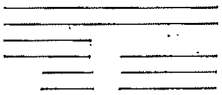
Let the trial parenthesis be $p\ \ q\ \ p\ \ q$.

Example 19. To prove that we have $(p \cdot q \cdot r) \supset [p \cdot (q \cdot r)]$.



Let the trial parenthesis be $p\ \ q\ \ r,\ p\ \ q\ \ r$.

Example 20. To prove that we have $[p \cdot (q \cdot r)] \supset (p \cdot q \cdot r)$.



Let the trial parenthesis be $p\ \ q\ \ r\ \ p\ \ q\ \ r$.

208 Howe Street,
Seattle, Washington.

SPLITTINGS OF A LATTICE.*

By PHILIP M. WHITMAN.

In another paper ([11] Lemma 1.1) the author has shown that in $FL(n)$—*i. e.*, the free lattice generated by $x_1, \cdots, x_n$; see also [10]—when n is finite, given any lattice polynomial b and index p, then either $x_p \leq b$ or $b \leq V_{i \neq p} x_i$, the two cases being mutually exclusive. This is a special case of the general situation considered in the present paper: we say a lattice is split if it is partitioned into two classes, an ideal and a dual ideal (detailed definitions are given below). Birkhoff and Stone have investigated this situation to some extent (cf. **3**), chiefly showing that any distributive lattice can be split in certain ways. Ward and Dilworth ([9] p. 332) have generalized the notion in the case of residuated lattices. In the present paper, we consider the properties of splittings and successions of splittings in general lattices.

In **2** we study their existence in terms of relations between the generating elements of the lattice. The relation of splittability to modularity and other properties is discussed in **3** and part of **5**. In **4** we make definite the intuitive notion of the "edge" along which the splitting seems geometrically to take place; under suitable conditions the two partition classes ("splinters") have isomorphic edges. In **5** we reverse the process of splitting and study the conditions under which two lattices may be combined to form a single lattice of which they are splinters. Here too we find that a principally split lattice is modular (distributive) if and only if each splinter has this property and the edges are convex in the splinters. In **7** we observe that the splinters from a splitting may themselves be splittable, leading to a discussion of successive splittings—or chippings, as they may appropriately be called.

For guidance in the preparation of this paper, the author is indebted to Prof. S. MacLane in the earlier stages and Prof. G. Birkhoff in the later stages.

1. Definitions and notation.

For general definitions and notation in lattice theory we refer the reader to Birkhoff's book [3]; for instance $\leq$ or $\geq$ denotes the order relation, $\frown$ and $\smile$ the meet and join of two elements, Λ and V the meet and join of more than two. $A + B$ is the set union of A and B, while $\subset$ means set inclusion. We may recall some pertinent specialized definitions:

---

* Received September 13, 1941; Presented in part to the American Mathematical Society September 10, 1940.

DEFINITION 1.1. In a lattice, an element a **covers**[1] an element b, denoted $a \gg b$, if $a > b$ but for no c is $a > c > b$.

DEFINITION 1.2. A **lattice polynomial** in $x_1, \cdots, x_n, \cdots$ is any finite combination of the x_i by $\cap$ and $\cup$, repetitions being allowed, e. g.,

$$(x_1 \cup x_1) \cap \{x_2 \cap [x_3 \cup (x_4 \cap x_1)]\}.$$

Cf. [3], p. 21.

DEFINITION 1.3. A **lattice** L **is generated** by elements $x_1, \cdots, x_n, \cdots$ if L consists of all lattice polynomials in the x_i.

The set of all elements of the lattice makes up one set of generators, but there may be much smaller sets. For instance (Birkhoff [1] p. 463-464, [2] p. 451; see also the author, [10] p. 330, [11] § 5), there are lattices with only three generators which have infinitely many elements.

DEFINITION 1.4. The **length** of a lattice polynomial b, denoted by $L(b)$, is the number of generators appearing in it, counting repetitions.

Thus $L(x_1) = 1$, $L([x_1 \cup x_1] \cap x_2) = 3$. Note that $L(b) = 1$ if and only if b is a generator. It may of course happen that $a = b$, yet $L(a) \neq L(b)$; the length is concerned only with the formal polynomial. This definition (cf. [10] p. 325, 326) is of technical use in proofs by induction; it of course presupposes a choice of generators.

DEFINITION 1.5. An **ideal** J in a lattice L is a set of elements of L such that $a \in J$, $b \in J$, and $c \in L$, imply $a \cup b \in J$ and $a \cap c \in J$. An ideal is **principal** if it is the set of all elements of L which are contained in (i. e., $\leq$) some fixed element p_0; we say that p_0 **generates the ideal.** An ideal J is **prime** if $a \cap b \in J$ implies $a \in J$ or $b \in J$. **Dual** prefixed to any of these means that $\cap$ and $\cup$, $\leq$ and $\geq$ are to be interchanged.[2]

Two equivalent properties ([3] p. 78) will illuminate the situation:

LEMMA 1.1. *An ideal J in a lattice L is prime if and only if the set complement of J in L (denoted by $L - J$) is a dual ideal.*

LEMMA 1.2. *An ideal J in a lattice L is prime if and only if J is the set of antecedents of* 0 *under some homomorphism of L on the two element lattice* $[0 < I]$.

DEFINITION 1.6. A lattice L is **split** if it is partitioned into an ideal J' and a dual ideal J'', each non-void; it is **principally split** if J' and J'' are principal.

---

[1] Some authors say, "is prime over."

[2] Dilworth [5] interchanges "ideal" and "dual ideal."

The appropriateness of the term "split" (Birkhoff [4] speaks of a "cleavage"; Ward and Dilworth, [9] p. 332, have used "cleavage" in extending the idea in a somewhat different direction) lies in a rough analogy with the case of a log or timber placed on its side and diagonally split. We call J' and J'' the bottom and top **splinters** of L, and the partition a **splitting** of L. If J' and J'' are principal we frequently denote them by $\boldsymbol{P}$ and $\boldsymbol{D}$ and the elements generating them by $\boldsymbol{p}_0$ and $\boldsymbol{d}_0$, and say that p_0 and d_0 split L. There exist lattices which cannot be split; e. g., the characteristic five-element modular non-distributive lattice (Birkhoff [3] p. 75, Corollary 2). On the other hand, the lemma cited in the introduction shows that $FL(n)$ can be split, for given p; then the set J' of all b such that $b \leq V_{i \neq p} x_i$ constitutes a principal ideal and the set J'' of those such that $x_p \leq b$ forms a dual principal ideal. By the lemma cited, J' and J'' are disjoint and exhaust $FL(n)$ as required by Definition 1. 6; we may also conclude by Lemma 1. 1 that each is prime.

2. Existence theorems.

In view of the preceding remarks, we seek conditions under which splittings will exist. For clarity we first state the results in case the lattice has a finite number of generators (Theorem 1) and then state and prove (Theorem 1′) a more general theorem. We observe that if a lattice L is split, the generators are thereby partitioned into two classes $\{x_j\} \subset J'$ and $\{y_i\} \subset J''$, and for all finite sets R and T, $\Lambda_{i \epsilon R} y_i \epsilon J''$, $V_{j \epsilon T} x_j \epsilon J'$, so that

$$\Lambda_{i \epsilon R} y_i \not\leq V_{j \epsilon T} x_j .$$

It turns out that we may essentially reverse this statement.

THEOREM 1. *Let L be a lattice with a finite number of generators, and let the generators be exhausted by two (possibly overlapping) non-void sets $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_s\}$. Then, given any element b of L, either* (1) $b \leq \overset{i=r}{\underset{i=1}{V}} x_i$ *or* (2) $\overset{j=s}{\underset{j=1}{\Lambda}} y_j \leq b$. *If moreover* $\overset{j=s}{\underset{j=1}{\Lambda}} y_j \not\leq \overset{i=r}{\underset{i=1}{V}} x_i$ *(then a fortiori $\{x_i\}$ and $\{y_j\}$ are disjoint) then the partition of L into the sets J' and J'' satisfying* (1) *and* (2) *respectively is a principal splitting of L.*

THEOREM 1′. *Let the generators of a lattice L be exhausted by two non-void sets $\{x_i\}$ and $\{y_j\}$. Given any element b of L, then* (A) *either* (1) $b \leq V_{i \epsilon R} x_i$ *for some finite set R, or else* (2) $\Lambda_{j \epsilon T} y_j \leq b$ *for some finite set T. Moreover* (B) *the set J' of all b for which* (1) *holds is an ideal and the set J'' of all b for which* (2) *holds is a dual ideal.* (C) *If for all finite sets R and T, $\Lambda_{j \epsilon T} y_j \not\leq V_{i \epsilon R} x_i$, then J' and J'' are disjoint and L is split into them.*

(D) *If besides the conditions of* (C), *also* $p_0 = V_{all\ i} x_i$ *and* $d_0 = \Lambda_{all\ j} y_j$ *exist and are in* J' *and* J'' *respectively* (e. g., *if the number of generators is finite*) *then* J' *and* J'' *are principal.*

Proof. That either (1) or (2) must hold is obvious for $L(b) = 1$; we proceed by induction on $L(b)$. If $b \equiv Vb_k$ and the statement were not true of b, then for all finite sets T, $\Lambda_{j \epsilon T} y_j \not\leq b$. Hence, given k, then $\Lambda_{j \epsilon T} y_j \not\leq b_k$ for all finite T. By induction, given k, $b_k \leq V_{i \epsilon R_k} x_i$ for some finite R_k. Consequently $b \leq V_{i \epsilon U} x_i$, where U is the set union of the R_k and so is again a finite set as desired. The case $b \equiv \Lambda b_k$ is dual. Thus (A) is proved. The other statements are then obvious.

Since in the free lattice $FL(n)$ the conditions of Theorem 1′ C hold by [10] Corollary (17) or [11] Theorem 1, we have:

COROLLARY 2. 1. *Any partition of the generators of* $FL(n)$ *into two non-void classes determines a splitting of* $FL(n)$ *with the elements of the first class in the bottom splinter. Hence* $FL(n)$ *has* $2^n - 2$ *splittings.*

Theorem 1 specializes Theorem 1′ to the case of a finite number of generators, yielding principal splittings. But we can discuss principal splittings even if there are infinitely many generators:

THEOREM 2. $p_0 = V_{all\ i} x_i$ *and* $d_0 = \Lambda_{all\ j} y_j$ *principally split a lattice* L *generated by the* x_i *and* y_j *if and only if they exist in* L *and* $d_0 \not\leq p_0$. *Whether or not these conditions hold, there are no other principal splittings of* L. *If the conditions do hold, then* $p_0 = V_{i \epsilon R} x_i$ *and* $d_0 = \Lambda_{j \epsilon T} y_j$ *for some finite sets* R *and* T.

Proof. In the first sentence, "if" follows from Theorem 1′, while "only if" is required by Definition 1. 6. Now consider any principal splitting, say by p and d into $P \equiv J'$ and $D \equiv J''$. Let $\{x_i\}$ be the generators of L which lie in P. By hypothesis $x_i \leq p$ for all i. But by the remarks at the beginning of 2, the first condition in Theorem 1′ is satisfied, so $p \leq V_{k \epsilon R} x_k$ for some finite R. Thus $x_i \leq V_{k \epsilon R} x_k$ for all i, so $V_{k \epsilon R} x_k$ is the least upper bound of the x_i, and

$$p_0 = V_{all\ i} x_i = V_{i \epsilon R} x_i = p$$

and dually in D. Thus the splitting is of the form specified, and so are p_0 and d_0.

d_0 and p_0 will exist, for instance, if the number of generators is finite or if the finite chain conditions hold or if J' and J'' are complete ([3] p. 17), but in general they need not, since L by definition consists of the *finite* combinations of the generators. For instance, they fail to exist in a chain of ordinal type $\omega + \omega^*$ partitioned into splinters of type ω and ω^*. We may extend § 4 of [10] and Theorem 3 of [11] slightly:

COROLLARY 2. 2. *In* $FL(n)$ *for* $k < n < \infty$,

$$(\bigvee_{i=1}^{k} x_i) \cup (\bigwedge_{i=k+1}^{n} x_i) \quad \textit{covers} \quad \bigvee_{i=1}^{k} x_i .$$

3. Relation of splittability to other properties. Splittability and modularity are completely independent.[3] Sometimes both hold, as with free modular lattices with more than one generator, especially that with two generators (4 elements), sometimes only splittability, as with free lattices with more than two generators, sometimes only modularity, as with the typical five-element modular non-distributive lattice ([3] p. 75), and sometimes neither as is the case if an extra element is interpolated in one of the chains of the lattice just cited. Likewise suitable examples show that splittability and the existence of complements are completely independent.

On the other hand any distributive lattice with more than one element can be split. This was shown by Birkhoff, [1] Theorem 21. 1 and [4] p. 449, and Stone, [8] Theorems 6-7. In fact, they show that given elements a and b such that $a \not\leq b$, then there is a splitting with $a \in J''$ and $b \in J'$. A much simpler proof[4] can however be given if we wish to show only the splittability.

4. Edges of the splinters. The geometrical interpretation of splitting pointed out in **1** suggests that the imaginary "line" along which the splitting takes place should in some sense represent an edge of the two splinters. For principal splittings, a natural sense seems to be this: **in this section, we suppose a lattice $\boldsymbol{L}$ is principally split into $\boldsymbol{P}$ and $\boldsymbol{D}$ by $\boldsymbol{p_0}$ and $\boldsymbol{d_0}$**, and make the following

DEFINITION 4. 1. An element a of D is in the **edge** of D, denoted $a \in \boldsymbol{E(D)}$, if a covers some element of P; dually for the edge of P. The edge of the splitting is the set union of $E(D)$ and $E(P)$.

The following lemma assures us that the edges as thus defined really exist—that there is not a pair of infinite chains of elements approaching the intuitive line of cleavage, one from each side, so as to avoid having any element of one cover any element of the other.

---

[3] In the sense that the presence or absence of one neither assures nor forbids that of the other. See however Theorems 5 and 6.

[4] Not published. See the author's thesis, Harvard University, 1940, p. 58.

LEMMA 4.1. *If* $a \in D$ *and* $b \in P$, *and* $b \leq a$, *then*

$$b \leq (b \cup d_0) \cap p_0 \ll b \cup d_0 = [(b \cup d_0) \cap p_0] \cup d_0 \leq a.$$

Thus $b \cup d_0 \in E(D)$ *and* $(b \cup d_0) \cap p_0 \in E(P)$. *Also dually.*

Proof. $b \in P$ implies $b \leq p_0$ by Definition 1.6, so $b \leq (b \cup d_0) \cap p_0$. Likewise $b \cup d_0 \leq a$. Obviously

$$[(b \cup d_0) \cap p_0] \cup d_0 \leq b \cup d_0 \leq [(b \cup d_0) \cap p_0] \cup d_0.$$

As for the covering relation stated, we have

$$(b \cup d_0) \cap p_0 < b \cup d_0$$

(equality cannot hold since they are in different splinters), and if

$$(1) \qquad (b \cup d_0) \cap p_0 < c < b \cup d_0$$

then either $c \in P$ or $c \in D$. In the former case however, $c \leq p_0$ so $c \leq (b \cup d_0) \cap p_0$, a contradiction, and in the latter case $d_0 \leq c$ and $b \leq p_0$ so by (1),

$$b \cup d_0 = [(b \cup d_0) \cap p_0] \cup d_0 \leq c \cup d_0 = c,$$

contrary to the sharp inequality in (1). Thus the covering relation is proved, and the rest of the lemma follows immediately.

COROLLARY 4.1. $b \in E(P)$ *is covered by exactly one element of* $E(D)$, *namely by* $b \cup d_0$, *and dually for* $b \in E(D)$.

LEMMA 4.2. (*i*) *If* a_1 *and* a_2 *are in* $E(D)$ *so is* $a_1 \cup a_2$. *The dual refers to* $E(P)$. (*ii*) *If* $a \in E(D)$ *and* $b \in P$, *then* $a \cup b \in E(D)$.

Proof. (*i*) Say a_i covers $b_i \in E(P)$. Now $b_1 \cup b_2 \in P$ since P is an ideal, so if $a_1 \cup a_2$ is not in $E(D)$ then

$$b_1 \cup b_2 < e < a_1 \cup a_2$$

for some $e \in E(D)$, by Lemma 4.1. Thus, by Corollary 4.1,

$$e < a_1 \cup a_2 = (d_0 \cup b_1) \cup (d_0 \cup b_2) = d_0 \cup (b_1 \cup b_2) \leq e$$

which is impossible. (*ii*) $b \cup d_0 \in E(D)$ by Lemma 4.1, so

$$a \cup b = (a \cup d_0) \cup b = a \cup (d_0 \cup b)$$

is in $E(D)$ by (*i*).

THEOREM 3. $E(D)$ *and* $E(P)$ *are lattices, though not generally sublattices of* D *and* P, *and as such are isomorphic.*

Proof. By Corollary 4.1, their elements are in a natural one-to-one correspondence. $E(D)$ and $E(P)$ are partially ordered by the inclusion relation of L. If a_1 and a_2 in $E(D)$ cover respectively b_1 and b_2 in $E(P)$, and $a_1 \leq a_2$ but $b_1 \nleq b_2$, then $b_2 < b_1 \cup b_2 \leq a_2$; but $b_1 \cup b_2 \in P$, so $b_1 \cup b_2 < a_2$ and a_2 could not cover b_2. Hence $E(D)$ and $E(P)$ are isomorphic as partially ordered sets. By Lemma 4.2, any two elements of $E(D)$ have a least upper bound in $E(D)$, namely the same as in L. Dually, any pair of elements of $E(P)$ have a greatest lower bound, so by the isomorphism as partially ordered sets, greatest lower bounds exist in $E(D)$, which is therefore a lattice. Dually for $E(P)$.

NOTE 4.1. d_0 is the zero of $E(D)$, $d_0 \cup p_0$ the unit, and dually for $E(P)$.

That the edges, though lattices, need not be sublattices of D and P (nor of L) is readily shown by an example.

COROLLARY 4.2. *If* $a \in D$, $c \in E(D)$, *and* $c \leq a$, *then* $c \leq (a \cap p_0) \cup d_0$ *which is in* $E(D)$.

Proof. By Lemma 4.1, if c covers $c' \in E(P)$, then $c' \leq a \cap p_0$; by the isomorphism, $c \leq (a \cap p_0) \cup d_0$.

5. Combination of splinters.

Having considered the existence of principal splittings and the properties of their edges, we turn now to the converse problem: given two lattices P and D, what about putting them together to get a lattice of which they are splinters?

DEFINITION 5.1. Let there be given disjoint partially ordered sets A and B, and subsets $E(A)$ and $E(B)$ thereof, and a correspondence (not necessarily one-to-one) between the elements of $E(A)$ and $E(B)$. Extend the inclusion relation of A and B to the set union $A + B$ by setting $c_1 \prec c_2$ if and only if

$c_1 \in A$, $c_2 \in A$, and $c_1 \leq c_2$ in A, or
$c_1 \in B$, $c_2 \in B$, and $c_1 \leq c_2$ in B, or

$c_1 \in A$, $c_2 \in B$, and there exist $e_1 \in E(A)$ and $e_2 \in E(B)$ such that $c_1 \leq e_1$ in A, $e_2 \leq c_2$ in B, and e_1 corresponds to e_2 under the given correspondence.

LEMMA 5.1. *Under* $\prec$, $A + B$ *is a partially ordered set, and* $\prec$ *and* $\leq$ *coincide between elements of* A, *likewise between elements of* B.

Proof. Note that $b \prec a$ is impossible if $a \in A$ and $b \in B$. It then follows from the construction that the postulates for partial ordering ([3] p. 5) hold and that $\prec$ coincides with $\leq$ in A (and in B).

THEOREM 4. *Suppose that disjoint lattices D and P and subsets $E(D)$ and $E(P)$ thereof are given, such that*

(1) *$E(D)$ and $E(P)$ are isomorphic partially ordered sets under the inclusion relations of D and P, and*

(2) *D has a zero d_0, and P has a unit p_0, and*

(3) *the zero of D is in $E(D)$, and given $b \in D$ then there is a greatest element c of $E(D)$ such that $c \leq b$ in D, and*

(3′) *the dual of (3) holds for $E(P)$.*

Then taking the correspondence which gives the isomorphism as the correspondence of Definition 5.1, the set union $L = P + D$ is a lattice under $\prec$, with top splinter D and bottom splinter P and edges $E(D)$ and $E(P)$.

Conversely, conditions (1)-(3) are necessary that L be principally split into D and P with those edges.

Proof. The necessity is shown by Theorem 3, the definition of principal, Note 4.1, and Corollary 4.2. To prove sufficiency, we first observe that L is partially ordered, by Lemma 5.1. To show L a lattice, we need to find a least upper bound for elements a and b.

Case 1. If a and b are both in P: Then $a \cup b$ (their lattice join in P) is an upper bound in L. In fact it is least, for suppose $a \prec e$ and $b \prec e$. If $e \in P$ then $a \cup b \prec e$ by construction and Lemma 5.1. Otherwise $e \in D$. Then of the $f \in E(D)$ for which $f \prec e$ there is by (3) a greatest, say e_1. Since $a \prec e$, then $f_1 \prec e$ for some $f_1 \in E(D)$, where f_1 corresponds under the isomorphism to $f'_1 \in E(P)$ and $a \prec f'_1$. Likewise $b \prec f'_2 \prec f_2 \prec e$ with similar conditions on f_2 and f'_2. By choice of e_1 we have $f_1 \prec e_1$ and $f_2 \prec e_1$. Say e_1 covers $e'_1 \in E(P)$; then by the isomorphism, $f'_1 \prec e'_1$ and $f'_2 \prec e'_1$ and all of these are in P, so $a \cup b \prec e'_1 \prec e$ as desired.

Case 2. a and b both in D. Then any upper bound is in D too, so $a \cup b$ serves also in L.

Case 3. $b \in D$ and $a \in P$. Let a_1 be the least element of $E(P)$ containing a, by (3′), and let it correspond to $a_1'' \in E(D)$. Then $a \prec a_1''$, and if $f \in D$ and $a \prec f$ then $a_1'' \prec f$ by the dual of the construction of e'_1 in Case 1. Then the lattice union in D, $b \cup a_1''$, is the least upper bound of a and b in L, for $b \prec g$ and $a \prec g$ imply $g \in D$ and $a_1'' \prec g$. Thus l. u. b. always exists; by duality L is a lattice.

L can be principally split into D and P, for let d_0 be the zero of D and p_0 the unit of P. By construction $a \in P$ and $b \in D$ imply $b \not\prec a$; hence $d_0 \not\prec p_0$.

Yet given a, then either $a \in D$ or $a \in P$, so $d_0 \prec a$ or $a \prec p_0$. This completes the proof.

Thus the top and bottom splinters and edges completely determine the lattice unless the edges admit automorphisms; in the latter case, if one correspondence of the edges yields a lattice, so does any correspondence resulting from an automorphism. However, the two lattices obtained may be non-isomorphic, as may be shown by suitable examples.

If S is a subset of L, then S is said ([3] p. 9) to be **convex** in L if whenever a and b are in S, c in L, and $a < c < b$, then c is also in S. If S is a convex set with zero a and unit b then we call S an **interval sublattice**, for it consists of all x in the interval $a \leq x \leq b$, and it will be a sublattice of L. We recall ([3] p. 34) that a lattice is modular if and only if it does not contain a certain five-element sublattice, graphed on page 35 of [3], which we shall call ***NM5***. Its elements obey the relations

$$(i)\ f < e < a \qquad (iii)\ e \cup c = e \cup b = a$$

$$(ii)\ f < c < b < a \qquad (iv)\ e \cap c = e \cap b = f.$$

THEOREM 5. *If a lattice L is principally split into P and D, then L is modular (or distributive, respectively) if and only if*

(I) *P and D are modular (distributive) and*

(II) *the edges are interval sublattices of the respective splinters.*[5]

Proof. Condition (I) is obviously necessary. As for (II), we note that d_0 is the zero of $E(D)$ and $d_0 \cup p_0$ the unit by Note 4.1. Thus to show (II) necessary, we suppose the contrary, that

$$d_0 < c < d_0 \cup p_0 \text{ but } c \text{ not in } E(D)$$

(or dually). Let $a = c \cap p_0$; then $a \in E(P)$ by Lemma 4.2 (ii) since $p_0 \in E(P)$. Let a be covered by $a'' \in E(D)$. Then we assert that a, p_0, a'', c, and $d_0 \cup p_0$ form $NM5$. As proof, we see that (i) $a < p_0 < d_0 \cup p_0$, since $a \in P$ implies $a \leq p_0$ and $a = p_0$ would give $d_0 \cup p_0 \leq c$ contrary to hypothesis, while $p_0 \neq p_0 \cup d_0$ since one is in P and one in D. (ii) $a < a'' < c < d_0 \cup p_0$ since $a'' \neq c$ (because one is in $E(D)$ and one is not) and yet $a'' = a \cup d_0 \leq c$. (iii) $p_0 \cup c = p_0 \cup a'' = d_0 \cup p_0$, since

$$d_0 \cup p_0 \leq a'' \cup p_0 \leq c \cup p_0 \leq d_0 \cup p_0$$

by construction. (iv) $p_0 \cap c = p_0 \cap a'' = a$, since

$$a \leq a'' \cap p_0 \leq c \cap p_0 = a$$

---

[5] We may replace "interval sublattices of" by "convex in," thus formally strengthening the sufficiency but weakening the other half of the theorem.

by construction. Thus if (II) fails, L contains $NM5$, so (II) is necessary for modularity, and a fortiori for distributivity.

To show sufficiency, suppose that (I) and (II) hold, but L is not modular. Then it has a sublattice $NM5$, consisting of elements a, b, c, e, f with the relations (i)-(iv). If all five were in D or all in P then (I) would be violated. If four lie in one splinter and one in the other, then the definition of the splinters as ideals is violated (e. g., if $f \epsilon P$ and the others are in D, then $b \epsilon D$, $c \epsilon D$, hence by definition of dual ideal $b \frown c = f$ is in D, a contradiction) and similarly the only possibility is that

$$(1) \qquad a, b, c \text{ are in } D;\ e, f \text{ are in } P$$

(or dually). In this case, we shall reach a contradiction to hypothesis (I), via steps (2) to (8) below. By Corollary 4. 2, we may

(2) let e' be the least element of $E(P)$ such that $e \leq e'$.

By Lemma 4. 1, $e' = (e \smile d_0) \frown p_0$ is covered by $e'' = e \smile d_0$; thus $e'' \leq a$. By construction $e'' \epsilon E(D)$ and

$$d_0 \leq c \frown e'' \leq b \frown e'' \leq e''$$

so by (II), $b \frown e''$ and $c \frown e''$ are in $E(D)$.

(3) Set $b'' = b \frown e''$ and $c'' = c \frown e''$, and suppose that they cover b' and c', in $E(P)$, respectively. $c'' \leq b'' \leq e''$ by construction.

$$(4) \qquad b' = b'' \frown e' = b \frown e' \quad \text{and} \quad c' = c'' \frown e' = c \frown e'.$$

Proof.

$$b' = b' \frown e' \leq b'' \frown e' \leq b'' \frown p_0 = b',$$

the first equality coming from the isomorphism of the edges and the fact that $b'' \leq e''$ by the construction of b''; similarly for c'.

(5) $a = e''$ and $c'' = b''$ cannot both hold.

Proof. If they did, then

$$c = c \frown a = c \frown e'' = c'' = b'' = b \frown e'' = b \frown a = b,$$

whereas $c < b$ by hypothesis.

(6) $c = c''$ implies $a = e''$.

Proof. Then

$$a = c \smile e \leq c \smile e'' = c'' \smile e'' = e'' \leq a.$$

(7) c'', b'', and e'' are all distinct.

Proof. $c'' \leq b'' \leq e''$. If $b'' = e''$ then $e \leq b$ whereas by hypothesis $b < a = e \cup b$, so $b'' \neq e''$. If $c'' = b''$ then by (5), $a \neq e''$, so by (6), $c \neq c''$. Then a, b, c, e'', and $c'' = b''$ form an $NM5$ in D, for

$$a = c \cup e \leq c \cup e'' \leq b \cup e'' \leq a$$

so that (*iii*) $b \cup e'' = c \cup e'' = a$ and (*iv*) $b \cap e'' = c'' = c \cap e''$. This contradicts (I). Hence $c'' < b'' < e''$ and

(8) $c' < b' < e'$, by the isomorphism of the edges. We proceed to show e', b', c', e, and f an $NM5$ in P.

(*i*) $f < e < e'$ and (*ii*) $f < c' < b' < e'$.

Proof. $f < e \leq e'$ and $f \leq c' < b' < e'$ by (4) and (8). If $e = e'$ then $f = c \cap e = c \cap e' = c'$ by (4). Conversely if $f = c'$ then $e = e'$ by (II), so if $e = e'$ or $f = c'$ then both hold and

$$c' = f = b \cap e = b \cap e' = b'$$

contrary to (8). Thus the inequalities are sharp as stated.

(*iii*) $b' \cup e = c' \cup e = e'$.

Proof. All are in $E(P)$ by (II) and contain e, and e' is the least such by choice, whereas $c' \cup e \leq b' \cup e \leq e'$.

(*iv*) $b' \cap e = c' \cap e = f$, for otherwise $f < b' \cap e \leq b \cap e$ whereas $b \cap e = f$ by hypothesis.

Thus we have $NM5$ in P contrary to (I), and so the theorem is proved in the modular case. It remains to show the sufficiency if P and D are distributive. Then L is at least modular, as we have just shown, and if not distributive it must have the standard five-element modular non-distributive sublattice of [3] p. 75, first part of Figure 10:

$$a \cup b = a \cup c = b \cup c = e, \qquad a \cap b = a \cap c = b \cap c = f.$$

But as in the first part of the present sufficiency proof, such a configuration must by definition of ideal have all its elements in P or all in D, contrary to the hypothesis that these are distributive. The proof is complete.

COROLLARY 5.1. *If the edges admit automorphisms, and if L, as constructed in Theorem 4, is modular or distributive for one correspondence, then it is so for all correspondences for which the automorphism holds.*

NOTE 5.1. If the edges of a principal splitting are convex in the splinters, then the edges are sublattices of the splinters. For the convexity and Note 4.1 imply that if a and b are in an edge so are $a \cap b$ and $a \cup b$.

LEMMA 5.2. *If a complemented lattice L is principally split, then the zero and unit of L are in the edges.*

Proof. d_0 has a complement d'_0. Now $d'_0 \in P$, for if $d'_0 \in D$ then $d_0 \frown d'_0 = 0 \in D$ by Definition 1.6, and P would be void. Hence by Lemma 4.2 (*ii*), $d_0 \smile d'_0 = I \in E(D)$. Dually $0 \in E(P)$.

THEOREM 6. *If a complemented modular lattice L can be principally split then L can be partitioned into two isomorphic interval sublattices.*

Proof. Theorem 5 (II) and Lemma 5.2 imply $D \equiv E(D)$ and $P \equiv E(P)$. Then Theorem 3 completes the proof.

Remark. We observe that by **3** every finite Boolean algebra satisfies the conditions of Theorem 6. In this case however the theorem is easily obtainable from the representation by point sets ([3] p. 89). In connection with complementation, we may add

THEOREM 7. *In the free lattice $FL(n)$, for $2 < n < \infty$, no element other than 0 and I has a complement.*

Proof. Suppose the contrary:

$$(1)\ a \smile b = I = \bigvee_{i=1}^{n} x_i; \qquad (2)\ a \frown b = 0 = \bigwedge_{i=1}^{n} x_i.$$

From [10] Corollary (17) and from hypothesis (1) it follows that, given i, either $x_i \leq a$ or $x_i \leq b$; likewise, given i, either $a \leq x_i$ or $b \leq x_i$. Say $x_1 \leq a$. *Case* 1: $a \leq x_1$. Then $x_1 = a$. But $x_i \leq x_j$ if and only if $i = j$, so for $i \neq 1$ we must have $x_2 \leq b$ and $b \leq x_2$. Thus $x_2 = b = x_3$, a contradiction. *Case* 2: $b \leq x_1$. Then $b \leq x_1 \leq a$, so $I = a \smile b = a$ and $0 = a \frown b = b$.

6. Non-principal splittings.

We seek now to generalize the results of the preceding sections to the case of non-principal splittings, though in doing so we must sacrifice some of the geometrical interpretation which inspired the nomenclature. Suppose the lattice L is split into J' and J''. Let $\mathfrak{J}'$ stand for the set of all ideals[6] in the bottom splinter J', $\mathfrak{J}''$ for the set of all dual ideals in the top splinter J''. We observe that in particular, if S is any fixed subset of the elements of J'', then the set of all elements of J' which are contained in every element of S is an ideal and hence is an element of $\mathfrak{J}'$.

---

[6] We may recall in this connection Dilworth's result ([5] Theorem 2.1): In a lattice L, let B be an ideal and (a) a principal ideal such that $B \leq (a)$ and $B \neq (a)$; then there exists an ideal C such that $B \leq C$ and (a) covers C in the lattice of all ideals of L. At first sight this would seem what was needed to generalize the idea of edges, but the author has had no success in this direction.

DEFINITION 6.1. F is in the **pseudo-edge** of J', denoted $F \in \mathfrak{E}(J')$, if F is the set of all elements of J' which are contained in every element of some (not necessarily proper) subset S of the elements of J''. Then we write $F = \mathfrak{J}'(S)$. $\mathfrak{E}(J'')$ is defined dually.

In view of the remarks above, we have these relations between sets: $\mathfrak{E}(J') \subset \mathfrak{J}'$ and $\mathfrak{E}(J'') \subset \mathfrak{J}''$. It is readily observed that

COROLLARY 6.1. *$\mathfrak{J}'(\mathfrak{J}''(T)) = T$ if and only if $T \in \mathfrak{E}(J')$; that is, if $T = \mathfrak{J}'(S)$ for some S. Also dually.*

We consider now the special closure operations $X \to (X^*)\dagger$ and $Y \to (Y\dagger)^*$ discussed by Birkhoff ([3] p. 24-25). He begins with a dyadic relation ρ defined between the members of any two abstract aggregates "I"[7] and "I^*". If X is any subset of "I", then he denotes by X^* the set of $x^* \in$ "I^*" such that $x \rho x^*$ for all $x \in X$; reciprocally, if Y is a subset of "I^*", $Y\dagger$ denotes the set of $x \in$ "I" such that $x \rho y$ for all $y \in Y$. In particular we may take J' as "I" and J'' as "I^*", and the inclusion relation of L as ρ. Then in the notation of the present section, X^* is $\mathfrak{J}''(X)$ and $Y\dagger$ is $\mathfrak{J}'(Y)$. By Birkhoff's Theorem 2.11, we may conclude that the lattices of subsets closed under the closure operations named above are dually isomorphic. What are these closed sets in the present case? By Corollary 6.1, they are precisely the members of $\mathfrak{E}(J')$ and $\mathfrak{E}(J'')$. This is reminiscent of Theorem 3. We may do even better: in the lattices of closed sets in Birkhoff's theorem, the inclusion relation is set inclusion; suppose we reverse the order relation in $\mathfrak{J}''$.

DEFINITION 6.2.[8] If F and G are in $\mathfrak{J}'$ and $F \subset G$, then $F \prec G$. If F and G are in $\mathfrak{J}''$ and $G \subset F$, then $F \prec G$.

Thus the dual isomorphism becomes an isomorphism, so

THEOREM 3′. *$\mathfrak{E}(J')$ and $\mathfrak{E}(J'')$ are isomorphic lattices under $\prec$.*

In case J' and J'' are principal (and then in accordance with earlier practice would ordinarily be denoted by P and D) we may apply this to prove Theorem 3, as follows:

LEMMA 6.1. *If J' and J'' are principal then each F which is in $\mathfrak{E}(J')$ or $\mathfrak{E}(J'')$ is principal, and its generating element is in $E(J')$ or $E(J'')$ respectively. Conversely, if J' and J'' are principal and $f' \in E(J')$ is covered by $f'' \in E(J'')$, then $\mathfrak{J}'(\{f''\})$ is generated by f'.*

---

[7] Not to be confused with the unit element of a lattice. Also "I^*" is not "I"* in the sense of his next sentence.

[8] The reader will observe below that the order relation of Definition 5.1 and Theorem 4 as applied to ideals is an extension of the relation in Definition 6.2.

Proof. Then d_0 and p_0 (as used in **2-5**) exist. Suppose $F = \mathfrak{J}'(S)$, s is any element of S, and J' is generated by p_0. Then F is the principal ideal generated by $p_0 \cap s$, for $p_0 \cap s \in F$ by definition, and if $f \in F$ then $f \leq s$ and $f \leq p_0$ so $f \leq p_0 \cap s$. Also $p_0 \cap s$ is in $E(J')$ by the dual of Lemma 4. 2 (*ii*). The converse part of the lemma is obvious.

NOTE 6. 1. If J' and J'' are principal then $F \prec G$ for elements of $\mathfrak{E}(J')$ generated by f and g if and only if $f \leq g$; likewise for $\mathfrak{E}(J'')$. Thus we may obtain Theorem 3 as a corollary of Theorem 3′ and Lemma 6. 1, though if the details of the proof of Theorem 3′ ([3] p. 24-25) are taken into account the proof is no shorter.

NOTE 6. 2. The ideals of any lattice form a complete lattice[9] under set inclusion, for their set intersection will serve as meet, and the ideal generated by their set union as join. This seems well known though seldom stated; Dilworth ([5] p. 329) essentially gives it (the set of all ideals in a lattice does have a unit, namely the whole lattice). We observe however that (unlike the ideals of a ring [3] p. 36) the ideals of a lattice need not form a modular lattice; this is shown by the ideals of the lattice $NM5$ used in **5.**

We now consider the generalization of Theorem 4, starting not with a splitting of a lattice but rather with two lattices which we seek to exhibit as splinters. We have at once, using the correspondence between the pseudo-edges given in Corollary 6. 1,

LEMMA 6. 2 (5. 1′). *If the construction of Definition* 5. 1 *is applied to* $\mathfrak{J}'$, $\mathfrak{J}''$, $\mathfrak{E}(J')$, *and* $\mathfrak{E}(J'')$ *then the inclusion relations of* $\mathfrak{J}'$ *and* $\mathfrak{J}''$ *are extended to the set union* $\mathfrak{L} = \mathfrak{J}' + \mathfrak{J}''$ *which is thereby partially ordered. If in* $J' + J''$ *we set* $a \prec b$ *when the elements of* $\mathfrak{L}$ *generated by* a *and* b *are in this relation, then* $L = J' + J''$ *is partially ordered and is isomorphic to* $\mathfrak{L}$ *or a subset thereof.*

THEOREM 4′. *If conditions* (1) *and* (3)-(3′) *of Theorem* 4 *hold for* $\mathfrak{J}''$ *and* $\mathfrak{J}'$ (*the sets respectively of dual ideals in a lattice* J'' *and of ideals in a lattice* J') *and subsets* $\mathfrak{E}(J'')$ *and* $\mathfrak{E}(J')$ *thereof, then the set union* $\mathfrak{L} = \mathfrak{J}' + \mathfrak{J}''$ *is a lattice under the construction of Definition* 5. 1, *taking* $\leq$ *of that definition as the* $\prec$ *of Definition* 6. 2.

Proof. $\mathfrak{J}'$ and $\mathfrak{J}''$ are principal with J' and J'' in the rôles of p_0 and d_0, and they are lattices by Note 6. 2. Then Theorem 4 applied to $\mathfrak{J}'$ and $\mathfrak{J}''$ gives the result stated.

---

[9] In the sense of having unrestricted meets and joins.

COROLLARY 6. 2. *If the conditions of Theorem 4′ are fulfilled and each element of $\mathfrak{E}(J')$ and $\mathfrak{E}(J'')$ is principal then L as constructed in Lemma 6. 2 is a lattice—the same lattice as obtained by applying Theorem 4 to $J' + J''$ directly, with suitable choice of $E(J')$ and $E(J'')$.*

Proof. Then in particular J' and J'' are principal. Take for $E(J')$ and $E(J'')$ the generators of the elements of $\mathfrak{E}(J')$ and $\mathfrak{E}(J'')$. Then Theorem 4 is applicable to $J' + J''$, and $a \prec b$ from Theorem 4 if and only if $a \prec b$ from Lemma 6. 2.

THEOREM 4″. *If a lattice L and a splitting thereof into J' and J'' are given, then the conditions of Theorem 4′ are satisfied. L is isomorphic to the sublattice of $\mathfrak{L}$ consisting of the elements of $\mathfrak{L}$ generated by the elements of L.*

Proof. That condition (1) of Theorem 4′ holds follows from Theorem 3′. As for (3), suppose that there is given $B \in \mathfrak{J}''$. Then $\mathfrak{J}''(\mathfrak{J}'(B))$ serves as the desired C. Also $J'' \in \mathfrak{E}(J'')$, for $J'' = \mathfrak{J}''(\mathfrak{J}'(J''))$.

It remains to show the stated isomorphism; that is, if a and b in L generate A and B in the lattice $\mathfrak{L}$ of Theorem 4′, then $a \frown b$ generates $A \frown B$ and dually. *Case* 1: A and B both in $\mathfrak{J}'$. Then by Note 6. 2, $A \frown B$ is just their set intersection. But $a \frown b$ is in $A \frown B$ since A and B are ideals, and if $c \in A \frown B$ then $c \in A$ and $c \in B$ so $c \leq a$, $c \leq b$, and $c \leq a \frown b$. Thus $A \frown B$ is generated by $a \frown b$. *Case* 2: A and B both in $\mathfrak{J}''$. Then by Theorem 4′ their meet $A \frown B$ in $\mathfrak{J}''$ (that is, the dual ideal generated by their set union) is likewise their meet in $\mathfrak{L}$. Then for $c \in A \frown B$ we have that $a \frown b \leq c$ by Dilworth ([5], Definition 2. 1).[10] Thus again $a \frown b$ generates $A \frown B$. *Case* 3: $A \in \mathfrak{J}'$, $B \in \mathfrak{J}''$. Then by Theorem 4′, $A \frown B = A \frown \mathfrak{J}'(B)$. Now $A \frown \mathfrak{J}'(B)$ is generated by $a \frown b$, for $a \frown b \in A \frown \mathfrak{J}'(B)$, and if $c \in A \frown \mathfrak{J}'(B)$ then $c \leq a$, $c \leq b$, so $c \leq a \frown b$.

THEOREM 5′. *For any splitting of a given lattice L, $\mathfrak{L} = \mathfrak{J}' + \mathfrak{J}''$ is modular (distributive) if and only if conditions* (I) *and* (II) *of Theorem* 5 *hold for $\mathfrak{E}(J')$ and $\mathfrak{E}(J'')$ in $\mathfrak{J}'$ and $\mathfrak{J}''$.*

Proof. From Theorem 4″ it follows that $\mathfrak{L}$ is split into $\mathfrak{J}'$ and $\mathfrak{J}''$. That the splitting is principal is shown by the proof of Theorem 4′. Thus Theorem 5 is applicable.

COROLLARY 6. 3. *If the lattice $\mathfrak{L}$ obtained from a given lattice L is modular (distributive) then so is L.*

Proof. By the isomorphism stated in Theorem 4″.

---

[10] This condition was discovered independently by the author (cf. footnote 4) but not published.

NOTE 6.3. In this connection we call attention to a result of Dilworth ([5] Lemma 2.2) in a slightly different situation: The lattice of all ideals of a fixed lattice L is modular (distributive) if and only if L is modular (distributive).

7. Chippings. Each splinter of a lattice L is again a (sub-)lattice under the original inclusion relation. Thus a splinter may itself be splittable, and so on. This leads us to consider the result of successive splittings S_1, S_2, etc. We view the S_i as operations on the lattices which they split and indirectly on L, and call the resulting partitions of the elements of L **chippings**, denoted $\boldsymbol{C}_m = (\boldsymbol{L}, \boldsymbol{S}_1, \boldsymbol{S}_2, \cdots, \boldsymbol{S}_m)$. We suppose m finite though perhaps unbounded.

A partition class c of the chipping C_m is called a **chip** of C_m, and we write $\boldsymbol{c} \in \boldsymbol{C}_m$. Each chip is a convex sublattice of L, but the converse is not true; cf. **3.** The operations S_i are obviously permutable if the chips which they split have no elements in common; thus the sequence of splittings—the **path**—by which a chipping can be obtained is not unique. The difference can be even greater, as may be seen by considering $FL(2)$. However, the number of chips of C_m (hence the number of splittings by which it is obtained) is invariant.

We shall say $\boldsymbol{C'} > \boldsymbol{C''}$ if and only if C'' can be obtained from C' by further splitting. This direction of the inclusion sign is contrary to the concept of C'' as a successor of C', but accords with customary usage for partitions (cf. [3] p. 6; [6]). Obviously, for chips $c' \in C'$ and $c'' \in C''$, the set intersection $c'c''$ is a convex sublattice of L; if $C' \geq C''$, then $c'c''$ is c'' or void. If $c \in C_m = (L, S_1, \cdots, S_k, \cdots, S_m)$, and k is given, we denote by $g_c\,(C_k, S_{k+1}, \cdots, S_m)$ the smallest number of the given splittings by which one can obtain from C_k a chipping with c as a chip. We may say that c is of the g_c-th generation after C_k. We write

$$g(C_k, S_{k+1}, \cdots, S_m) = \max_{c \in C_m} g_c(C_k, S_{k+1}, \cdots, S_m).$$

By definition, $g_c(C_k, S_{k+1}, \cdots, S_m)$ and $g(C_k, S_{k+1}, \cdots, S_m)$ are independent of the path by which C_k is obtained, but they may depend on $S_{k+1}, \cdots, S_m$. Obviously

LEMMA 7.1. $g(C_k, \cdots, S_{m+j}) \leq g(C_k, \cdots, S_m) + g(C_m, \cdots, S_{m+j})$.

LEMMA 7.2. *Given* $C' = (L, S_1, \cdots, S_k, S'_{k+1}, \cdots, S'_m)$ *and* $C'' = (L, S_1, \cdots, S_k, S''_{k+1}, \cdots, S''_n)$, *then there exists* $C''' = (L, \cdots, S'_m, S'_{m+1}, \cdots, S'_p) = (L, \cdots, S''_n, S''_{n+1}, \cdots, S''_p)$ *such that* (1) *the chips of* C''' *are the non-void set intersections* $c'c''$ *where* $c' \in C'$ *and* $c'' \in C''$, *and*

(2) $g(C', \cdots, S'_p) \leq g(C_k, \cdots, S''_n)$ *and* $g(C'', \cdots, S''_p) \leq g(C_k, \cdots, S'_m)$.

We denote C''', which is uniquely determined by (1), by $C' \frown C''$. Condition (2) serves for induction.

Proof. The intended paths being obvious, we abbreviate $g(C', \cdots, S'_p)$ to $g(C'/C'_p)$. The lemma is easily verified if $g(C_k/C')$ or $g(C_k/C'')$ is zero or both are 1. Then we proceed by induction on $z = g(C_k/C') + g(C_k/C'')$. Suppose the lemma is true for $z < z_0$ and that we have C' and C'' with $z = z_0$. If this has not already been taken care of, then say $g(C_k/C'') = p > 1$. It is easily seen that there is another path to C'', say $C'' = (L, \cdots, S_k, S^*_{k+1}, \cdots, S^*_n)$, such that for some j, $g(C_k/C^*_{k+j}) = 1$ and $g(C^*_{k+j}/C'') = p - 1$. Denote C^*_{k+j} by C^*. Now

$$g(C_k/C') + g(C_k/C^*) = g(C_k/C') + 1 < z_0,$$

so by induction there exists $C' \frown C^*$, and $g(C^*/C' \frown C^*) \leq g(C_k/C')$. Then

$$g(C^*/C'') + g(C^*/C' \frown C^*) \leq p - 1 + g(C_k/C') < z_0,$$

so again by induction there exists $(C' \frown C^*) \frown C''$, which we call C'''. One can verify that (1) and (2) hold. Q. E. D. We may remark that it would not suffice to make an induction on the length of the path, for this does not obey (2).

THEOREM 8. *The set of all chippings of a lattice is a lattice.*

Proof. We use the inclusion relation defined above. The $C' \frown C''$ of Lemma 7.2 is a lower bound to C' and C'' by construction; that it is greatest follows from (1) of that lemma. As for least upper bound, we observe that the ascending chain condition holds and that the set of chippings has a unit, namely (L). Then least upper bounds exist by an argument like the proof of Theorem 2.2 of [3] (p. 17).

Condition (1) of Lemma 7.2 shows that $C' \frown C''$ is just the common refinement of C' and C'' as partitions; however, examples (using the convexity of chips) show that the join of C' and C'' as partitions need not be a chipping. Thus the lattice of chippings is not a sublattice of the lattice of all partitions. In fact, though the latter is a matroid lattice (Birkhoff [3] p. 67) the former need not be, as is seen in the case of $FL(2)$.

From Theorem 8 we see that the unsplittable chips of a lattice are uniquely determined. From **3** it follows that any finite distributive lattice has a chipping whose chips consist of single elements. Also, we might consider only principal splittings. The above arguments (e. g., Theorem 8) can equally well be carried out under this restriction. In this case we can define the edge of a chipping as the collection of all the elements of the edges of the splittings involved, with proper multiplicity. By examining simple cases as in the proof of Lemma 7.2, and induction as in the proof of the Jordan-Hölder Theorem, it can be shown that the edge of a chipping C is independent of the path by which C is reached.

8. Unsolved problems. (1) Characterize lattices of partitions[11] ([3] p. 146). This would be partly done if one could show that $FL(n)$ could be represented as such a lattice. (2) Characterize $C' \vee C''$ (7). (3) Can Stone's and Birkhoff's work (cited in **3**) on splitting distributive lattices be generalized? We know from **3** that not every modular lattice can be split. Ward and Dilworth have generalized this work in a slightly different direction ([9] p. 332). (4) What further properties has the lattice of all chippings of a fixed lattice?

HARVARD UNIVERSITY
AND
THE UNIVERSITY OF PENNSYLVANIA.

BIBLIOGRAPHY

[1] G. Birkhoff, "On the combination of subalgebras," *Proceedings of the Cambridge Philosophical Society,* Vol. 29 (1933), pp. 441-464.
[2] ———, "On the structure of abstract algebras," *ibid.,* Vol. 31 (1935), pp. 433-454.
[3] ———, "Lattice Theory," *American Mathematical Society Colloquium Publications* XXV, New York (1940).
[4] ———, "Rings of sets," *Duke Mathematical Journal,* Vol. 3 (1937), pp. 443-454.
[5] R. P. Dilworth, "Ideals in Birkhoff lattices," *Transactions of the American Mathematical Society,* Vol. 49 (1941), pp. 325-353.
[6] P. Dubreil and M.-L. Dubreil-Jacotin, "Théorie algébrique des relations d'équivalence," *Jour. de Math.,* Vol. 18 (1939), pp. 63-95.
[7] M. H. Stone, "The theory of representations for Boolean algebras," *Transactions of the American Mathematical Society,* Vol. 40 (1936), pp. 37-111.
[8] ———, "Topological representations of distributive lattices and Brouwerian logics," *Casopis pro Pestovani Mat. a Fys.,* Vol. 67 (1937), pp. 1-25.
[9] M. Ward and R. P. Dilworth, "Evaluations over residuated structures," *Annals of Mathematics* (2), Vol. 40 (1939), pp. 328-337.
[10] P. M. Whitman, "Free lattices," *Annals of Mathematics* (2), Vol. 42 (1941), pp. 325-330.
[11] ———, "Free lattices II," *ibid.,* Vol. 43 (1942), pp. 104-115.

---

[11] Added in proof: In this connection see O. Ore, "Theory of equivalence relations," *Duke Mathematical Journal,* Vol. 9 (1942), pp. 573-628.

ON A PROBLEM OF GEÖCZE.*

By Tibor Radó.

1. Introduction.

1. 1. The Lebesgue area $A(S)$ of a continuous surface S is defined by the formula

$$A(S) = gr.\, l.\, b.\, \underline{\lim}\, E(\Pi_n),$$

where the greatest lower bound is taken with respect to all sequences of polyhedra Π_n converging to S in the Fréchet sense, and $E(\Pi_n)$ is the area of Π_n in the elementary sense.[1]

1. 2. Let us denote by $A^*(S)$ the quantity obtained by requiring, in the preceding definition, that the polyhedra Π_n be *inscribed* in S. By restricting the polyhedra, we certainly do not decrease the greatest lower bound. Hence $A^*(S) \geqq A(S)$. Obviously then, we have surely $A^*(S) = A(S)$ if $A(S) = \infty$.

1. 3. It may be argued that it would be more natural to consider $A^*(S)$, defined in terms of *inscribed* polyhedra, as the area of S. Still, $A(S)$ is generally accepted as the area, probably because very little is known about $A^*(S)$. The problem of determining whether $A^*(S) = A(S)$ may properly be called *the problem of Geöcze*, because Geöcze was the first to study this question in a substantial manner. Several of his many profound papers on the area of surfaces are devoted to this subject.[2] The purpose of the present paper is to improve the result obtained by Geöcze concerning this problem.

1. 4. We shall be concerned with a surface S of the special form

$$(1) \qquad S:\ z = f(x, y), \quad (x, y) \in I_0,$$

where I_0 is the unit square

$$I_0:\ 0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1,$$

and $f(x, y)$ is *continuous* in I_0. For $0 \leqq x \leqq 1$, let $L(x)$ denote the length of the arc $x = \text{const.}$, $0 \leqq y \leqq 1$, $z = f(x, y)$. Similarly, let us denote, for

---

* Received March 9, 1942; Presented to the American Mathematical Society at the meeting in Chicago, April 1942.

[1] For general information concerning the Lebesgue area of a surface, see for example Radó [1], Chapter I.

[2] See Geöcze [1], [2], [3].

$0 \leqq y \leqq 1$, by $M(y)$ the length of the arc $y = \text{const.}$, $0 \leqq x \leqq 1$, $z = f(x, y)$. Geöcze proved the equality $A^*(S) = A(S)$ under the following assumptions.[3]

(i) The functions $L(x)$, $M(y)$ are bounded.

(ii) $f(x, y)$ satisfies a uniform Lipschitz condition with respect to one of the variables, say x. That is, there exists a finite constant L_0 such that

$$| f(x_2, y) - f(x_1, y) | \leqq L_0 \, | x_2 - x_1 |$$

for all $0 \leqq x_1 \leqq 1$, $0 \leqq x_2 \leqq 1$, $0 \leqq y \leqq 1$.

1. 5. We shall extend this result by proving the following theorem.

THEOREM. *Suppose that the function* $f(x, y)$, *defining the continuous surface* (1), *is absolutely continuous with respect to* x *for a. e.* y *in* $0 \leqq y \leqq 1$ (*or, alternatively, that* $f(x, y)$ *is absolutely continuous with respect to* y *for a. e.* x *in* $0 \leqq x \leqq 1$). *Then* $A^*(S) = A(S)$.

1. 6. Let us recall that $A^*(S) = A(S)$ if $A(S) = \infty$ (see **1. 2**). In proving the preceding theorem we can therefore assume that $A(S) < \infty$. But then, by a result of Geöcze,[4] the functions $L(x)$, $M(y)$ of **1. 4.** are summable. Hence, in proving our theorem, we shall really use the following assumptions.

(i)* The functions $L(x)$, $M(y)$ are summable.

(ii)* For a. e. y in $0 \leqq y \leqq 1$, the function $f(x, y)$ is absolutely continuous with respect to x.

Stated in this manner, our assumptions may be more readily compared with those of Geöcze (see **1. 4**).

1. 7. Geöcze stated that he found a proof of $A^*(S) = A(S)$, for surfaces of the form (1), without any restriction upon $f(x, y)$ beyond *mere continuity*.[5] He also stated that his proof depended upon a thorough study of Riemann-type approximations to Lebesgue integrals and also upon a topological study of the sections $z = \text{const.}$ of the surface $z = f(x, y)$. While he did publish extensive and profound investigations in both of these subjects,[6] the present writer was unable to find in his published work either a general proof of the theorem $A^*(S) = A(S)$ or sufficient information to construct such a proof. The proof presented in this paper for the partial result stated in **1. 5** makes essential use of the work of Geöcze on Riemann-type approximations, but the author was unable to utilize the topological results of Geöcze on the sections

---

[3] See Geöcze [1].

[4] See Geöcze [4].

[5] See Geöcze [2].

[6] See Geöcze [2], [3].

$z =$ const. of the surface which he stated were essential for his proof. For this reason, it is unlikely that our proof is similar to the one Geöcze had in mind for the general theorem.

1.8. The method followed in this paper, in proving the theorem of **1.5**, is essentially a refinement of the method used by Geöcze[7] to prove the result stated in **1.4**. Various changes were necessary on account of the greater generality of our assumptions, and substantial simplifications were achieved by properly modifying some of the auxiliary quantities introduced by Geöcze (see especially **3.7**). However, a definite effort was made to disturb as little as possible the original line of thought of Geöcze.

1.9. The literature on the area of surfaces contains many beautiful results concerning approximations by inscribed polyhedra. As far as the author is aware, no result of a generality comparable to that of our theorem in **1.5.** has been previously obtained. For this reason, we restrict ourselves to a reference to the work of Kempisty [1, 2], because he proved several results of great interest beyond the mere equality $A^*(S) = A(S)$. On the other hand, his assumptions are far more restrictive than ours (cf. Kempisty [1]).

2. On Riemann-type approximations.

2.1. In studying the continuous surface S given by (1), we can and shall assume that $A(S) < \infty$ (cf. **1.6**). As a consequence, condition (i)* of **1.6** is satisfied. Furthermore, condition (ii)* in **1.6** will be assumed throughout the paper.

2.2. X will designate a subdivision of the interval $0 \leqq x \leqq 1$ of the form

$$X\colon \quad 0 = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = 1.$$

It will be convenient to describe the generic interval of X by writing $x' \leqq x \leqq x''$. Henceforth, x' will always refer to the left-hand end-point and x'' to the right-hand end-point of a generic interval of X. The following symbols will be used. $\| X \|$ will designate the norm of X, that is the maximum length of the sub-intervals of X, $q(X)$ the number of intervals of X, and $p(X)$ the parameter of regularity of X, that is the ratio of the longest to the shortest interval of X. Thus always $p(X) \geqq 1$. We shall also use subdivisions Y of the interval $0 \leqq y \leqq 1$ and the corresponding symbols y', y'', $\| Y \|$, $q(Y)$, $p(Y)$. A pair of subdivisions X, Y determine, in an obvious manner, a subdivision into rectangles of the unit square I_0: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. This subdivision will be denoted by XY.

---

[7] See Geöcze [1].

2.3. Let $\psi(\Delta)$ be an interval-function defined for all closed sub-intervals Δ in $0 \leqq x \leqq 1$. Then X_ψ will denote the sum of the values of $\psi(\Delta)$ for the sub-intervals of X. The symbol Y_ψ will be used in a similar sense. If $\psi(r)$ is a rectangle-function defined for all closed rectangles $r: x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ in the unit square I_0, then XY_ψ will denote the sum of the values of $\psi(r)$ for the rectangles of the subdivision XY.

2.4. Let us consider now the function $L(x)$ of **1.4**. Since $L(x)$ is summable in $0 \leqq x \leqq 1$, it is finite a.e. in this interval. We agree once and for all to use only subdivisions X such that $L(x)$ is finite at every interior point of division of X. Of course, $L(0)$ and $L(1)$ may be infinite. For this reason, it may happen that in certain summations X_ψ the terms depending upon $x' = 0$ or $x'' = 1$ are not finite. In such a case, we shall write X^*_ψ, instead of X_ψ, to indicate that such exceptional terms are to be discarded, and hence their influence should be studied separately. Similar conventions will apply to the symbols Y, Y^* in relation to the function $M(y)$ of **1.4**.

An important example is furnished by the summations

$$X^*L(x')(x''-x'), \qquad X^*L(x'')(x''-x'),$$
$$Y^*M(y')(y''-y'), \qquad Y^*M(y'')(y''-y').$$

It will be a matter of great importance for us to know that by proper choice of X and Y these summations may be brought as close as we wish to

$$\int_0^1 L(x)\,dx, \qquad \int_0^1 M(y)\,dy$$

respectively, and that furthermore X, Y may be made to satisfy a variety of further conditions necessary in the sequel. The required facts will be summarized presently.[8]

2.5. Let there be given two constants ϵ, p such that $0 < \epsilon < 1$, $p > 1$. Then we have an $\eta > 0$, $0 < \eta < 1$, such that the following statements hold. Given any measurable subset e_x of $0 \leqq x \leqq 1$ such that $|e_x| < \eta$, and any measurable subset e_y of $0 \leqq y \leqq 1$ such that $|e_y| < \eta$, there exists a sequence of subdivisions $X_\nu Y_\nu$ of the unit square I_0, depending upon ϵ, p, e_x, e_y and the functions $L(x)$, $M(y)$, with the following properties.

(a) $\|X_\nu\| \to 0, \qquad \|Y_\nu\| \to 0.$

(b) $q(X_\nu) = q(Y_\nu), \qquad (\nu = 1, 2, \cdots).$

(c) $p(X_\nu) < p, \qquad p(Y_\nu) < p, \qquad (\nu = 1, 2, \cdots).$

---

[8] Except for slight changes in wording, these facts are proved in Geöcze [3]. Subsequently, these results of Geöcze were extended in Hahn [1].

(d) For each ν, $X_{\nu+1}$ is a refinement of X_ν and $Y_{\nu+1}$ is a refinement of Y_ν.

(e) $L(x)$ is finite at every interior point of division of X_ν, and $M(y)$ is finite at every interior point of division of Y_ν.

(f) $$\underset{x'\in e_x}{X^*_\nu} L(x')(x''-x') < \epsilon, \quad \underset{x''\in e_x}{X^*_\nu} L(x'')(x''-x') < \epsilon,$$
$$\underset{y'\in e_y}{Y^*_\nu} M(y')(y''-y') < \epsilon, \quad \underset{y''\in e_y}{Y^*_\nu} M(y'')(y''-y') < \epsilon.$$

(g) $$| X^*_\nu L(x')(x''-x') - \int_0^1 L(x)\,dx \,| < \epsilon,$$
$$| X^*_\nu L(x'')(x''-x') - \int_0^1 L(x)\,dx \,| < \epsilon,$$
$$| Y^*_\nu M(y')(y''-y') - \int_0^1 M(y)\,dy \,| < \epsilon,$$
$$| Y^*_\nu M(y'')(y''-y') - \int_0^1 M(y)\,dy \,| < \epsilon.$$

2. 6. In the sequel, ϵ and p will be arbitrarily assigned and then kept fixed. Next, a corresponding η, in the sense of **2. 5**, will be taken and then kept fixed. Then the sets e_x and e_y will be selected in the following manner. $L(x)$ being summable, we can select e_x to satisfy the following conditions. (1) $| e_x | < \eta$. (2) The complement E_x of e_x, with respect to $0 \leqq x \leqq 1$, is a closed set. (3) $L(x)$ is finite and continuous on E_x.[9]

2. 7. The selection of e_y is more complicated, since we shall bring into play condition (ii)* of **1. 6**. Since $M(y)$ is summable, we can first select a set e'_y in $0 \leqq y \leqq 1$ to satisfy the following conditions. (1) $| e'_y | < \eta/3$. (2) The complement E'_y of e'_y with respect to $0 \leqq y \leqq 1$ is a closed set. (3) $M(y)$ is finite and continuous on E'_y. Next, since $f(x,y)$ is absolutely continuous with respect to x for a. e. y in $0 \leqq y \leqq 1$, we can select a subset e_y'' of E_y' with the following properties. (1) $| e_y'' | < \eta/3$. (2) The set $E_y'' = E_y' - e_y''$ is closed. (3) For every $y \in E_y''$, $f(x,y)$ is absolutely continuous with respect to x. Take now a $y \in E_y''$. Since then $f(x,y)$ is absolutely continuous with respect to x, for every interval $x' \leqq x \leqq x''$ the length s of the arc $y = y$, $x' \leqq x \leqq x''$, $z = f(x,y)$ is given by

$$(2) \qquad s = \int_{x'}^{x''} [1 + (\partial f/\partial x)^2]^{1/2} dx.$$

Consider then any finite system of such arcs, with lengths $s_1, s_2, \cdots$, corresponding to the given $y \in E_y''$ and to a system of non-overlapping intervals

---

[9] See Saks [1], p. 72.

$x_1' \leqq x \leqq x_1''$, $x_2' \leqq x \leqq x_2''$, $\cdots$ of total length $\leqq 1/k$, where k is a positive integer. Let $\omega_k(y)$ designate the least upper bound of $s_1 + s_2 + \cdots$ for all such systems, with y and k fixed. By the absolute continuity of the indefinite Lebesgue integral, it follows from (2) that for fixed $y \in E_y''$

$$\omega_k(y) \to 0 \text{ for } k \to \infty.$$

Further, for fixed k, $\omega_k(y)$ is clearly a lower semi-continuous, and hence measurable, function of $y \in E_y''$. Hence we have a subset e_y''' of E_y'' with the following properties. (1) $| e_y''' | < \eta/3$. (2) The set $E_y''' = E_y'' - e_y'''$ is closed. (3) $\omega_k(y)$ is continuous on E_y''', $k = 1, 2, \cdots$. Thus we have, on the closed set E_y''', a decreasing sequence of continuous functions $\omega_k(y)$ converging to zero. By the theorem of Dini, the convergence of such a sequence is *uniform*.[10] Hence if we define, for $\psi > 0$,

$$\omega(\psi) = \underset{\substack{y \in E_y''' \\ 1/k \leqq \psi}}{\text{l. u. b.}}\, \omega_k(y)$$

then $\omega(\psi) \to 0$ for $\psi \to 0$.

2.8. We put now $e_y = e_y' + e_y'' + e_y'''$. Then the complement of e_y, with respect to $0 \leqq y \leqq 1$, is the set E_y''' which we shall denote by E_y, and the following statements hold.

(1) $| e_y | < \eta$.

(2) E_y, the complement of e_y with respect to $0 \leqq y \leqq 1$, is closed.

(3) $M(y)$ is finite and continuous on E_y.

(4) We have a function $\omega(\psi)$, defined for $\psi > 0$, such that $\omega(\psi) \to 0$ for $\psi \to 0$, and such that the following holds. If $y_0 \in E_y$, and if $x_1' \leqq x \leqq x_1''$, $x_2' \leqq x \leqq x_2''$, $\cdots$ are non-overlapping intervals in $0 \leqq x \leqq 1$ with total length $\leqq \psi$, then the sum of the lengths of the arcs

$$\begin{aligned} y &= y_0, x_1' \leqq x \leqq x_1'', \quad & z &= f(x, y_0), \\ y &= y_0, x_2' \leqq x \leqq x_2'', \quad & z &= f(x, y_0), \\ &\cdots\cdots\cdots & &\cdots\cdots \end{aligned}$$

is $\leqq \omega(\psi)$.

2.9. For given $0 < \epsilon < 1$ and $p > 1$, and a corresponding $\eta > 0$, we have now selected the sets e_x, e_y, whose complements, with respect to $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$ respectively, we denoted by E_x, E_y. Thereupon we select a sequence $X_\nu Y_\nu$ as described in **2.5.** Until further notice, ϵ, p, e_x, e_y, E_x, E_y, $\omega(\psi)$, $X_\nu Y_\nu$ will be kept fixed. In the sequel, we shall use the Landau symbol O in writing relations of the form $A = O(B)$, with the following meaning:

---

[10] This is a well-known theorem of Dini. See any standard text on Real Variables.

there exists some finite constant $\Gamma > 0$, depending only upon the given constant $p > 1$ and the given function $z = f(x, y)$, but *independent* of ϵ, η, e_x, e_y, E_x, E_y, $\omega(\psi)$, $X_\nu Y_\nu$, such that $A \leqq \Gamma B$.

2.10. For the sequence $X_\nu Y_\nu$ we have (see **2.5**, (b)) $q(X_\nu) = q(Y_\nu)$. Let us designate this common value by q_ν. From the conditions (a), (b), (c) in **2.5** we infer easily the following relations for the generic intervals $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ of X_ν, Y_ν respectively.

$$x'' - x' = O(1/q_\nu), \quad y'' - y' = O(1/q_\nu), \quad (x'' - x')(y'' - y') = O(1/q_\nu^2),$$

$$\frac{1}{x'' - x'} = O(q_\nu), \quad \frac{1}{y'' - y'} = O(q_\nu), \quad \frac{1}{(x'' - x')(y'' - y')} = O(q_\nu^2),$$

$$x'' - x' = O(y'' - y'), \quad y'' - y' = O(x'' - x').$$

2.11. The sequence $X_\nu Y_\nu$ gives rise to two sequences of functions $l_\nu(x)$, $m_\nu(y)$ defined as follows:

$$l_\nu(x) = Y_\nu[(y'' - y')^2 + (f(x, y'') - f(x, y'))^2]^{1/2}, \qquad 0 \leqq x \leqq 1,$$

$$m_\nu(y) = X_\nu[(x'' - x')^2 + (f(x'', y) - f(x', y))^2]^{1/2}, \qquad 0 \leqq y \leqq 1.$$

Thus $l_\nu(x)$, $m_\nu(y)$ represent the lengths of polygons inscribed in the sections $x =$ const., $y =$ const. respectively of the surface (1). By condition (d) in **2.5** we have therefore

$$l_1(x) \leqq l_2(x) \leqq \cdots, \tag{3}$$

and, by the geometrical meaning of $l_\nu(x)$,

$$l_\nu(x) \to L(x) \quad \text{for} \quad \nu \to \infty. \tag{4}$$

Since $l_\nu(x)$ is obviously a continuous function of x, and since $L(x)$ is continuous on the closed set E_x, (3) and (4) imply, by the theorem of Dini, that $l_\nu(x)$ converges to $L(x)$ uniformly on E_x.[11] Similarly, it follows that $m_\nu(y)$ converges to $M(y)$ uniformly on E_y. Therefore, if we define

$$\delta_\nu = \max_{x \in E_x} (L(x) - l_\nu(x)) + \max_{y \in E_y} (M(y) - m_\nu(y)),$$

then

$$\delta_\nu \to 0 \quad \text{for} \quad \nu \to \infty \tag{5}$$

and

$$\begin{aligned} L(x) - l_\nu(x) &\leqq \delta_\nu, & x &\in E_x, \\ M(y) - m_\nu(y) &\leqq \delta_\nu, & y &\in E_y. \end{aligned} \tag{6}$$

2.12. Let d_ν designate the greatest one of the oscillations of $f(x, y)$ in the various rectangles of $X_\nu Y_\nu$. From (a) in **2.5** it follows that $d_\nu \to 0$ for $\nu \to \infty$.

---

[11] See [10].

2.13. By **2.6**, **2.8** the functions $L(x)$, $M(y)$ are finite and continuous on the closed sets E_x, E_y respectively. We can define therefore

$$L = \max_{x \in E_x} L(x), \qquad M = \max_{y \in E_y} M(y).$$

3. Auxiliary rectangle functions.

3.1. We consider the general term of the sequence of subdivisions $X_\nu Y_\nu$ (see **2.5** to **2.11**) and a generic rectangle $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ of $X_\nu Y_\nu$. The reader is requested to consult **2** whenever in doubt about the meaning of the various symbols used.

3.2. We shall associate with the rectangle

$$r: \quad x' \leqq x \leqq x'', \qquad y' \leqq y \leqq y'' \tag{7}$$

the following four points of the surface (1).

$$\begin{aligned}
A&: \quad x = x', \quad y = y', \quad z = f(x', y'), \\
B&: \quad x = x', \quad y = y'', \quad z = f(x', y''), \\
C&: \quad x = x'', \quad y = y'', \quad z = f(x'', y''), \\
D&: \quad x = x'', \quad y = y', \quad z = f(x'', y').
\end{aligned}$$

The symbols ABC, ACD will denote the areas, taken with the *positive* sign, of the triangles with the vertices A, B, C and A, C, D respectively. The orthogonal projections of the points A, B, C, D upon the planes xy, yz, zx respectively will be denoted by A_i, B_i, C_i, D_i, $i = 1, 2, 3$. The areas of the triangles $A_iB_iC_i$, $A_iC_iD_i$, $i = 1, 2, 3$, each taken with a *convenient* sign to be explained in a moment, will be denoted by $A_iB_iC_i$, $A_iC_iD_i$. Whatever signs we shall select for these areas, we shall have

$$ABC = \| \mathfrak{x}_1 \|, \qquad ACD = \| \mathfrak{x}_2 \|, \tag{8}$$

where $\mathfrak{x}_1$, $\mathfrak{x}_2$ are vectors, defined by their components as follows.

$$\begin{aligned}
\mathfrak{x}_1 &= (A_1B_1C_1, A_2B_2C_2, A_3B_3C_3), \\
\mathfrak{x}_2 &= (A_1C_1D_1, A_2C_2D_2, A_3C_3D_3).
\end{aligned} \tag{9}$$

The quantity

$$P_\nu = X_\nu Y_\nu (ABC + ACD)$$

represents then the area, in the elementary sense, of a polyhedron Π_ν inscribed in the surface S. By condition (a) in **2.5**, the sequence Π_ν converges to the surface (1) in the Fréchet sense, and hence we have, by the definition of $A^*(S)$ in **1.2**,

$$\varliminf_{\nu\to\infty} P_\nu \geqq A^*(S) \geqq A(S).$$

3.3. We make the following agreements concerning the signs of $A_iB_iC_i$, $A_iC_iD_i$.

$$A_1B_1C_1 = \tfrac{1}{2}(x''-x')(y''-y'), \qquad A_1C_1D_1 = \tfrac{1}{2}(x''-x')(y''-y'),$$
$$A_2B_2C_2 = \tfrac{1}{2}(y''-y')(f(C_1)-f(B_1)), \quad A_2C_2D_2 = \tfrac{1}{2}(y''-y')(f(D_1)-f(A_1)),$$
$$A_3B_3C_3 = \tfrac{1}{2}(x''-x')(f(B_1)-f(A_1)), \quad A_3C_3D_3 = \tfrac{1}{2}(x''-x')(f(C_1)-f(D_1)).$$

We use, for brevity, $f(A_1)$ instead of $f(x',y')$ and so on.

3.4. We define

$$\begin{aligned} a &= (x''-x')(y''-y'),\\ b &= \tfrac{1}{2}(y''-y')(f(D_1)+f(C_1)-f(B_1)-f(A_1)),\\ c &= \tfrac{1}{2}(x''-x')(f(B_1)+f(C_1)-f(A_1)-f(D_1)),\\ n &= (a^2+b^2+c^2)^{1/2}. \end{aligned}$$

If (a,b,c) denotes the vector with components a, b, c, then we have (cf. **3.2**, **3.3**)

$$(a,b,c) = \mathfrak{x}_1+\mathfrak{x}_2, \qquad n = \|\mathfrak{x}_1+\mathfrak{x}_2\|.^{12}$$

3.5. The inscribed polyhedron Π_ν (see **3.2**) can be represented by an equation of the form

$$\Pi_\nu\colon \quad z = \phi_\nu(x,y), \quad (x,y)\in I_0.$$

We have then, for example, for $x'\leqq x\leqq x''$, $y=y''$ the formula

$$\phi_\nu(x,y'') = f(B_1) + \frac{f(C_1)-f(B_1)}{x''-x'}(x-x').$$

Similarly, $\phi_\nu(x,y)$ is a linear function on each of the sides of the rectangle (7) of $X_\nu Y_\nu$. From this remark we obtain the formulas

$$b = \int_{y'}^{y''} (\phi_\nu(x'',y)-\phi_\nu(x',y))\,dy,$$
$$c = \int_{x'}^{x''} (\phi_\nu(x,y'')-\phi_\nu(x,y'))\,dx.$$

3.6. We define

$$\begin{aligned} \beta &= \int_{y'}^{y''} (f(x'',y)-f(x',y))\,dy,\\ \gamma &= \int_{x'}^{x''} (f(x,y'')-f(x,y'))\,dx,\\ \tau &= (a^2+\beta^2+\gamma^2)^{1/2}. \end{aligned}$$

---

[12] $\|\mathfrak{x}\|$ designates the length of the vector $\mathfrak{x}$.

3.7. In defining β, γ we deviate from Geöcze[13] who works with

$$\bar{\beta} = \int_{y'}^{y''} | f(x'',y) - f(x',y) | \, dy,$$
$$\bar{\gamma} = \int_{x'}^{x''} | f(x,y'') - f(x,y') | \, dx,$$
$$\bar{\tau} = (\alpha^2 + \bar{\beta}^2 + \bar{\gamma}^2)^{1/2}.$$

That is, we omit the absolute value signs appearing in the Geöcze expressions $\bar{\beta}$, $\bar{\gamma}$. The advantages of this modification were noticed in previous literature (cf. the last footnote on p. 626 in Radó [3]; see also Morrey [1]). The quantities β, γ will permit us to dispense with a considerable amount of most ingenious but quite cumbersome geometrical detail in the method of Geöcze. We have to satisfy ourselves, however, that certain important properties of $\bar{\beta}$, $\bar{\gamma}$, $\bar{\tau}$ are shared by β, γ, τ.

3.8. We shall need the following facts.

(a) $\operatorname{Lim} XY\tau$ exists, for every sequence of subdivisions XY of I_0, provided only that $\| X \| \to 0$, $\| Y \| \to 0$, and the limit is independent of the choice of the sequence XY.

(b) This limit, to be denoted by T, is $\leqq A(S)$.[14]

(c) These statements remain valid for sequences of subdivisions of any sub-rectangle $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$. The corresponding limit will be denoted by t.

(d) For every sub-rectangle we have $\tau \leqq t$.

(e) $XYt = T$ for every subdivision XY of I_0.

(f) As a direct consequence of the preceding statements, we have $XY\tau \leqq T$ for every subdivision XY of I_0, and $XY(t - \tau) \to 0$ for every sequence of subdivisions XY such that $\| X \| \to 0$, $\| Y \| \to 0$.

3.9. The proofs of these statements are precisely the same as in the case of the original Geöcze expressions $\bar{\beta}$, $\bar{\gamma}$, $\bar{\tau}$. The reader is therefore referred to the excellent presentation in Saks [1], Chapter V, especially Theorem 3.9, p. 168, or to Morrey [1].

3.10. We define, for the rectangle (7),

$$H = f(A_1) - f(B_1) + f(C_1) - f(D_1). \tag{10}$$

---

[13] Geöcze [1].

[14] In fact, $T = A(S)$ for every continuous surface (1). This theorem was surmised by Geöcze who proved it however only in special cases. The general theorem was proved by the author (Radó [2]). See also the excellent presentation in Saks [1]. These authors used the original Geöcze quantities, but the proofs remain valid, with obvious modifications, for the modified quantities β, γ, τ.

Further, $s_{x'}$, $s_{x''}$, $s_{y'}$, $s_{y''}$ will denote the lengths of the arcs

$$\begin{aligned} x &= x', & y' \leqq y \leqq y'', && z = f(x', y), \\ x &= x'', & y' \leqq y \leqq y'', && z = f(x'', y), \\ y &= y', & x' \leqq x \leqq x'', && z = f(x, y'), \\ y &= y'', & x' \leqq x \leqq x'', && z = f(x, y''). \end{aligned}$$

$\sigma_{x'}$, $\sigma_{x''}$, $\sigma_{y'}$, $\sigma_{y''}$ will denote the lengths of the chords of these arcs. We also define

$$\theta_{x'} = \int_{y'}^{y''} | f(x', y) - \phi_\nu(x', y) | \, dy, \quad \theta_{x''} = \int_{y'}^{y''} | f(x'', y) - \phi_\nu(x'', y) | \, dy,$$

$$\theta_{y'} = \int_{x'}^{x''} | f(x, y') - \phi_\nu(x, y') | \, dx, \quad \theta_{y''} = \int_{x'}^{x''} | f(x, y'') - \phi_\nu(x, y'') | \, dx.$$

3.11. Put $\xi = \frac{1}{2}(x' + x'')$. We define then

$$\rho = \int_{x'}^{\xi} (\phi_\nu(x, y') - f(x, y'))\,dx + \int_{\xi}^{x''} (f(x, y') - \phi_\nu(x, y'))\,dx + \\ + \int_{x'}^{\xi} (f(x, y'') - \phi_\nu(x, y''))\,dx + \int_{\xi}^{x''} (\phi_\nu(x, y'') - f(x, y''))\,dx.$$

3.12. From (10), **3.3**, (9) we obtain the identity

$$\mathfrak{x}_1 - \mathfrak{x}_2 = (0, \tfrac{1}{2}(y'' - y')H, -\tfrac{1}{2}(x'' - x')H).$$

3.13. We subdivide now the rectangle (7) into two rectangles by the line $x = \xi = \frac{1}{2}(x' + x'')$. Let us use the superscript 1 to refer to the rectangle $x' \leqq x \leqq \xi$, $y' \leqq y \leqq y''$ and the superscript 2 to refer to the rectangle $\xi \leqq x \leqq x''$, $y' \leqq y \leqq y''$. We obtain then from **3.6**, (10), **3.11** the identity

$$H = \frac{4}{x'' - x'} [\gamma^{(2)} - \gamma^{(1)} + \rho].$$

3.14. Since $A(S) < \infty$ by assumption, the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ exist a. e. in I_0 and are summable there (see Saks [1], p. 181). By the absolute continuity of the indefinite Lebesgue integral we have therefore, for $\psi > 0$, a function $\Omega(\psi)$ such that the following holds.

(a) $\Omega(\psi) \to 0$ for $\psi \to 0$.

(b) If $r_1, r_2, \cdots, r_j, \cdots$ is any system of non-overlapping rectangles in I_0 with total area $\leqq \psi$, then

$$\sum_j \iint_{r_j} | \partial f/\partial x | \, dxdy \leqq \Omega(\psi),$$

$$\sum_j \iint_{r_j} | \partial f/\partial y | \, dxdy \leqq \Omega(\psi),$$

4. Inequalities.

4.1. If $\mathfrak{a}_1$, $\mathfrak{a}_2$ are vectors, in Euclidean three-space, then it follows from the definition of $\mathfrak{a}_1 + \mathfrak{a}_2$ that

$$u = \| \mathfrak{a}_1 \|, \quad v = \| \mathfrak{a}_2 \|, \quad w = \| \mathfrak{a}_1 + \mathfrak{a}_2 \|$$

are the lengths of the sides of a triangle whose area we shall denote by Δ.

4.2. From the preceding remark the following inequalities can be derived immediately (the first three being, of course, well known).

$$\| \mathfrak{a}_1 + \mathfrak{a}_2 \| \leqq \| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \|,$$

$$\big| \| \mathfrak{a}_1 \| - \| \mathfrak{a}_2 \| \big| \leqq \| \mathfrak{a}_1 - \mathfrak{a}_2 \|, \tag{11}$$

$$\| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \| \leqq \| \mathfrak{a}_1 + \mathfrak{a}_2 \| + \| \mathfrak{a}_1 - \mathfrak{a}_2 \|, \tag{12}$$

$$\| \mathfrak{a}_1 \times \mathfrak{a}_2 \| \leqq 2[\| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \|]^{3/2}[\| \mathfrak{a}_1 \| + \| \mathfrak{a}_2 \| - \| \mathfrak{a}_1 + \mathfrak{a}_2 \|]^{1/2}. \tag{13}$$

In the last inequality, $\mathfrak{a}_1 \times \mathfrak{a}_2$ designates the vector product of $\mathfrak{a}_1$, $\mathfrak{a}_2$.

4.3. The proof of (13) is as follows. By a well-known elementary formula we have

$$\Delta^2 = \mu(\mu - u)(\mu - v)(\mu - w),$$

where $\mu = (u + v + w)/2$. Thus $\mu \leqq u + v$. Hence

$$\Delta^2 \leqq \mu^3(u + v - w) \leqq (u + v)^3(u + v - w),$$

$$\Delta \leqq (u + v)^{3/2}(u + v - w)^{1/2}. \tag{14}$$

Since $\| \mathfrak{a}_1 \times \mathfrak{a}_2 \| = 2\Delta$, (13) is proved.

4.4. Let there be given, in $x_1 \leqq x \leqq x_2$, a continuous function $f(x)$. We assume that the length s of the arc

$$x_1 \leqq x \leqq x_2, \qquad y = f(x) \tag{15}$$

is finite. We denote by σ the length of the chord of this arc. Let

$$x_1 \leqq x \leqq x_2, \qquad y = \phi(x) \tag{16}$$

be the equation of the chord. We assert the inequalities

$$\int_{x_1}^{x_2} | f(x) - \phi(x) | \, dx \leqq s(x_2 - x_1), \tag{17}$$

$$\int_{x_1}^{x_2} | f(x) - \phi(x) | \, dx \leqq 2s^{3/2}(s - \sigma)^{1/2}. \tag{18}$$

If we drop the assumption that s is finite we have the inequality

$$\int_{x_1}^{x_2} | f(x) - \phi(x) | \, dx \leqq 2(x_2 - x_1) d, \tag{19}$$

where d is the oscillation of $f(x)$ in $x_1 \leqq x \leqq x_2$.

Proof. (19) is obvious. (17) and (18) can be derived either geometrically, by enclosing the area between the arc (15) and its chord (16) in a parallelogram and in an ellipse respectively, or else by the following more analytic argument. Let F denote the maximum of $| f(x) - \phi(x) |$ in $x_1 \leqq x \leqq x_2$. Then

$$\int_{x_1}^{x_2} | f(x) - \phi(x) | \, dx \leqq (x_2 - x_1) F, \tag{20}$$

and we have a ξ, $x_1 < \xi < x_2$, such that

$$| f(\xi) - \phi(\xi) | = F. \tag{21}$$

Let us denote by P, Q, R, S the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(\xi, f(\xi))$, $(\xi, \phi(\xi))$. From the triangle PQR we see that $F = \overline{RS}$ cannot exceed $\overline{PR} + \overline{RQ}$. Hence

$$F \leqq \overline{PR} + \overline{RQ} \leqq s. \tag{22}$$

(20), (21), (22) prove (17). Next, observe that $(x_2 - x_1) F$ is twice the area of the triangle PQR. Hence, by (14),

$$(x_2 - x_1) F \leqq 2(\overline{PR} + \overline{RQ})^{3/2} (\overline{PR} + \overline{RQ} - \overline{PQ})^{1/2}. \tag{23}$$

But $\overline{PR} + \overline{RQ} \leqq s$, $\overline{PQ} = \sigma$. Hence (18) follows from (20) and (23).

We shall list now, for easier reference, a number of inequalities for the auxiliary quantities defined in **3**. The cross-references indicate the sections containing the information needed in the proofs.

4. 5. $\| \mathfrak{x}_1 - \mathfrak{x}_2 \| = O(| H | / q_\nu)$. See (9) in **3. 2**, (10) in **3. 10**, **3. 12**, **2. 10**.

4. 6. $| n - \tau | \leqq | b - \beta | + | c - \gamma |$. See **3. 4**, **3. 6**, (11) in **4. 2**.

4. 7. $| c - \gamma | \leqq \theta_{y'} + \theta_{y''}$, $| b - \beta | \leqq \theta_{x'} + \theta_{x''}$. See **3. 4**, **3. 5**, **3. 6**, **3. 10**.

4. 8. We have the relations

$$\theta_{x'} = O(s_{x'} / q_\nu), \tag{24}$$

$$\theta_{x'} = O[s_{x'}^{3/2} (s_{x'} - \sigma_{x'})^{1/2}], \tag{25}$$

and analogous relations for $\theta_{x''}$, $\theta_{y'}$, $\theta_{y''}$. See **3. 10**, **4. 4**, **2. 10**.

4. 9. The preceding estimates for $\theta_{x'}$ may be meaningless if $x'=0$, because then possibly $s_{x'}=\infty$. For $x'=0$ we have the estimate

$$\theta_{x'} = O(d_v/q_v)$$

with analogous statements for $\theta_{x''}$, $\theta_{y'}$, $\theta_{y''}$. See **3. 10**, (19) in **4. 4**, **2. 10**, **2. 12**.

4. 10. Let us return now to **3. 13**. We shall use the superscripts 1, 2 in the sense explained there. We introduce the vectors (see **3. 6**, **3. 13**)

$$\mathfrak{y}^{(1)} = (\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}), \qquad \mathfrak{y}^{(2)} = (\alpha^{(2)}, \beta^{(2)}, \gamma^{(2)}).$$

Then (see **3. 6**, **3. 13**)

$$\tau = \| \mathfrak{y}^{(1)} + \mathfrak{y}^{(2)} \|. \tag{26}$$

By the definition of the vector product we have

$$| \alpha^{(1)}\gamma^{(2)} - \alpha^{(2)}\gamma^{(1)} | \leqq \| \mathfrak{y}^{(1)} \times \mathfrak{y}^{(2)} \|. \tag{27}$$

By **3. 13** we have

$$\alpha^{(1)} = \alpha^{(2)} = \tfrac{1}{2}\alpha. \tag{28}$$

(27) and (28) yield

$$| \gamma^{(2)} - \gamma^{(1)} | \leqq 2/\alpha \, \| \mathfrak{y}^{(1)} \times \mathfrak{y}^{(2)} \|. \tag{29}$$

By **3. 13**, **3. 6**, **3. 8**, (26) we have

$$\| \mathfrak{y}^{(1)} \| + \| \mathfrak{y}^{(2)} \| = \tau^{(1)} + \tau^{(2)} \leqq t^{(1)} + t^{(2)} = t, \tag{30}$$

$$\| \mathfrak{y}^{(1)} \| + \| \mathfrak{y}^{(2)} \| - \| \mathfrak{y}^{(1)} + \mathfrak{y}^{(2)} \| = \tau^{(1)} + \tau^{(2)} - \tau \leqq t - \tau. \tag{31}$$

(29), (13) in **4. 2**, (30), (31), **3. 4**, **2. 10** yield now

$$| \gamma^{(2)} - \gamma^{(1)} | = O[q_v^2 t^{3/2} (t-\tau)^{1/2}]. \tag{32}$$

4. 11. $|\rho| \leqq \theta_{y'} + \theta_{y''}$. See **3. 10**.

4. 12. $|H| = O[q_v^3 t^{3/2}(t-\tau)^{1/2} + q_v(\theta_{y'} + \theta_{y''})]$. See **3. 13**, (32) in **4. 10**, **4. 11**.

4. 13. $ABC + ACD - n = O[q_v^2 t^{3/2}(t-\tau)^{1/2} + \theta_{y'} + \theta_{y''}]$. See (8) in **3. 2**, **3. 4**, (12) in **4. 2**, **4. 5**, **4. 12**.

4. 14. From the definition of H (see **3. 10**) it is obvious that

$$|H| \leqq s_{y'} + s_{y''}.$$

Hence (see references in **4. 13**)

$$ABC + ACD - n = O\left(\frac{s_{y'} + s_{y''}}{q_v}\right).$$

4.15. This last estimate may become meaningless if $y'=0$ or $y''=1$, since then possibly $s_{y'}=\infty$ or $s_{y''}=\infty$. We derive therefore an alternative estimate. By **3.10** and **2.12** it is obvious that

$$|H|=O(d_\nu).$$

By the same cross-references as in **4.14** we obtain then the estimate

$$ABC+ACD-n=O(d_\nu/q_\nu).$$

4.16. Let there be given now a positive constant K. For each rectangle r of $X_\nu Y_\nu$ we define then a quantity λ_K as follows. $\lambda_K=1$ if for the rectangle r *all* the inequalities

$$s_{x'},\ s_{x''},\ s_{y'},\ s_{y''}\leqq K/q_\nu, \qquad t\leqq K/q_\nu^2$$

hold. Otherwise $\lambda_K=0$ for that rectangle. We propose to show that

$$X_\nu Y_\nu(1-\lambda_K)\alpha=O(1/K+1/q_\nu), \tag{33}$$

where α denotes the area of the generic rectangle r of $X_\nu Y_\nu$.

4.17. Let us consider first those rectangles of $X_\nu Y_\nu$ for which

$$s_{x'}>K/q_\nu \quad \text{and} \quad x'\neq 0. \tag{34}$$

Let $N(x')$ designate the number of such rectangles for fixed x'. Then (see **1.4**, **2.10**, **2.4**, **2.5**)

$$\frac{N(x')K}{q_\nu}\leqq L(x'),$$

$$N(x')\leqq\frac{q_\nu^2}{K}\ \frac{L(x')}{q_\nu}=O[(q_\nu^2/K)L(x')(x''-x')],$$

$$\begin{aligned}X^*{}_\nu N(x') &= O[(q_\nu^2/K)X^*{}_\nu L(x')(x''-x')]\\ &= O[(q_\nu^2/K)(\int_0^1 L(x)dx+\epsilon)]=O(q_\nu^2/K).\end{aligned}$$

Thus the number of all rectangles satisfying (34) is $O(q_\nu^2/K)$ and hence their total area is $O(1/K)$. The sum of the areas of those rectangles of $X_\nu Y_\nu$ for which $x'=0$ is $O(1/q_\nu)$ (cf. **2.10**). Hence, the sum of the areas of those rectangles of $X_\nu Y_\nu$ for which $s_{x'}>K/q_\nu$ is $O(1/K+1/q_\nu)$. Applying the same reasoning to $s_{x''}$, $s_{y'}$, $s_{y''}$, it follows that the sum of the areas of those rectangles of $X_\nu Y_\nu$ for which the inequalities

$$s_{x'},\ s_{x''},\ s_{y'},\ s_{y''}\leqq K/q_\nu$$

do *not* hold simultaneously, is $O(1/K+1/q_\nu)$.

4.18. Let now N designate the number of those rectangles of $X_\nu Y_\nu$ for which $t > K/q_\nu^2$. Then (see **3.8**)

$$NK/q_\nu^2 \leqq T,$$

and hence

$$N = O(q_\nu^2/K).$$

Thus the sum of the areas of these rectangles is $O(1/K)$. Hence (see **4.16**, **4.17**)

$$X_\nu Y_\nu(1-\lambda_K)\alpha = O(1/K + 1/q_\nu) + O(1/K) = O(1/K + 1/q_\nu),$$

and (33) is proved.

4.19. $\underset{x' \in e_x}{X^*_\nu Y_\nu s_{x'}/q_\nu}$, $\underset{x'' \in e_x}{X^*_\nu Y_\nu s_{x''}/q_\nu}$, $\underset{y' \in e_y}{X_\nu Y^*_\nu s_{y'}/q_\nu}$, $\underset{y'' \in e_y}{X_\nu Y^*_\nu s_{y''}/q_\nu} = O(\epsilon)$.

Let us consider the first summation. By **2.4** to **2.10**

$$\underset{x' \in e_x}{X^*_\nu Y_\nu s_{x'}/q_\nu} = O[\underset{x' \in e_x}{X^*_\nu Y_\nu s_{x'}(x''-x')}] = O[\underset{x' \in e_x}{X^*_\nu L(x')(x''-x')}] = O(\epsilon).$$

4.20. We have

$$\underset{\substack{y' \in E_y \\ s_{y'} > K^{1/2}/q_\nu}}{X_\nu Y_\nu s_{y'}/q_\nu}, \qquad \underset{\substack{y'' \in E_y \\ s_{y''} > K^{1/2}/q_\nu}}{X_\nu Y_\nu s_{y''}/q_\nu} = O[\omega(pM/K^{1/2})]. \tag{35}$$

Proof. It is sufficient to consider the first summation (see **2.6**, **2.8**, **2.5**, **2.13**, **4.16** for notations). Let $N(y')$ designate the number of arcs $s_{y'}$, for fixed y', that contribute to the first summation. Then by **2.13**

$$\frac{N(y')K^{1/2}}{q_\nu} \leqq M, \qquad N(y') \leqq \frac{Mq_\nu}{K^{1/2}}.$$

For fixed y', the sum of the intervals $x' \leqq x \leqq x''$ corresponding to these arcs is therefore (see **2.5**, (c))

$$\leqq N(y')p/q_\nu \leqq pM/K^{1/2}.$$

By **2.8**, (4) it follows that, for fixed y', the sum of the lengths of these arcs $s_{y'}$ is $\leqq \omega(pM/K^{1/2})$. Since y' takes on not more than q_ν values, (35) is established.

4.21. $X_\nu Y^*_\nu(1-\lambda_K)s_{y'}/q_\nu$, $X_\nu Y^*_\nu(1-\lambda_K)s_{y''}/q_\nu$

$$= O[\epsilon + \omega(pM/K^{1/2}) + 1/K^{1/2} + K^{1/2}/q_\nu].$$

Proof. For notations see **2.4**, **4.16**, **2.5**, **2.8**, **2.13**. It is sufficient to consider the first summation. We split it into three summations Σ_1, Σ_2, Σ_3 as follows. Σ_1 is relative to those terms for which $y' \in e_y$ (see **2.7**). Σ_2 is relative to those terms for which

$$y' \epsilon E_y \quad \text{and} \quad s_{y'} > K^{1/2}/q_\nu.$$

Σ_3 is relative to those terms for which

$$y' \epsilon E_y \quad \text{and} \quad s_{y'} \leqq K^{1/2}/q_\nu.$$

By **4. 19**, $\Sigma_1 = \theta(\epsilon)$. By **4. 20**, $\Sigma_2 = \theta(\omega(pM/K^{1/2}))$. Finally, by **2. 10**, **4. 16**,

$$\Sigma_3 \leqq X_\nu Y_\nu(1-\lambda_K)K^{1/2}/q_\nu^2 = K^{1/2}O[X_\nu Y_\nu(1-\lambda_K)\alpha]$$
$$= K^{1/2}O[1/K + 1/q_\nu] = O[1/K^{1/2} + K^{1/2}/q_\nu].$$

4. 22. $\underset{x'\epsilon c_x}{X_\nu} Y_\nu\theta_{x'}, \underset{x''\epsilon e_x}{X_\nu} Y_\nu\theta_{x''}, \underset{y'\epsilon e_y}{X_\nu} Y_\nu\theta_{y'}, \underset{y''\epsilon c_y}{X_\nu} Y_\nu\theta_{y''} = O(\epsilon + d_\nu).$

See (24) in **4. 8**, **4. 19**, **4. 9**.

4. 23. $\underset{s_{x'}\leqq K/q_\nu}{X_\nu} Y_\nu\theta_{x'}, \underset{s_{x''}\leqq K/q_\nu}{X_\nu} Y_\nu\theta_{x''}, \underset{s_{y'}\leqq K^{1/2}/q_\nu}{X_\nu} Y_\nu\theta_{y'}, \underset{s_{y''}\leqq K^{1/2}/q_\nu}{X_\nu} Y_\nu\theta_{y''} = O(\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2}).$

Proof. It is sufficient to consider the first summation. We split it into two summations Σ_1, Σ_2, where Σ_1 is relative to those terms for which $x' \epsilon e_x$, and Σ_2 is relative to those terms for which $x' \epsilon E_x$. By **4. 22**, $\Sigma_1 = O(\epsilon + d_\nu)$. By (25) in **4. 8**, the inequality of Schwartz, and (6) in **2. 11**,

$$\Sigma_2 = O[\underset{\substack{x'\epsilon E_x \\ s_{x'}\leqq K/q_\nu}}{X_\nu} Y_\nu s_{x'}{}^{3/2}(s_{x'} - \sigma_{x'})^{1/2}] = O[(K^{3/2}/q_\nu^{3/2}) \underset{x'\epsilon E_x}{X_\nu} (Y_\nu 1^{1/2}(s_{x'} - \sigma_{x'})^{1/2})]$$
$$= O[(K^{3/2}/q_\nu^{3/2}) \underset{x'\epsilon E_x}{X_\nu} q_\nu^{1/2}(L(x') - l_\nu(x'))^{1/2}]$$
$$= O[K^{3/2}/q_\nu^{3/2} \cdot q_\nu^{1/2} \cdot q_\nu\delta_\nu^{1/2}] = O(K^{3/2}\delta_\nu^{1/2}].$$

4. 24. We have

$$(36) \quad X_\nu Y_\nu\lambda_K \,|\, ABC + ACD - \tau \,| = O[\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2} + K^{3/2}(T - X_\nu Y_\nu\tau)^{1/2}].$$

Proof. We start with

$$|\, ABC + ACD - \tau \,| \leqq |\, ABC + ACD - n \,| + |\, n - \tau \,|.$$

By **4. 6**, **4. 7**, **4. 23**

$$(37) \qquad X_\nu Y_\nu\lambda_K \,|\, n - \tau \,| = O[\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2}].$$

By **4. 13**, **4. 23**

$$(38) \quad X_\nu Y_\nu\lambda_K \,|\, ABC + ACD - n \,|$$
$$= O[X_\nu Y_\nu\lambda_K q_\nu^2 t^{3/2}(t-\tau)^{1/2}] + O[\epsilon + d_\nu + K^{3/2}\delta_\nu^{1/2}].$$

By **4. 16**, **3. 8**, and the inequality of Schwartz,

$$(39) \quad X_\nu Y_\nu\lambda_K q_\nu^2 t^{3/2}(t-\tau)^{1/2} \leqq X_\nu Y_\nu\lambda_K q_\nu^2(K^{3/2}/q_\nu^3)(t-\tau)^{1/2}$$
$$\leqq (K^{3/2}/q_\nu)X_\nu Y_\nu 1^{1/2}(t-\tau)^{1/2}$$
$$\leqq (K^{3/2}/q_\nu)(q_\nu^2)^{1/2}[X_\nu Y_\nu(t-\tau)]^{1/2} = K^{3/2}(T - X_\nu Y_\nu\tau)^{1/2}.$$

(37), (38), (39) imply (36).

4.25. We have

$$(40)\quad X_\nu Y_\nu(1-\lambda_K)\,|ABC+ACD-\tau|$$
$$= O\{\Omega(1/K+1/q_\nu)+d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu\}.$$

Proof. By **4.6**, **4.7**

$$(41)\quad X_\nu Y_\nu(1-\lambda_K)\,|n-\tau| \leqq X_\nu Y_\nu(1-\lambda_K)(|\beta|+|b|+\theta_{y'}+\theta_{y''}).$$

By (33) in **4.16**

$$X_\nu Y_\nu(1-\lambda_K)\alpha = O(1/K+1/q_\nu),$$

and hence, by **3.14**, **3.6**, **2.1**,

$$(42)\qquad X_\nu Y_\nu(1-\lambda_K)\,|\beta| \leqq \Omega[O(1/K+1/q_\nu)].$$

From the definition of b (see **3.4**) it is clear that

$$|b| \leqq \tfrac{1}{2}(y''-y')(s_{y'}+s_{y''}) = O(s_{y'}/q_\nu+s_{y''}/q_\nu),$$

and (see **2.12**)

$$|b| \leqq (y''-y')d_\nu = O(d_\nu/q_\nu).$$

Hence, by **2.4**, **4.21**,

$$(43)\quad X_\nu Y_\nu(1-\lambda_K)\,|b| = O(d_\nu)+O[X_\nu Y^*_\nu(1-\lambda_K)(s_{y'}/q_\nu+s_{y''}/q_\nu)]$$
$$= O[d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu].$$

By **4.8**, **4.9**, **4.21**,

$$(44)\quad X_\nu Y_\nu(1-\lambda_K)(\theta_{y'}+\theta_{y''}) = O(d_\nu)+O[X_\nu Y^*_\nu(1-\lambda_K)(s_{y'}/q_\nu+s_{y''}/q_\nu)]$$
$$= O[d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu]$$

By (41), (42), (43), (44),

$$(45)\quad X_\nu Y_\nu(1-\lambda_K)\,|n-\tau| = O\{\Omega[O(1/K+1/q_\nu)]$$
$$+d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu\}.$$

By **4.14**, **4.15**, **4.21** we obtain similarly

$$(46)\quad X_\nu Y_\nu(1-\lambda_K)\,|ABC+ACD-n|$$
$$= O[d_\nu+\epsilon+\omega(pM/K^{1/2})+1/K^{1/2}+K^{1/2}/q_\nu].$$

(45), (46) imply (40).

4.26. As a direct consequence of **4.24** and **4.25** we obtain

$$X_\nu Y_\nu\,|ABC+ACD-\tau|$$
$$= O\{\Omega[O(1/K+1/q_\nu)]+\omega(pM/K^{1/2})+\epsilon+\delta_\nu+1/K^{1/2}+K^{1/2}/q_\nu$$
$$+K^{3/2}\delta_\nu^{1/2}+K^{3/2}(T-X_\nu Y_{\nu T})^{1/2}\}.$$

5. Proof of the theorem.

5.1. Lemma. *For the sequence $X_\nu Y_\nu$, described in* **2.5** *to* **2.9** *we have*

$$\overline{\lim_{\nu\to\infty}} X_\nu Y_\nu \mid ABC + ACD - \tau \mid \leqq \Gamma\epsilon, \tag{47}$$

where Γ *is a constant that depends* only *upon the given surface* (1) *and the constant* p *of* **2.5**.

Proof. In **4.26** keep K fixed and make $\nu \to \infty$. By **2.5**, **2.12**, (5) in **2.11**, (e) and (f) in **3.8** we obtain

$$\overline{\lim_{\nu\to\infty}} X_\nu Y_\nu \mid ABC + ACD - \tau \mid = O\{\Omega[O(1/K)] + \omega(pM/K^{1/2}) + \epsilon + 1/K^{1/2}\}.$$

This relation holds for every $K > 0$. Hence, making $K \to \infty$, we obtain, by **3.14**, **2.8**,

$$\overline{\lim_{\nu\to\infty}} X_\nu Y_\nu \mid ABC + ACD - \tau \mid = O(\epsilon).$$

By the meaning of the symbol O (see **2.9**), this last relation proves (47).

5.2. By **3.2**, we have

$$X_\nu Y_\nu (ABC + ACD) = E(\Pi_\nu),$$

where $E(\Pi_\nu)$ is the elementary area of an inscribed polyhedron

$$\Pi_\nu: \quad z = \phi_\nu(x, y), \quad (x, y) \in I_0,$$

such that $\phi_\nu(x, y) \to f(x, y)$ uniformly in I_0. Since (cf. **3.8**)

$$\begin{aligned} \mid E(\Pi_\nu) - T \mid &= \mid X_\nu Y_\nu (ABC + ACD) - X_\nu Y_\nu \tau + X_\nu Y_\nu (\tau - t) \mid \\ &\leqq X_\nu Y_\nu \mid ABC + ACD - \tau \mid + X_\nu Y_\nu (t - \tau) \end{aligned}$$

the relation (47) implies that (cf. **3.8**, (f))

$$\overline{\lim_{\nu\to\infty}} \mid E(\Pi_\nu) - T \mid \leqq \Gamma\epsilon. \tag{48}$$

On the other hand, as noted above,

$$\max_{(x,y)\in I_0} \mid f(x, y) - \phi_\nu(x, y) \mid \underset{\nu\to\infty}{\to} 0. \tag{49}$$

We recall that ϵ, p were assigned arbitrarily, and that Γ in (48) is *independent* of ϵ and of the sequence $X_\nu Y_\nu$. Hence, for any positive integer j, we can select first p to satisfy

$$1 < p < 1 + 1/j.$$

Once p has been so selected, the constant Γ in (48) is determined. Then we select $\epsilon > 0$ to satisfy

$$\Gamma\epsilon < 1/j.$$

After ϵ has been so selected, we have a sequence $X_\nu Y_\nu$ as described in **2.5** to **2.9**, and (48) and (49) hold. For ν large enough, we have then, by (48) and (49),

$$(50) \qquad |E(\Pi_\nu) - T| < 1/j, \qquad \max_{(x,y)\,\epsilon\, I_0} |f(x,y) - \phi_\nu(x,y)| < 1/j.$$

We take such a ν, and we designate the corresponding X_ν, Y_ν, ϕ_ν, Π_ν by $X^{(j)}$, $Y^{(j)}$, $\phi^{(j)}$, $\Pi^{(j)}$ to stress their dependence upon the positive integer j. We obtain in this manner the following lemma.

5.3. LEMMA. *We have a sequence of subdivisions* $X^{(j)}Y^{(j)}$, *and a corresponding sequence of inscribed polyhedra*

$$\Pi^{(j)}: \quad z = \phi^{(j)}(x,y), \quad (x,y)\,\epsilon\, I_0,$$

such that the following holds (cf. **2.5** *and* (50)).

$$\text{(a)} \quad \phi^{(j)}(x,y) \underset{j\to\infty}{\longrightarrow} f(x,y) \textit{ uniformly in } I_0.$$
$$\text{(b)} \quad q(X^{(j)}) = q(Y^{(j)}), \; j = 1, 2, \cdots.$$
$$\text{(c)} \quad p(X^{(j)}) \to 1, \; p(Y^{(j)}) \to 1 \textit{ for } j \to \infty.$$
$$\text{(d)} \quad E(\Pi^{(j)}) \underset{j\to\infty}{\longrightarrow} T.$$

5.4. The proof of the theorem of **1.5** is now immediate. By the definition of $A^*(S)$ in **1.2** we have

$$(51) \qquad \underline{\lim}\, E(\Pi^{(j)}) \geqq A^*(S).$$

By **3.8**, (b) we have

$$(52) \qquad T \leqq A(S).$$

By **1.2**

$$(53) \qquad A(S) \leqq A^*(S).$$

(51), (52), (53) and (d) in **5.3** imply $A(S) = A^*(S)$.

5.5. In fact we have proved more than the mere equality $A(S) = A^*(S)$. Indeed, we also proved that we have sequences of inscribed polyhedra as described in **5.3**. The conditions stated there express the fact that the approximating inscribed polyhedron $\Pi^{(j)}$ is such that its projection upon the xy plane consists of a number of triangles obtained by subdividing the unit square I_0 first into rectangles that are *practically congruent squares,* and then drawing one diagonal in each of these rectangles.

THE OHIO STATE UNIVERSITY.

BIBLIOGRAPHY.

Geöcze, Z. de

[1] "Quadrature des surfaces courbes," *Math. u. Naturw. Berichte aus Ungarn,* vol. 26 (1908), pp. 1-88.

[2] "Contributions to the quadrature of the surface $z = f(x, y)$" (in Hungarian), *Math. Term. Tud. Ért.,* vol. 26 (1908), pp. 475-512.

[3] "Sur la fonction semi-continue," *Bull. de la Soc. Math. de France,* vol. 39 (1911), pp. 256-295.

[4] "A necessary and sufficient condition for the finiteness of the area of a surface" (in Hungarian), *Math. Phys. Lapok,* vol. 25 (1916), pp. 61-81.

Hahn, H.

[1] "Über Annäherung an Lebesguesche Integrale durch Riemannsche Summen," *Sitzungsberichte d. k. Akad. Wien,* vol. 123 (1914), pp. 713-743.

Kempisty, S.

[1] "Sur la méthode triangulaire du calcul de l'aire d'une surface courbe," *Bull. Soc. Math. France,* vol. 64 (1936), pp. 119-132.

[2] "Application des fonctions de triangle à la théorie de l'aire d'une surface courbe," *Travaux Soc. Sc. et Lettres,* Wilno, vol. 12 (1937), pp. 1-8.

Morrey, C. B.

[1] "A class of representations of manifolds," part I, *American Journal of Mathematics,* vol. 55 (1933), pp. 683-707.

Radó, T.

[1] "On the problem of Plateau," *Ergebnisse der Mathematik,* vol. 2, no. 2 (1933).

[2] "Sur le calcul de l'aire des surfaces courbes," *Fund. Math.,* vol. 10 (1927), pp. 197-210.

[3] "Über zweidimensionale reguläre Variationsprobleme," *Mathematische Annalen,* vol. 101 (1929), pp. 620-632.

Saks, S.

[1] *Theory of the Integral,* 1937.

INTERPOLATED DENUMERANTS AND LAMBERT SERIES.*

By E. T. Bell.

1. Notation, summary. The letters denoting integers, $r > 1$; $a_1, \cdots, a_r$ are constant and > 0; $x_1, \cdots, x_r$ are variable and $\geqq 0$. The number $D(n|a_1, \cdots, a_r)$ of sets $(x_1, \cdots, x_r)$ of solutions of

$$(1.1) \qquad a_1x_1 + \cdots + a_rx_r = n$$

is (in Sylvester's terminology) the denumerant of n with respect to $a_1, \cdots, a_r$. To avoid trivial modifications in the notation, the case $a_1 = \cdots = a_r = a$ is excluded; $D(n|a, \cdots, a)$ is the coefficient of x^n in $(1 - x^a)^{-r}$, $= 1$ if $n = 0$, $= 0$ if $a \nmid n$, $= \frac{a^{1-r}}{(r-1)!} \prod_{j=1}^{r-1} (n + ja)$ if $a|n$. When there can be no ambiguity, we abbreviate $D(n|a_1, \cdots, a_r)$ to $D(n)$, and, if necessary, state the associated equation (1.1).

The L. C. M. of $a_1, \cdots, a_r$ will be denoted by a. $n_1, \cdots, n_r$ are any r different non-negative constant integers. Hence

$$(1.2) \qquad D(an_j + b_j), \qquad 0 \leqq b_j < a, \qquad (j = 1, \cdots, r)$$

are constants.

In Sylvester's method,[1]

$$D(n) = A(n) + U$$

where $A(n)$ is a polynomial in n of degree $r - 1$, and U, the 'undulant' part, contains roots of unity. Sylvester states [2] that

$$A(n) = \sum_{j=0}^{r-1} A_{r-1-j}n^j/j!.$$

where A_s is the coefficient of x^s in the power series expansion of $\prod_{j=1}^{r} (1 - e^{-a_jx})^{-1}$. Cayley's method leads to similar conclusions, the periodic terms in $D(n)$ being collected in his 'circulators.' Our object is to get rid of both undulants

---

* Received January 14, 1942.

[1] For the history of denumerants to 1919, see L. E. Dickson, *History of the Theory of Numbers*, vol. 2 (1920), chap. 3, and for an exposition of Sylvester's and Cayley's methods, either the papers in vol. 2 of each of these authors' works or P. Bachmann, *Niedere Zahlentheorie*, zweiter Teil (1910), Kap. 3. The results stated by Sylvester in 1857 (*loc. cit.*, No. 17) are repeated, and most are proved, in several later papers.

[2] *Loc. cit.*, 87, last footnote.

and circulators, as these are responsible for the most awkward part of the calculations, by suitably classifying the arguments n in $D(n)$.

It will be shown (2) that $D(an+b)$, for any constant integer b such that $0 \leqq b < a$, is a polynomial in n of degree $r-1$,

$$(1.3) \qquad D(an+b) = c_0 + c_1 n + \cdots + c_{r-1} n^{r-1},$$

with constant (independent of n) rational number coefficients which may be explicitly evaluated in terms of the constants (1.2) with $b_j = b$.

If in (1.3) we put $n = n_1, \cdots, n_r$, and from the resulting equations and (1.3) eliminate $c_0, \cdots, c_{r-1}$, we get

$$\begin{vmatrix} D(an+b) & 1 & n & \cdots & n^{r-1} \\ D(an_1+b) & 1 & n_1 & \cdots & n_1^{r-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ D(an_r+b) & 1 & n_r & & n_r^{r-1} \end{vmatrix} = 0.$$

The coefficients of the elements in the first column in the expanded form of the left are determinants of a familiar type (alternants), whose expansions as products of differences are immediate. Or we may apply Lagrange's interpolation formula to the polynomial in (1.3):

$$F(x) \equiv \prod_{j=1}^{r} (x - n_j), \qquad F_j(x) \equiv F(x)/(x - n_j);$$

$$(1.4) \qquad D(an+b) = \sum_{j=1}^{r} \frac{F_j(n)}{F_j(n_j)} D(an_j + b).$$

The constants n_j are at our disposal. For the choices $n_j = j - 1$, or $n_j = j$, (1.4) takes a simple form. If $n_j = j$,

$$(1.5) \quad D(an+b) = (-1)^r (n-1)(n-2)\cdots(n-r)\phi_n(a,b),$$

$$\phi_n(a,b) \equiv \sum_{j=1}^{r} \frac{(-1)^j D(ja+b)}{(j-1)!\,(r-j)!\,(n-j)},$$

which may be written

$$(1.6) \quad (-1)^r r!\, D(an+b) = (n-1)\cdots(n-r) \sum_{j=1}^{r} (-1)^j j \binom{r}{j} \frac{D(ja+b)}{n-j}.$$

It follows that the r constants c_j in (1.3) are linear homogeneous functions, with constant rational number coefficients, of the r constants $D(an_j + b)$ for (1.4), and of $D(ja+b)$ for (1.6), $j = 1, \cdots, r$. The complete set of formulas $D(n)$ for a particular equation (1.1) therefore requires the calculation of the numerical values of ra constants of the kind (1.2). This may be done directly or by the classical methods cited.[1]

It will be seen in the proof of (1.3) that b may be any non-negative integer. Hence in (1.6) we may take $n = 0$ and in the result replace b by the arbitrary non-negative integer n,

$$(1.7) \qquad D(n) + \sum_{j=1}^{r} (-1)^j \binom{r}{j} D(n + ja) = 0.$$

For $n = 0$ this enables us to calculate $D(ra)$ when $D(ja)$, $j = 1, \cdots, r-1$ are known,

$$(1.8) \qquad 1 + \sum_{j=1}^{r} (-1)^j \binom{r}{j} D(ja) = 0.$$

From (1.6) we have

$$(1.9) \qquad D(an + b) = O(n^{r-1}),$$

the constant implied in the order function being

$$\frac{(-1)^r}{r!} \sum_{j=1}^{r} (-1)^j j \binom{r}{j} D(ja + b).$$

The enumeration of compound partitions being reducible to that of simple partitions, there are formulas corresponding to the preceding for compound denumerants.

To illustrate (1.5), let $r = 3$. Then a is the L. C. M. of a_1, a_2, a_3, and $D(m)$ refers to

$$a_1x_1 + a_2x_2 + a_3x_3 = m:$$

$$\begin{aligned} 2D(an + b) = (n-2)(n-3)D(a + b) - 2(n-1)(n-3)D(2a + b) \\ + (n-1)(n-2)D(3a + b), \qquad 0 \leqq b < a. \end{aligned}$$

For large values of n, this $D(an + b)$ is approximately

$$\tfrac{1}{2}n^2[D(a + b) - 2D(2a + b) + D(3a + b)].$$

2. Proof of (1.3). Everything will be proved if (1.3) is. With the notation as in 1, let α be a primitive a-th root of 1. Then if $a_j > 1$, there is a least integer $s_j > 0$ such that α^{s_j} is a primitive a_j-th root of 1. In the set $\alpha^{s_j t_j}$ $(t_j = 1, \cdots, a_j - 1;\ j = 1, \cdots, r)$ let $\gamma, \delta, \cdots, \tau$ be all the unequal elements of the set, and let these occur precisely $g, d, \cdots, t$ times respectively. (If $s_j t_j \equiv s_i t_i$, mod a, the corresponding elements are equal.) Then, $|x| < 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} D(n)x^n &= \prod_{j=1}^{r} (1 - x^{a_j})^{-1}, \\ &= (1 - x)^{-r}(x - \gamma)^{-g}(x - \delta)^{-d} \cdots (x - \tau)^{-t}. \end{aligned}$$

No $1 - x^{a_j}$ has a repeated root, and there are precisely r such binomials. Hence none of $g, d, \cdots, t$ can exceed r.

The resolution of the above generating function into partial fractions may be written

$$\begin{aligned} &R_1(1-x)^{-1}+R_2(1-x)^{-2}+\cdots+R_r(1-x)^{-r}\\ &-G_1(x-\gamma)^{-1}+G_2(x-\gamma)^{-2}+\cdots+(-1)^rG_r(x-\gamma)^{-r}\\ &-D_1(x-\delta)^{-1}+D_2(x-\delta)^{-2}+\cdots+(-1)^rD_r(x-\delta)^{-r}\\ &-\cdots \qquad \cdots \qquad \cdots\\ &-T_1(x-\tau)^{-1}+T_2(x-\tau)^{-2}+\cdots+(-1)^rT_r(x-\tau)^{-r}, \end{aligned}$$

in which the capital letters denote constants, and since $g\leqq r$, $d\leqq r,\cdots,t\leqq r$, some of the constants $G, D,\cdots,T$ may be identically zero. The coefficient of y^n in $(1-y)^{-s}$, s an integer >0, is $(n)_s$ where $(n)_1=n$, and

$$(s-1)!\,(n)_s\equiv(n+1)(n+2)\cdots(n+s-1), \qquad s>1.$$

If λ is any one of $\gamma,\delta,\cdots,\tau$,

$$(-1)^j(x-\lambda)^{-j}=\lambda^{-j}(1-\lambda^{-1}x)^{-j},$$

in which the coefficient of x^n is $\lambda^{-n-j}(n)_j$. Hence

$$D(n)=\sum_{j=1}^{r}(n)_j[R_j+G_j\gamma^{-n-j}+\cdots+T_j\tau^{-n-j}].$$

Since $\lambda^{ma}=1$ for all integers m,

$$D(an+b)=\sum_{j=1}^{r}(an+b)_j[R_j+G_j\gamma^{-b-j}+\cdots+T_j\tau^{-b-j}].$$

The expression in square brackets is a constant, say k_j. Hence

$$D(an+b)=\sum_{j=1}^{r}k_j(an+b)_j,$$

which may be written in the form (1.3), since $(an+b)_j$ is a polynomial of degree $j-1$ in n with coefficients independent of n.

3. An r-fold Lambert series. One possible generalization of a (1-fold) Lambert series

$$\sum_{n=1}^{\infty}\frac{f(n)x^n}{1-x^n}, \qquad =\sum_{n=1}^{\infty}{}'[\sum_{d|n}f(d)]x^n,$$

is suggested by (1.3). If in this series x is replaced by x^c, where c is a constant integer >0, the effect is merely that of a change of variable with an obvious modification of the convergence criteria.[3] It will be assumed that $f(n)$ is a numerical function of n such that all series in which it occurs here are convergent.

---

[3] As stated, for example, by K. Knopp, *Journal für Mathematik*, vol. 142 (1913), pp. 283-315, especially 287, Satz 1.

With $a_1, \cdots, a_r$, a, b, $D(n)$ as in 1, and $s \equiv a_1 + \cdots + a_r$, we define an r-fold Lambert series $L(x; a_1, \cdots, a_r; f) \equiv L(x)$ for the numerical function f and the parameters $a_1, \cdots, a_r$ by

$$L(x) \equiv \sum_{n=1}^{\infty} \frac{f(n)x^{sn}}{(1-x^{a_1 n}) \cdots (1-x^{a_r n})}.$$

To expand $L(x)$ in a power series we multiply the generating function of $D(n)$ by $f(t)x^s$, replace x by x^t in the result, and sum over all integers $t > 0$. Thus

$$L(x) = \sum_{t=1}^{\infty} [f(t)x^{st} + \sum_{n=1}^{\infty} f(t)D(n)x^{(n+s)t}],$$

$$= \sum_{t=1}^{\infty} f(t)x^{st} + \sum_{m=s+1}^{\infty} [\sum f(\delta)D(d-s)]x^m,$$

in which $\sum$ refers to all pairs (d, δ) of positive integers such that $d\delta = m$, $d > s$. If there are none such for a particular m, $\sum$ is vacuous and there is no term x^m.

We now reduce $D(d-s)$ by (1.3), noting that c_j in (1.3) depends on b, say $c_j \equiv c_j(b)$, so that

$$D(n) = \sum_{j=0}^{r-1} c_j(b)n^j.$$

For b fixed let (d_b, δ_b) be a (d, δ) such that

$$d_b \equiv s + b \bmod \mathrm{a}, \qquad d_b > s + b, \qquad d_b\delta_b = m.$$

Then $(d_b - s - b)/a$ is an integer > 0, and by (1.3),

$$D(d_b - s) \equiv g_b(d_b) = a^{1-r} \sum_{j=0}^{r-1} c_j(b) a^{r-j-1}(d_b - s - b)^j,$$

and hence

$$\sum f(\delta)D(d-s) = \sum_{b=0}^{a-1} g_b(d_b)f(\delta_b).$$

Finally, then, the complete expansion is

$$L(x) = f(1)x^s + \sum_{m=s+1}^{\infty} [f(m/s) + \sum_{b=0}^{a-1} g_b(d_b)f(\delta_b)]x^m,$$

in which $f(y) = 0$ if y is not an integer > 0.

California Institute of Technology.

ON THE BLOCH-LANDAU CONSTANT.*

By Hans Rademacher.

1. In 1929 Landau, in a paper[1] on the Bloch constant, introduced besides the Bloch constant $\mathfrak{B}$ two other constants $\mathfrak{L}$ and $\mathfrak{A}$ of which only the first will interest us here. The definition of the Bloch constant is the following:

There exists a constant $\mathfrak{B}$ such that to every $\epsilon > 0$ and to every function $w = f(z)$ which is regular for $|z| < 1$ with $f'(0) = 1$ there exists a closed region $G = G_{f,\epsilon}$ in the interior of the unit-circle which is mapped by the function $w = f(z)$ in a one-to-one manner on a circle of radius $\mathfrak{B} - \epsilon$ in the w-plane; whereas there exist functions of the same specifications by which no subregion of the unit-circle is mapped in a one-to-one manner on a circle of radius $\mathfrak{B} + \epsilon$.

By dropping the condition of bi-uniqueness Landau defines another constant which we shall call the Bloch-Landau constant $\mathfrak{L}$: There exists a constant $\mathfrak{L}$ such that every function $w = f(z)$ which is regular in $|z| < 1$ with $f'(0) = 1$ assumes all values of a certain circle of radius $\mathfrak{L} - \epsilon$ in the w-plane, for any given positive ϵ; whereas to every $\epsilon > 0$ there exists a function of the same specifications whose values do not completely fill any circle of radius $\mathfrak{L} + \epsilon$.

Since every "Bloch circle" in the w-plane is a "Landau-circle," but not conversely, it is clear that we have $\mathfrak{B} \leqq \mathfrak{L}$. Landau, in the paper cited, proves also

$$0.39 < \mathfrak{B}, \qquad 0.43 \leqq \mathfrak{L} \leqq 2^{2/3}\pi\Gamma(1/3)^3\Gamma(1/4)^{-4},$$

where the latter upper bound is established by a certain example of $f(z)$ which does not permit a greater Landau circle. Landau gave no separate upper estimate for $\mathfrak{B}$. This was done some years ago by Ahlfors and Grunsky who proved[2] by a suitable example that

$$\mathfrak{B} \leqq (1 + 3^{1/2})^{1/2}\Gamma(1/3)\Gamma(3/4)\Gamma(1/12)^{-1} = 0.4719 \cdots.$$

I shall show that their method leads also to an improved upper estimate of $\mathfrak{L}$.

---

* Received March 21, 1942.

[1] E. Landau, "Über die Blochsche Konstante und zwei verwandte Weltkonstanten," *Mathematische Zeitschrift*, vol. 30 (1929), pp. 608-634.

[2] L. V. Ahlfors and H. Grunsky, "Über die Blochsche Konstante," *Mathematische Zeitschrift*, vol. 42 (1937), pp. 671-673.

2. The interior of the unit circle $|\zeta| < 1$ is mapped on an equilateral equiangular circular triangle of angles $\alpha\pi$ $(0 \leqq \alpha < 1)$ by means of the function

$$(1) \quad w = \phi_\alpha(\zeta) = \zeta \cdot \frac{\int_0^1 t^{-1/2-\alpha/2}(1-t)^{-1/6+\alpha/2}(1-\zeta^3 t)^{-5/6+\alpha/2}dt}{\int_0^1 t^{-1/2-\alpha/2}(1-t)^{-5/6+\alpha/2}(1-\zeta^3 t)^{-1/6+\alpha/2}dt}$$

$$= \frac{\Gamma(2/3)\Gamma(5/6+\alpha/2)}{\Gamma(4/3)\Gamma(1/6+\alpha/2)} \cdot \zeta \cdot \frac{F(5/6-\alpha/2, 1/2-\alpha/2, 4/3; \zeta^3)}{F(1/6-\alpha/2, 1/2-\alpha/2, 2/3; \zeta^3)},$$

where $F(a, b, c; z)$ is that branch of the hypergeometric function which is, for $|z| < 1$, represented by the Gaussian hypergeometric series. In the mapping established by (1) the points $1, \rho, \rho^2$ $(\rho = e^{2\pi i/3})$ of the ζ-plane correspond to the vertices of the circular triangle in the w-plane.

The formula (1) is given by Ahlfors and Grunsky for $\alpha = 1/k$, $k > 0$. But it can be seen that the arguments which they set forth to establish its validity hold also for $\alpha = 0$. The case $\alpha = 1/3$ means the mapping of the unit circle on an equilateral straight triangle. We can, therefore, by combining two conformal mappings of the sort (1), map a zero-angled equilateral triangle on an equilateral straight triangle. If we introduce the inverse function

$$\zeta = \Phi_\alpha(w)$$

of $w = \phi_\alpha(\zeta)$ then

$$(2) \quad w = f(z) = C_1 \phi_{1/3}(\Phi_0(C_2 z))$$

will produce that conformal mapping; the constants C_1 and C_2 can be used to adjust the sizes of the triangles.

Now the conformal mapping (2) can be extended by reflection with respect to the sides of the triangles. The repeated images of the zero-angled triangle in the z-plane will just fill its orthogonal circle, which is its circumcircle, whereas the repeated images of the equilateral triangle in the w-plane will build up a Riemann surface in which the vertices of the triangle and of its reflections form a regular point lattice, and where all these vertices become branch-points of infinite order. The function $w = f(z)$ therefore maps, by this process of analytic continuation, the interior of a whole circle K on the described Riemann surface. We can normalize the mapping so that K is the unit circle $|z| < 1$ and so that $f'(0) = 1$.

Since, then, the image of any circle $|z| \leqq 1 - \epsilon$ will not cover any of the branch-points in the Riemann surface over the w-plane a Landau circle cannot be greater than the circumcircle of the equilateral straight triangle which was the element out of which the Riemann surface was constructed.

The normalization just mentioned requires

$$\Phi_0(C_2) = 1$$

or

$$C_2 = \phi_0(1) \tag{3}$$

and

$$f'(0) = C_1 \frac{\phi'_{1/3}(0)}{\phi'_0(0)} C_2 = 1. \tag{4}$$

Equation (3) has to be understood as

$$C_2 = \lim_{\xi\to 1} \phi_0(\xi) = \lim_{\xi\to 1} \frac{\Gamma(5/6)\Gamma(2/3)F(5/6, 1/2, 4/3; \xi)}{\Gamma(4/3)\Gamma(1/6)F(1/6, 1/2, 2/3; \xi)}.$$

Here we have hypergeometric functions $F(a, b, c; \xi)$ with $a + b = c$, in which case

$$F(a, b, c; \xi) \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log \frac{1}{1-\xi}, \qquad \xi \to 1.$$

Hence

$$C_2 = 1,$$

and therefore formula (4) furnishes

$$C_1 = \frac{\phi'_0(0)}{\phi'_{1/3}(0)} = \frac{\Gamma(5/6)\Gamma(1/3)}{\Gamma(1/6)}.$$

The upper bound of the radii of all Landau circles, i. e. the circumradius of the equilateral triangle in the w-plane, is

$$f(1) = C_1\phi_{1/3}(1) = C_1 = \Gamma(5/6)\Gamma(1/3)\Gamma(1/6)^{-1} = 2\pi\Gamma(1/3)\Gamma(1/6)^{-2}.$$

We have therefore our result:

An upper bound of the Bloch-Landau constant $\mathfrak{L}$ is given by

$$\mathfrak{L} \leqq 2\pi\Gamma(1/3)\Gamma(1/6)^{-2}. \tag{5}$$

This result compares numerically with Landau's estimate quoted above as follows:

$$0.54325 < 2\pi\Gamma(1/3)\Gamma(1/6)^{-2} < 0.0.54326,^{3}$$

$$0.55488 < 2^{2/3}\pi\Gamma(1/3)^3\Gamma(1/4)^{-4} < 0.55489.$$

---

[3] In a letter of May 30, 1942, Professor R. M. Robinson informs me that he found the same upper bound for $\mathfrak{L}$ in 1937. His result is briefly mentioned in a footnote to Ahlfor's paper "An extension of Schwarz's Lemma," *Transactions of the American Mathematical Society*, vol. 43 (1938), p. 364 which I had overlooked: "In the other direction R. M. Robinson has proved $\mathfrak{L} < .544$. This result has not been published."

3. The configuration of the zero-angled circular triangle together with all its reflections is, of course, the well-known modular figure; and if it had not been for the convenience of the Ahlfors-Grunsky formula, I would have adhered to my first redaction of the present paper, which was based on the modular function $k^2(\tau) = \lambda(\tau)$ into which $\Phi_0(w)$ can be transformed. Landau in his paper quotes previous results of Carathéodory and Hartogs, which are also obtained from the theory of modular functions. An analysis of Landau's procedure shows that he has actually used the conformal mapping of the zero-angled triangle with the vertices 1, i, -1 on the equilateral triangle. It is through a gain in symmetry of our example over Landau's that we obtained the improved estimate (5).

Ahlfors and Grunsky in their paper make it plausible that their appraisal of Bloch's constant actually gives its true value. A similar reasoning would apply to the above example. The symmetry of the discussed conformal mapping suggests strongly that (5) may contain the best value of the Bloch-Landau constant $\mathfrak{L}$.

It is noteworthy that the Ahlfors-Grunsky example, as well as ours, exhibits functions which have the unit circle as natural boundary. This is in agreement with R. M. Robinson's statements about "Bloch functions,"[4]

THE UNIVERSITY OF PENNSYLVANIA.

---

[4] Raphael M. Robinson, "Bloch functions," *Duke Mathematical Journal*, vol. 2 (1936), pp. 453-459.

THE RECTILINEAR MOTION OF A GAS.*

By Monroe H. Martin.

Introduction. In the rectilinear motion of a gas the particles of the gas move along straight lines. The lines are all parallel to each other and it is assumed that the velocity and density of the gas at any given instant is constant over any plane perpendicular to the straight lines along which the motion takes place. Letting x denote the distance along any one of these lines measured from some point as origin, and denoting the time by t, the physical problem we consider may be formulated as follows. *Given the velocity* u *and the density* ρ *of the gas as a function of* x *for* $t = 0$, *what can be said about the subsequent behavior of the gas, i. e., how shall* u, ρ *be expressed as functions of* x, t? In addition to assuming that the gas moves rectilinearly we shall suppose that it is perfect and that its pressure and density are connected by a characteristic equation $p = p(\rho)$.

This type of motion has been studied by Riemann in a celebrated memoir.[1] In it he reduces the problem of determining the motion of the gas under certain conditions to finding a solution of a linear, hyperbolic partial differential equation of second order of the type $w_{rs} - a(r,s)(w_r - w_s) = 0$, taking prescribed values for its first partial derivatives along a given arc.[2] The solution of this initial value problem given by Riemann has become classic and is known today as "Riemann's method." Essentially the method consists in determining a two parameter family of solutions of the adjoint partial differential equation subject to certain initial conditions. This two-

---

* Received February 21, 1942; Revised August 22, 1942.

[1] "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite," *Göttingen Abhandlungen*, vol. 8 (1858-59), pp. 245-264 or *Werke*, Leipzig (1892), 2nd Edition, pp. 156-175. See also H. Lamb, *Hydrodynamics*, 5th Ed., Cambridge 1924, pp. 456-459, where both the methods of Riemann and Earnshaw are discussed. For further work on the problem see Rayleigh's *Theory of Sound*, vol. 2, p. 38 and vol. 3, pp. 214-216 of Durand's *Aerodynamic Theory*. An excellent bibliography on the problem is given in the article on Compressible Fluids by H. Bateman in Bulletin 84 of the National Research Council, pp. 543-551. An admirable treatment of the subject will be found in *Die partiellen Differentialgleichungen der Mathematischen Physik* prepared from Riemann's lectures by Heinrich Weber, Braunschweig 1919, vol. 2, pp. 503-562.

An error in § 4 of Riemann's memoir is pointed out in the report of Christoffel, *Fortschritte der Physik*, vol. 15, p. 123. For comments on Lord Rayleigh's objection to Riemann's work see pp. 559-562 of Weber's book cited above.

[2] *Werke*, pp. 160-162 and pp. 169-170.

parameter family of solutions is the so-called "Riemann's function" and, once it has been obtained, the determination of the solution of the original initial value problem is reduced to a quadrature. Since Riemann's time his method has been extended to the most general linear hyperbolic partial differential equation of the second order.[3]

Sections **1** and **2** of the present paper contain no new results but are devoted to a resumé of certain results[4] due to Riemann which the author has felt advisable to include in order to properly prepare the ground for the new developments which follow.

In section **3** the solution of the initial value problem mentioned above is reduced to the determination of what is termed the resolvent of the partial differential equation. The resolvent is analogous to Riemann's function and is a two parameter family of solutions $v = \bar{v}(r, s;\ \bar{r}, \bar{s})$ of the partial differential equation $v_{rs} + a(r, s)(v_r - v_s) = 0$ satisfying the initial conditions

$$\bar{v}(\bar{r}, s;\ \bar{r}, \bar{s}) = \bar{s} - s, \qquad \bar{v}(r, \bar{s};\ \bar{r}, \bar{s}) = r - \bar{r}.$$

As in Riemann's method, the solution of the initial value problem is reduced to a quadrature, once the resolvent is obtained. The new method offers two advantages. The initial conditions on the resolvent are simpler than those for Riemann's function and the final expression for the solution as a quadrature is simpler in form. On the other hand the method does not appear to permit extension to the general linear, hyperbolic partial differential equation.

The method developed in **3** finds an application in **4** to the solution of the problem of the rectilinear motion of a gas with which Riemann occupied himself in § 10 of his memoir. The culminating point of his work in this paragraph is contained in his equations (2). These equations involve a function ω of four arguments r, s, r', s'. However we do not find any formula in Riemann's paper in which the nature of the dependence of ω upon its four arguments is explicitly set forth.

In the theory developed in the present paper the resolvment $\bar{v}$ plays the role of the function ω of Riemann and in **5**, **6** we calculate $\bar{v}$ explicitly for the important case in which the gas is moving adiabatically. In the determination of $\bar{v}$ we are led to the generalized integral equation of Abel, the solution of which yields a formula for $\bar{v}$ involving Appell's hypergeometric function $F_1(\alpha; \beta, \beta'; \gamma; x, y)$ of two variables.

We have selected in **7** what appears to be mathematically the simplest problem to exemplify the application of the methods of this paper to the actual computation of the rectilinear motion of a gas. The application of the method to more complicated cases in which a shock wave develops is reserved for a later paper.

---

[3] Cf., for example, D. Hilbert und R. Courant, *Methoden der Mathematischen Physik*, II, Berlin (1937), pp. 311-315.

[4] *Werke*, pp. 168-175.

1. The partial differential equations of the problem. If it be assumed that the particles of a gas move in straight lines parallel to the x-axis of a rectangular coördinate system in three dimensional Euclidean space and that the velocity and density of the gas are constant over any plane $x = \text{const.}$ at any given instant t, the partial differential equations for the velocity $u = u(x, t)$ and density $\rho = \rho(x, t)$ are[5]

$$(1) \qquad \rho(u_t + u_x u) + G^2 \rho_x = 0, \qquad \rho_t + \rho_x u + \rho u_x = 0,$$

it being understood that there are no external forces. Here

$$(2) \qquad G = G(\rho) = (p')^{\frac{1}{2}}, \qquad p = p(\rho),$$

the latter being the characteristic equation of the gas,[6] so that G is the "local velocity of sound." If a new dependent variable f, defined by

$$(3) \qquad f = f(\rho) = \int G d\rho/\rho,$$

is introduced in place of ρ, the partial differential equations take the more symmetric form

$$u_t + u_x u + G f_x = 0, \qquad f_t + f_x u + G u_x = 0,$$

which, in turn, may be replaced by

$$(4) \qquad r_t + (u + G) r_x = 0, \qquad s_t + (u - G) s_x = 0,$$

on taking

$$(5) \qquad u = r + s,\ f = r - s; \qquad 2r = u + f,\ 2s = u - f.$$

Thus u and ρ are functions of $r + s$, $r - s$ respectively. Therefore G, a function of ρ, is likewise a function of $r - s$ and we have

$$(6) \qquad u_r = u_s = 1, \qquad G_r = - G_s.$$

Let us consider a solution

$$(7) \qquad r = r(x, t), \qquad s = s(x, t),$$

---

[5] Literal subscripts denote partial differentiation. Equations (1) follow immediately from the hydrodynamical equations and the equation of continuity on setting

$$X = Y = Z = 0, \qquad v = w = 0, \qquad u_y = u_z = 0, \qquad \rho_y = \rho_z = 0.$$

[6] The prime denotes differentiation with respect to ρ. It is assumed that $p(\rho)$ is a monotone increasing function of ρ possessing a continuous derivative with respect to ρ.

of (4) defined and of class C' throughout a region X of the (x,t)-plane and such that the Jacobian

$$J = \frac{\partial(r,s)}{\partial(x,t)} = 2Gr_x s_x,$$

does not vanish in X. *In addition we suppose the region* X *is mapped in a one-to-one manner upon a region* R *in the* (r,s)*-plane by* (7) and denote by

$$(7)' \qquad x = x(r,s), \qquad t = t(r,s),$$

the inverse transformation from R to X. Now

$$r_t = -Jx_s, \; r_x = Jt_s; \qquad s_t = Jx_r, \; s_x = -Jt_r,$$

and substituting for r_t, r_x, s_t, s_x in (4) we see that the functions (7)$'$ are solutions of two partial differential equations, linear in the unknown functions x, t

$$(4)' \qquad x_r - (u - G)t_r = 0, \qquad x_s - (u + G)t_s = 0.$$

A solution (7) of (4) enables one to compute the *state of the gas*, that is, the velocity and density of the gas, on a plane $\bar{x}$ at time $\bar{t}$; provided, of course, that the point $(\bar{x}, \bar{t})$ lies in X. On the other hand, the solution (7)$'$ of (4)$'$ enables one to compute the location and time of occurrence of a preassigned state; provided, naturally, that the point $(\bar{r}, \bar{s})$ corresponding by (5) to the state in question lies in R. *Either aspect is accordingly available for the description of the motion of the gas.*

Choosing the functions x, t in (7)$'$ to be solutions of (4)$'$, the results in (6) may be used to infer that the line integral

$$w = \int\{[x - (u + G)t]dr + [x - (u - G)t]ds\},$$

is independent of the path. Moreover, if $w = w(r,s)$ is the function of r and s thereby defined, we have

$$(7)'' \qquad x - (u + G)t = w_r, \qquad x - (u - G)t = w_s.$$

Subtracting the two equations, and differentiating the first partially with respect to s, we obtain

$$(8) \qquad w_r - w_s = -2Gt, \qquad w_{rs} = (G_r - 1)t,$$

and w *is accordingly a solution of the partial differential equation of the second order*

$$(9) \qquad w_{rs} - \frac{1 - G_r}{2G}(w_r - w_s) = 0.$$

Conversely, if w *is a solution of* (9), *equations* (7)$''$, *when solved for*

x, t yield functions (7)′ of r, s which are solutions of (4)′. This follows directly upon differentiating the first equation in (7)″ partially with respect to s and the second equation partially with respect to r, the relations (6) being kept in mind. The integration of (4)′ is therefore reduced to the integration of (9).

Differentiating the first equation in (7)″ partially with respect to r, the second partially with respect to s, and using (4)′, one obtains

$$-2Gt_r = w_{rr} + (1 + G_r)t, \qquad 2Gt_s = w_{ss} + (1 + G_r)t.$$

Eliminating t by means of (8), the Jacobian

$$J^{-1} = \frac{\partial(x, t)}{\partial(r, s)} = -2Gt_r t_s,$$

is expressible in terms of w as follows,

$$J^{-1} = \frac{1}{2G}\left[w_{rr} - \frac{1 + G_r}{2G}(w_r - w_s)\right]\left[w_{ss} - \frac{1 + G_r}{2G}(w_r - w_s)\right].$$

By hypothesis the Jacobian J is finite and not equal to zero in the region X. It follows that the Jacobian J^{-1} is not equal to zero and is finite in R. Thus if w denotes the solution of (9) corresponding to the solution (7) of (4), the region R contains none of the loci

$$w_{rr} - \frac{1 + G_r}{2G}(w_r - w_s) = \epsilon, \qquad w_{ss} - \frac{1 + G_r}{2G}(w_r - w_s) = \epsilon, \quad (\epsilon = 0 \text{ or } \infty).$$

2. The initial conditions. If the region X over which the solution (7) of (4) is defined is assumed to contain the segment $\alpha < x < \beta$ of the x-axis, this segment is transformed by (7) into an arc in the (r, s)-plane

(10) $\quad C_0\colon\ r = r_0(x) = r(x, 0), \quad s = s_0(x) = s(x, 0), \quad \alpha < x < \beta,$

which, since J cannot vanish in X by hypothesis, is met by a line $r = \text{const.}$ (or $s = \text{const.}$) in at most a single point, or, in other words, C_0 is a *monotone* arc.

Thus the initial velocity u_0 and initial density ρ_0 of the gas are assumed to satisfy the inequalities

$$2r'_0 = u'_0 + \frac{G(\rho_0)}{\rho_0}\rho'_0 \neq 0, \quad 2s'_0 = u'_0 - \frac{G(\rho_0)}{\rho_0}\rho'_0 \neq 0, \quad G(\rho_0) \neq 0,$$

along the segment $\alpha < x < \beta$ of the x-axis. Conversely, if the above inequalities are met, the arc C_0 is a monotone arc.

One implication of the assumption that the Jacobian J does not vanish may now be pointed out. If the gas is initially at rest, the density must be a monotone function of x and the case of wave motion is ruled out.[7] The case in which the Jacobian vanishes identically has been discussed by Riemann[8] and various other writers.

In view of the preceding results, in particular (7)″, the problem of constructing a solution (7) of (4) subject to the initial conditions (10) is equivalent to the following boundary value problem: *to construct a solution w of* (9) *such that the first partial derivatives take values prescribed by*

$$w_r = x, \quad w_s = x, \tag{11}$$

on the monotone arc C_0.

3. Conjugate partial differential equations. Two partial differential equations of the type

$$L(u) \equiv u_{xy} - a(u_x - u_y) = 0, \quad M(v) \equiv v_{xy} + a(v_x - v_y) = 0, \quad a = a(x, y),$$

are said to be *conjugate to each other*. Taking u, v to be solutions of $L(u) = 0$, $M(v) = 0$ respectively, it is readily verified that

$$(u_x v_x)_y = (u_y v_y)_x,$$

and the integral

$$\int \{u_x v_x dx + u_y v_y dy\},$$

taken around any closed path is accordingly equal to zero.[9] Applying this to the closed path formed by two straight lines $x = \bar{x}$, $y = \bar{y}$ and a monotone arc $\bar{X}\bar{Y}$ connecting a point $\bar{X}$ of $x = \bar{x}$ to a point $\bar{Y}$ of $y = \bar{y}$, one has

$$\int_{\bar{P}}^{\bar{X}} u_y v_y dy + \int_{\bar{X}}^{\bar{Y}} \{u_x v_x dx + u_y v_y dy\} + \int_{\bar{Y}}^{\bar{P}} u_x v_x dx = 0,$$

---

[7] In case the density is not a monotone function of x, one may apply the method to the intervals in which the density is monotone. However this will not suffice in general to define the motion of the gas for all x, t. An interesting problem of apparently some difficulty arises when one seeks to prolong the solution over the entire (x, t)-plane.

[8] *Werke*, pp. 162-164. Durand, *Aerodynamic Theory*, *loc. cit.* See in particular H. Weber, *op. cit.*, pp. 516-522.

[9] Naturally, this statement may require qualification if the closed path bounds a region containing points at which u_x, u_y, v_x, v_y possess singularities. The referee points out that the relation of the method used in the text to classical theory is brought out by the identity

$$(v_x - v_y)L(u) + (u_x - u_y)M(v) = P_y - Q_x, \qquad P = u_x v_x, \qquad Q = u_y v_y,$$

analogous to the well-known identity

$$vL(u) - uM(v) = P_y - Q_x,$$

where M is adjoint to L. Assuming that $L(u) = M(v) = 0$, it is apparent that $\int \{u_x v_x dx + u_y v_y dy\}$ taken around any closed path is zero.

where $\bar{P}$ is the point $(\bar{x}, \bar{y})$. Consequently if a solution $\bar{v}$ of $M(v) = 0$ be taken so that

$$\text{(12)} \qquad \text{on } \bar{P}\bar{X}: \bar{v} = \bar{y} - y, \qquad \text{on } \bar{P}\bar{Y}: \bar{v} = x - \bar{x},$$

one finds the value $u(\bar{P})$ of the solution u at the point $\bar{P}$ expressed in terms of its values at $\bar{X}$ and $\bar{Y}$, and the values of u_x, u_y along the monotone arc $\bar{X}\bar{Y}$, by the formula

$$\text{(13)} \qquad u(\bar{P}) = \tfrac{1}{2}[u(\bar{X}) + u(\bar{Y})] - \tfrac{1}{2} \int_{\bar{X}}^{\bar{Y}} \{u_x \bar{v}_x dx + u_y \bar{v}_y dy\}.$$

The solution $\bar{v}$ of $M(v) = 0$ employed above obviously depends on the coördinates $\bar{x}$, $\bar{y}$ of the point $\bar{P}$, a fact which we signalize in the notation

$$\bar{v} = \bar{v}(x, y; \bar{x}, \bar{y}) = \bar{v}(P; \bar{P}).$$

The function $\bar{v}$ is termed the *resolvent* of $L(u) = 0$ and the problem of integrating $L(u) = 0$ has been reduced to its determination.

The resolvent $\bar{u}$ of $M(v) = 0$ is a solution of $L(u) = 0$ equal to $y^* - y$ on $x = x^*$ and to $x - x^*$ on $y = y^*$. It is connected with the resolvent $\bar{v}$ of $L(u) = 0$ by the relation

$$\text{(14)} \qquad \bar{u}(\bar{P}; P^*) + \bar{v}(P^*; \bar{P}) = 0,$$

analogous to the Principle of Reciprocity in the theory of the adjoint partial differential equation. To prove this relation, consider the parallelogram with the four vertices

$$\bar{P}(\bar{x}, \bar{y}),\ \bar{X}(\bar{x}, y^*),\ \bar{Y}(x^*, \bar{y}),\ P^*(x^*, y^*).$$

Replacing u by $\bar{u}$ and the arc $\bar{X}\bar{Y}$ in (13) by the broken line $\bar{X}P^*\bar{Y}$, one finds

$$\bar{u}(\bar{P}; P^*) = \tfrac{1}{2}[\bar{u}(\bar{X}; P^*) + \bar{u}(\bar{Y}; P^*)] - \tfrac{1}{2} \int_{\bar{X}}^{P^*} \bar{u}_x(P; P^*) \bar{v}_x(P; \bar{P}) dx$$

$$- \tfrac{1}{2} \int_{P^*}^{\bar{Y}} \bar{u}_y(P; P^*) \bar{v}_y(P; \bar{P}) dy,$$

and, since by definition

$$\bar{u}_x(P; P^*) = 1 \text{ on } P^*\bar{X}, \qquad \bar{u}_y(P; P^*) = -1 \text{ on } P^*\bar{Y},$$

the relation is established, inasmuch as

$$\bar{u}(\bar{X}; P^*) = \bar{x} - x^*,\ \bar{u}(\bar{Y}; P^*) = y^* - \bar{y},\ \bar{v}(\bar{X}; \bar{P}) = \bar{y} - y^*,\ \bar{v}(\bar{Y}; \bar{P}) = x^* - \bar{x}.$$

Among the relations to be obtained from (14) by partial differentiation with respect to $\bar{x}$, $\bar{y}$, x^*, y^* we shall set down only two

$$(15)\quad \bar u_{\bar x}(\bar P;P^*)+\bar v_{\bar x}(P^*;\bar P)=0,\qquad \bar u_{\bar y}(\bar P;P^*)+\bar v_{\bar y}(P^*;\bar P)=0,$$

as important for later application.

Returning to the solution (13) and differentiating it partially with respect to $\bar x$ and $\bar y$, it is found that

$$(16)\qquad u_{\bar x}=\tfrac{1}{2}u_x(\bar X)[1+\bar v_x(\bar X)]-\tfrac{1}{2}\int_{\bar X}^{\bar Y}\{u_x\bar v_{x\bar x}dx+u_y\bar v_{y\bar x}dy\},$$

$$(17)\qquad u_{\bar y}=\tfrac{1}{2}u_y(\bar Y)[1-\bar v_y(\bar Y)]-\tfrac{1}{2}\int_{\bar X}^{\bar Y}\{u_x\bar v_{x\bar y}dx+u_y\bar v_{y\bar y}dy\}.$$

It will be recalled that (14) was derived from (13) by replacing u by $\bar u$ and the arc $\bar X\bar Y$ by the broken line $\bar XP^*\bar Y$. If the same procedure is applied to (16) and (17), it is found that

$$\begin{aligned}\bar v_x(\bar X;\bar P)+\bar v_{\bar x}(\bar X;\bar P)&=2[\bar u_{\bar x}(\bar P;P^*)+\bar v_{\bar x}(P^*;\bar P)],\\ \bar v_y(\bar Y;\bar P)+\bar v_{\bar y}(\bar Y;\bar P)&=2[\bar u_{\bar y}(\bar P;P^*)+\bar v_{\bar y}(P^*;\bar P)],\end{aligned}$$

inasmuch as

$$(18)\qquad \bar v_{\bar x}(\bar Y;\bar P)=-1,\qquad \bar v_{\bar y}(\bar X;\bar P)=1.$$

Referring to (15) it is apparent that

$$\bar v_x(\bar X;\bar P)+\bar v_{\bar x}(\bar X;\bar P)=0,\qquad \bar v_y(\bar Y;\bar P)+\bar v_{\bar y}(\bar Y;\bar P)=0,$$

which, since $\bar X$ may be taken arbitrarily on $x=\bar x$ and $\bar Y$ may be taken arbitrarily on $y=\bar y$, leads to the following principle. *The resolvent* $\bar v=\bar v(x,y;\ \bar x,\bar y)$ *satisfies the relations*

$$(19)\qquad \bar v_x+\bar v_{\bar x}=0,\qquad \bar v_y+\bar v_{\bar y}=0,$$

respectively along the lines $x=\bar x$ *and* $y=\bar y$.

4. Rectilinear motion of a gas. In the first two sections of this paper the problem of determining the rectilinear motion of a gas under the hypothesis stated therein has been reduced to the boundary value problem formulated at the close of **2**. For any equation of state $p=p(\rho)$ the partial differential equation (9) takes the form

$$(20)\qquad w_{rs}-a(w_r-w_s)=0,\qquad a=a(r,s),$$

and, granting that the resolvent,

$$\bar v=\bar v(r,s;\ \bar r,\bar s),$$

of this partial differential equation has been determined, the value of the solution w at a point $(\bar r, \bar s)$ is, from (13), computed by

$$w(\bar r, \bar s) = \tfrac{1}{2}[w(\bar R) + w(\bar S)] - \tfrac{1}{2} \int_{\bar R}^{\bar S} \{w_r \bar v_r dr + w_s \bar v_s ds\},$$

in which $\bar R$, $\bar S$ denote the points where[10] the lines $r = \bar r$, $s = \bar s$, respectively, intersect the monotone arc C_0.

What is really needed, however, in (7″), is not the value of the function w at the point $(\bar r, \bar s)$, but the values of its partial derivatives w_r, w_s at this point. For these we refer to (16), (17) and obtain

$$w_{\bar r} = \tfrac{1}{2} w_r(\bar R)[1 + \bar v_r(\bar R)] - \tfrac{1}{2} \int_{\bar R}^{\bar S} \{w_r \bar v_{r\bar r} dr + w_s \bar v_{s\bar r} ds\},$$

$$w_{\bar s} = \tfrac{1}{2} w_s(\bar S)[1 - \bar v_s(\bar S)] - \tfrac{1}{2} \int_{\bar R}^{\bar S} \{w_r \bar v_{r\bar s} dr + w_s \bar v_{s\bar s} ds\},$$

which, when the initial values (11) for w_r, w_s on C_0 are inserted, become

$$w_{\bar r} = \tfrac{1}{2} x(\bar R)[1 + \bar v_r(\bar R)] - \tfrac{1}{2} \int_{\bar R}^{\bar S} x\, d\bar v_{\bar r}, \quad w_{\bar s} = \tfrac{1}{2} x(\bar S)[1 - \bar v_s(\bar S)] - \tfrac{1}{2} \int_{\bar R}^{\bar S} x\, d\bar v_{\bar s},$$

where $x(\bar R)$, $x(\bar S)$ denote the abscissae of the points on the interval $\alpha < x < \beta$ of the x-axis which map into the points $\bar R$, $\bar S$ under the transformation (10). After integrating by parts and applying (18) and (19′) it is found that

$$w_r = \tfrac{1}{2}[x(\bar R) + x(\bar S)] + \tfrac{1}{2} \int_{x(\bar R)}^{x(\bar S)} \bar v_{\bar r} dx, \quad w_{\bar s} = \tfrac{1}{2}[x(\bar R) + x(\bar S)] + \tfrac{1}{2} \int_{x(\bar R)}^{x(\bar S)} \bar v_{\bar s} dx.$$

The motion of the gas is consequently to be determined from the following equations

$$(21) \qquad \bar x - (\bar u + \bar G)\bar t = \tfrac{1}{2}[x(\bar R) + x(\bar S)] + \tfrac{1}{2} \int_{x(\bar R)}^{x(\bar S)} \bar v_{\bar r} dx,$$

$$\bar x - (\bar u - \bar G)\bar t = \tfrac{1}{2}[x(\bar R) + x(\bar S)] + \tfrac{1}{2} \int_{x(\bar R)}^{x(\bar S)} \bar v_{\bar s} dx,$$

obtained by substituting for the partial derivatives in (7)″.

---

[10] The values of w at the points $\bar R$, $\bar S$ are indicated by $w(\bar R)$, $w(\bar S)$ respectively.

The use of these formulae may best be understood with the aid of Figure 1. Suppose the initial velocity and density u_0, ρ_0 of the gas are known along the segment $\alpha < x < \beta$ of the x-axis. To fix the ideas, let us suppose that along this segment u_0, ρ_0 are chosen so that the arc C_0 of **2** is the monotone arc $A\bar{R}\bar{S}B$ of the figure. The basic idea in the use of these formulae is that they yield a mapping of the "triangular region" ABC of the (r, s)-plane upon the "triangular region" $\alpha\beta\gamma$ of the (x, t)-plane. Thus the mapping brings to each point $(\bar{x}, \bar{t})$ inside $\alpha\beta\gamma$ a pair of values $\bar{r}, \bar{s}$ from which the velocity $\bar{u}$ and density $\bar{\rho}$ of the gas at the point $(\bar{x}, \bar{t})$ is computed from (5) and (3).

The details of the application of these formulae will be taken up in **7**, where they are applied to a specific problem in the rectilinear motion of a gas.

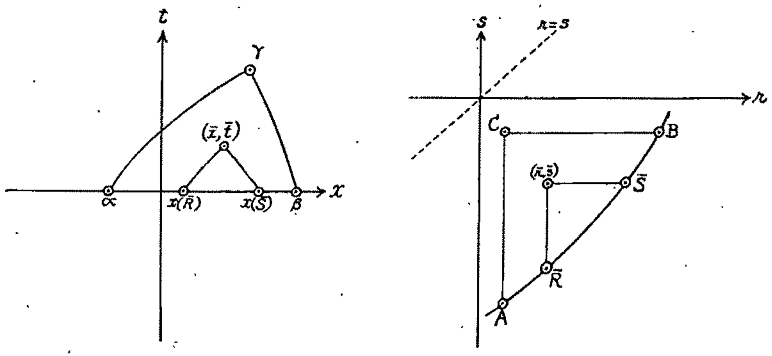


Figure 1.

5. The adiabatic case. The results obtained so far are valid independent of the choice made for the function $p(\rho)$ expressing the dependence of the pressure of a gas upon its density. From now on we shall suppose that the motion of the gas proceeds adiabatically. The characteristic equation of the gas takes the well-known form

$$p = k^2 \rho^\gamma,$$

where γ is the ratio of the two specific heats of the gas and k is a constant. It is readily verified that

$$G = \frac{\gamma - 1}{2}(r - s),$$

and that the partial differential equation (9) becomes

$$(r - s)w_{rs} - m(w_r - w_s) = 0, \qquad m = \frac{3 - \gamma}{2(\gamma - 1)}. \tag{22}$$

This type of partial differential equation has received intensive study[11] since the time of Euler.

Equations (21) controlling the motion of the gas take the form

$$(23) \qquad \bar{x} - \left(\frac{1+\gamma}{2}\bar{r} + \frac{3-\gamma}{2}\bar{s}\right)\bar{t} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \bar{v}_{\bar{r}}\,dx,$$

$$\bar{x} - \left(\frac{3-\gamma}{2}\bar{r} + \frac{1+\gamma}{2}\bar{s}\right)\bar{t} = \tfrac{1}{2}[x(\bar{R}) + x(\bar{S})] + \tfrac{1}{2}\int_{x(\bar{R})}^{x(\bar{S})} \bar{v}_{\bar{s}}\,dx,$$

and, once the resolvent $\bar{v}$ of (22) has been found, may be used to compute the motion of the gas as explained in **4**.

According to the kinetic theory of gases[12]

$$\gamma = 1 + \frac{2}{3+n},$$

where $n = 0, 2$ for monatomic and diatomic gases respectively, a result which is in good agreement with experiment. The corresponding value of m is

$$m = \tfrac{1}{2}(n+2),$$

so that $m = 1$ for monatomic gases and $m = 2$ for diatomic gases. There appears[12] to be no known law for n in the case of more complex gases, and we are accordingly led to consider the problem of integrating (22) for arbitrary constant values of m.

The general solutions of (22) for monatomic and diatomic gases are

$$w = \frac{R-S}{r-s}, \qquad w = \frac{\partial^2}{\partial r \partial s}\,\frac{R-S}{r-s},$$

where R denotes an arbitrary function of r and S an arbitrary function of s. Although the general solution of (22) is available[11] for arbitrary constant values of m, our problem is to find a solution subject to the initial conditions (11). Presumably one might achieve this by properly specializing the arbitrary elements in the general solution. However it appears more feasible to proceed along the lines of **4** and we accordingly take up the problem of determining the resolvent of (22).

6. The resolvent. The resolvent $\bar{v}$ of (22) is a two parameter family $\bar{v} = \bar{v}(r, s; \bar{r}, \bar{s})$ of solutions of the conjugate equation

---

[11] See, for example, G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, 2nd Ed., Paris (1915), pp. 54-70.

[12] J. H. Jeans, *The Dynamical Theory of Gases*, 4th Ed., Cambridge (1925), pp. 191-193. Recent work by D. H. Andrews has enabled fairly accurate estimates of γ to be made for molecules containing up to 20 atoms.

$$(r-s)v_{rs}+m(v_r-v_s)=0, \tag{24}$$

such that

$$\bar v(\bar r, s;\ \bar r, \bar s)=\bar s-s, \qquad \bar v(r, \bar s;\ \bar r, \bar s)=r-\bar r. \tag{25}$$

It is easily verified[13] that a one parameter family of solutions of (24) is given by

$$v=(r-a)^m(a-s)^m,$$

a being the parameter of the family. This one parameter family offers the starting point for our construction of the resolvent.

To begin with we observe in (3) that f is a positive function of ρ and therefore it follows from (5) that $r>s$. We may accordingly confine our attention to the half plane lying under the line $r=s$. Taking $\bar r>\bar s$, we shall construct the resolvent of (22) under the following restrictions

$$r>\bar r>0,\ s<\bar s; \qquad m=-\lambda \qquad (0<\lambda<1). \tag{26}$$

Eventually we shall show how these restrictions may be removed.

By forming the solution[14]

$$v=\int_s^r \phi(a)(r-a)^{-\lambda}(a-s)^{-\lambda}da,$$

where $\phi(a)$ is an arbitrary function we gain a solution of (24) containing an arbitrary function. Taking $\phi(a)=0$ for $\bar s\leqq a\leqq\bar r$, we arrive at a two-parameter family of solutions

$$v=\int_{\bar r}^{r} \phi(a)(r-a)^{-\lambda}(a-s)^{-\lambda}da+\int_{s}^{\bar s} \phi(a)(r-a)^{-\lambda}(a-s)^{-\lambda}da, \tag{27}$$

involving an arbitrary function. If this arbitrary function can now be determined so that the conditions laid down in (25) are met, the resolvent of (22) will have been determined.

Thus we are to determine $\phi(a)$ so that

$$\int_{\bar r}^{r} \phi(a)(a-\bar s)^{-\lambda}(r-a)^{-\lambda}da=r-\bar r, \quad \int_{\bar s}^{s} \phi(a)(\bar r-a)^{-\lambda}(a-s)^{-\lambda}da=s-\bar s \tag{28}$$

Let us consider the first of these integral equations. If we define

$$\psi(t)=\Psi(t)=0,\ 0\leqq t\leqq\bar r; \qquad \psi(t)=\phi(t)(t-\bar s)^{-\lambda},\ \Psi(t)=t-\bar r,\ t\geqq\bar r,$$

this integral equation is recognized as the generalized integral equation of Abel

$$\int_0^r \psi(t)(r-t)^{-\lambda}dt=\Psi(t).$$

---

[13] Cf. G. Darboux, *op. cit.*, p. 58.

[14] Cf. G. Darboux, *op. cit.*, pp. 66-68.

The solution of this integral equation is[15]

$$\psi(a) = \frac{\sin \pi\lambda}{\pi} \int_0^a \Psi'(t)\,(a-t)^{\lambda-1}dt = \frac{\sin \pi\lambda}{\pi} \int_{\bar r}^a (a-t)^{\lambda-1}dt,$$

from which it is readily ascertained that

$$\phi(a) = \frac{\sin \pi\lambda}{\pi\lambda}\,(a-\bar r)^{\lambda}(a-\bar s)^{\lambda}. \tag{29}$$

The second integral equation goes over into the first when the variables in the pairs (r, s), $(\bar r, \bar s)$ are interchanged. It follows that the solution is also given by (29).

When $\phi(a)$ is substituted in (27) and a new integration variable t is introduced by setting

$$a = \bar r + t(r-\bar r), \qquad a = \bar s + t(s-\bar s),$$

in the first and second integrals respectively, the resolvent $\bar v$ is obtained in the form

$$\bar v = \frac{\sin \pi\lambda}{\pi\lambda} \Big\{ (r-\bar r) \int_0^1 \frac{t^{\lambda}[\bar r-\bar s+t(r-\bar r)]^{\lambda}}{(1-t)^{\lambda}[\bar r-s+t(r-\bar r)]^{\lambda}}\,dt$$
$$+ (\bar s-s) \int_0^1 \frac{t^{\lambda}[\bar r-\bar s-t(s-\bar s)]^{\lambda}}{(1-t)^{\lambda}[r-\bar s-t(s-\bar s)]^{\lambda}}\,dt \Big\}.$$

The resolvent will now be expressed in terms of Appell's hypergeometric function of two variables[16]

$$F_1(\alpha;\,\beta,\beta';\,\gamma;\,x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)\,(\beta, m)\,(\beta', n)}{(\gamma, m+n)\,(1, m)\,(1, n)}\,x^m y^n, \tag{30}$$
$$(\alpha, r) = \alpha(\alpha+1)\cdots(\alpha+r-1).$$

Indeed if $\Re(\alpha) > 0$, $\Re(\gamma-\alpha) > 0$, Picard has shown[17] that

$$\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}dt = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\,F_1(\alpha;\beta,\beta';\gamma;x,y).$$

If we put $\alpha = 1+\lambda$, $\beta = -\lambda$, $\beta' = \lambda$, $\gamma = 2$, in this formula and observe that

$$\frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(2)} = \frac{\pi\lambda}{\sin \pi\lambda},$$

we find on writing

---

[15] The solution of this integral equation is well known. Cf., for example, G. Kowalewski, *Integralgleichungen*, Leipzig (1930), pp. 11-13.

[16] P. Appell, "Sur les fonctions hypergéométriques de plusieurs variables," *Memorial des Sciences Mathématiques*, Fasc. III (1925), p. 4.

[17] See Appell, *op. cit.*, p. 9.

$$x=\frac{r-\bar{r}}{\bar{s}-\bar{r}},\ y=\frac{r-\bar{r}}{s-\bar{r}};\quad \text{and}\quad x=\frac{s-\bar{s}}{\bar{r}-\bar{s}},\ y=\frac{s-\bar{s}}{r-\bar{s}},$$

that the resolvent may be expressed as follows

$$(31)\quad \bar{v}=(r-\bar{r})\left(\frac{\bar{r}-\bar{s}}{\bar{r}-s}\right)^{\lambda}F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{r-\bar{r}}{\bar{s}-\bar{r}},\frac{r-\bar{r}}{s-\bar{r}}\right)$$
$$-(s-\bar{s})\left(\frac{\bar{r}-\bar{s}}{r-\bar{s}}\right)^{\lambda}F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{s-\bar{s}}{\bar{r}-\bar{s}},\frac{s-\bar{s}}{r-\bar{s}}\right).$$

It will be recalled that the resolvent was derived under the restrictions (26). These restrictions we now proceed to remove. Availing ourselves of the formula[18]

$$F_1(\alpha;\beta,\beta';\gamma;x,y)=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha;\beta,\beta';\gamma;\frac{x}{x-1},\frac{y}{y-1}\right),$$

we find alternative expressions for the hypergeometric functions in (31), namely,

$$F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{r-\bar{r}}{\bar{s}-\bar{r}},\frac{r-\bar{r}}{s-\bar{r}}\right)$$
$$=\left(\frac{r-\bar{s}}{\bar{r}-\bar{s}}\right)^{\lambda}\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{r-\bar{r}}{r-\bar{s}},\frac{r-\bar{r}}{r-s}\right),$$

$$F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{s-\bar{s}}{\bar{r}-\bar{s}},\frac{s-\bar{s}}{r-\bar{s}}\right)$$
$$=\left(\frac{\bar{r}-s}{\bar{r}-\bar{s}}\right)^{\lambda}\left(\frac{r-\bar{s}}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{s-\bar{s}}{s-\bar{r}},\frac{s-\bar{s}}{s-r}\right),$$

and thus we are led to consider three additional forms for $\bar{v}$:

$$(32)\quad \bar{v}=(r-\bar{r})\left(\frac{r-\bar{s}}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda,2;\frac{r-\bar{r}}{r-\bar{s}},\frac{r-\bar{r}}{r-s}\right)$$
$$-(s-\bar{s})\left(\frac{\bar{r}-\bar{s}}{r-\bar{s}}\right)^{\lambda}F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{s-\bar{s}}{\bar{r}-\bar{s}},\frac{s-\bar{s}}{r-\bar{s}}\right),$$

$$(33)\quad \bar{v}=(r-\bar{r})\left(\frac{\bar{r}-\bar{s}}{\bar{r}-s}\right)^{\lambda}F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{r-\bar{r}}{\bar{s}-\bar{r}},\frac{r-\bar{r}}{s-\bar{r}}\right)$$
$$-(s-\bar{s})\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda,2;\frac{s-\bar{s}}{s-\bar{r}},\frac{s-\bar{s}}{s-r}\right),$$

$$(34)\quad \bar{v}=(r-\bar{r})\left(\frac{r-\bar{s}}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda,2;\frac{r-\bar{r}}{r-\bar{s}},\frac{r-\bar{r}}{r-s}\right),$$
$$-(s-\bar{s})\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda,2;\frac{s-\bar{s}}{s-\bar{r}},\frac{s-\bar{s}}{s-r}\right).$$

---

[18] See, for example, W. N. Bailey, "Generalized hypergeometric series," *Cambridge Tracts in Mathematics and Physics*, No. 32, Cambridge (1935), p. 78.

Appell[19] has shown that the power series (30) converges for *arbitrary real or complex values of* α, β, γ inside the unit square $|x| < 1$, $|y| < 1$, provided, of course, γ is not a negative integer. The various formulae obtained for $\bar{v}$ accordingly define the resolvent in the following regions respectively:

$$(31)' \; |r-\bar{r}| < |\bar{r}-\bar{s}|, \; |r-\bar{r}| < |s-\bar{r}|, \; |s-\bar{s}| < |\bar{r}-\bar{s}|, \; |s-\bar{s}| < |r-\bar{s}|;$$

$$(32)' \; |r-\bar{r}| < |r-\bar{s}|, \; |r-\bar{r}| < |r-s|, \; |s-\bar{s}| < |\bar{r}-\bar{s}|, \; |s-\bar{s}| < |r-\bar{s}|;$$

$$(33)' \; |r-\bar{r}| < |\bar{r}-\bar{s}|, \; |r-\bar{r}| < |s-\bar{r}|, \; |s-\bar{s}| < |s-\bar{r}|, \; |s-\bar{s}| < |s-r|;$$

$$(34)' \; |r-\bar{r}| < |r-\bar{s}|, \; |r-\bar{r}| < |r-s|, \; |s-\bar{s}| < |s-\bar{r}|, \; |s-\bar{s}| < |s-r|.$$

The four forms (31), (32), (33), (34) obtained for $\bar{v}$ are analytic continuations of each other and taken together suffice to define the resolvent throughout the region

$$s < \bar{r}, \quad r > s, \quad r > \bar{s},$$

for any real value of λ. By hypothesis the arc $\bar{R}\bar{S}$ in (21) or (23) connecting a point of $r = \bar{r}$ to a point of $s = \bar{s}$ is a monotone arc lying in the region $r > s$ and therefore cannot leave the above region. It follows that $\bar{v}$ is adequately determined for use in (21) or (23).

The formula[20]

$$F_1(\alpha; \beta, \beta'; \gamma; x, x) = F(\alpha, \beta + \beta', \gamma; x),$$

reducing the hypergeometric function F_1 to an ordinary hypergeometric function affords a partial check on our computation, inasmuch as, taking $r = \bar{r}$ and $s = \bar{s}$ in (31) yields

$$\bar{v}(\bar{r}, s; \bar{r}, \bar{s}) = (\bar{s} - s) F\left(1 + \lambda, 0, 2; \frac{s - \bar{s}}{\bar{r} - \bar{s}}\right) = \bar{s} - s,$$

$$\bar{v}(r, \bar{s}; \bar{r}, \bar{s}) = (r - \bar{r}) F\left(1 + \lambda, 0, 2; \frac{r - \bar{r}}{\bar{s} - \bar{r}}\right) = r - \bar{r},$$

which agrees with (25).

A further check is provided by the monatomic gases, the partial differential equation (22) becoming

$$(35) \qquad (r - s) w_{rs} - (w_r - w_s) = 0.$$

Taking $\lambda = -1$ in (31), the resolvent of (35) is calculated with the aid of (30) and found to be

$$(36) \qquad \bar{v} = \frac{(r - \bar{r})(\bar{r} - s) - (s - \bar{s})(r - \bar{s})}{\bar{r} - \bar{s}},$$

---

[19] *Op. cit.*, p. 4.

[20] P. Appell, *op. cit.*, pp. 3-4.

and a direct computation verifies that this function $\bar{v}$ is a solution of the conjugate equation to (35) and that $\bar{v}$ fulfills the conditions required in (25).

7. The motion of a monatomic gas. Of all the gases the motion of a monatomic gas appears to be most susceptible to mathematical investigation. Thus the resolvent $\bar{v}$ has the particularly simple form (36). One readily verifies that placing this resolvent in (23) with $\gamma = 5/3$ leads to the following explicit formulae:

$$\bar{x} = (\bar{r} - \bar{s})^{-3}\{\bar{r}(\bar{r}^2 + \bar{r}\bar{s} + 4\bar{s}^2)x(\bar{R}) - \bar{s}(4\bar{r}^2 + \bar{r}\bar{s} + \bar{s}^2)x(\bar{S})$$

$$- \int_{x(\bar{R})}^{x(\bar{S})} [3(\bar{r} + \bar{s})rs - 2(\bar{r}^2 + \bar{r}\bar{s} + \bar{s}^2)(r + s)]dx\},$$

$$\bar{t} = 3(\bar{r} - \bar{s})^{-3}\{\bar{r}\bar{s}[x(\bar{R}) - x(\bar{S})] - \int_{x(\bar{R})}^{x(\bar{S})} [rs - \tfrac{1}{2}(\bar{r} + \bar{s})(r + s)]dx\}.$$

To illustrate the application of the method to a concrete problem let us investigate the rectilinear motion of a monatomic gas which initially is at rest everywhere, a vacuum ($\rho = 0$) being present for negative x and $\rho = (2x/15k^2)^{3/2}$ for positive x, under the assumption that the gas moves adiabatically. The apparent artificiality in the choice of the initial density due to an attempt on our part to find an example which illustrates the method without burdening the reader with tedious and excessive computation.

Referring to **1**, one verifies that $f = \sqrt{15}\, k\rho^{1/3}$ and that the parametric equations (10) of the arc C_0 in **2** (the arc $A\bar{R}\bar{S}B$ in Figure 1) are

$$r = (x/2)^{1/2}, \qquad s = -(x/2)^{1/2}.$$

Substituting $x(\bar{R}) = 2\bar{r}^2$, $x(\bar{S}) = 2\bar{s}^2$ and carrying out the integration in the above formulae, we find on dropping the bars from all letters that the region $r \geqq s$ of the (r, s)-plane is mapped upon a region in the (x, t)-plane by

$$x = -(r^2 + 4rs + s^2), \qquad t = -3(r + s).$$

These formulas take on greater physical significance if the variables u, f are employed in place of r, s. In terms of these variables, the mapping is given by

$$x = \tfrac{1}{2}(f^2 - 3u^2), \qquad t = -3u. \tag{37}$$

It is apparent that the loci in the (x, t)-plane of points where the gas has a constant velocity comprise the lines $t = \text{const.}$, while the loci of points where the gas has constant density are the parabolas $x = \frac{1}{2}(f^2 - t^2/3)$, $f = \text{const.}$ These curves are graphed below and form a network over a region in the (x, t)-plane, from which it is possible to read off the velocity and density of the gas at any position and at any time. Outside this region both the density and velocity of the gas is zero, i. e., physically one has a vacuum. As a check on our computation, one finds on solving (37) for u, f in terms of x, t, that

$$u = -t/3, \qquad \rho = \left(\frac{2x + t^2/3}{15k^2}\right)^{3/2},$$

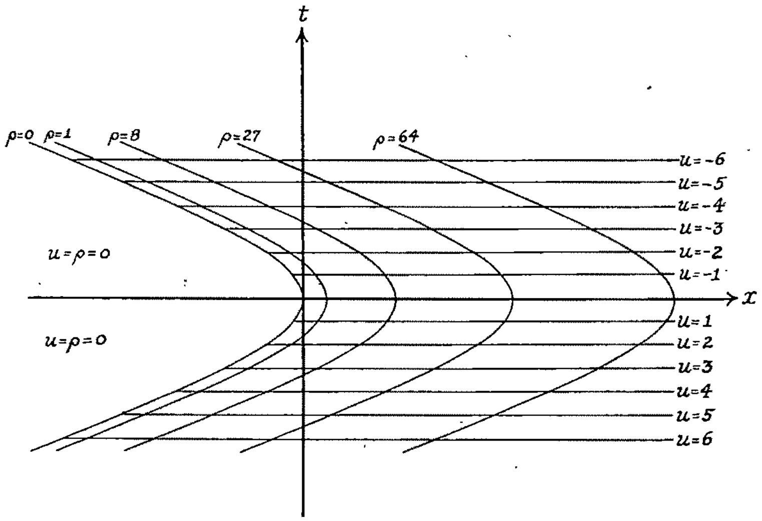


Figure 2.

yield a solution of the hydrodynamical equations (1). It will be noted that a discontinuity in the velocity presents itself along the parabola $\rho = 0$ of Figure 2. The discontinuity meets, however, the compatibility, impulse, and energy conditions.[21]

UNIVERSITY OF MARYLAND,
COLLEGE PARK, MD.

---

[21] For a formulation of these conditions, see Weber, *op. cit.*, p. 511.

A CLASS OF SOLUTIONS FOR THE HEAT EQUATION AND ASSOCIATED BOUNDARY VALUE PROBLEMS.*

By F. G. Dressel and E. R. Elliott.

1. Introduction. Associated with the heat equation

$$\partial^2 u/\partial x^2 = \partial u/\partial y, \tag{1}$$

we have the region D bounded by the characteristics whose ordinates are h and g $(h < g)$, and the curves

$$x = \chi_1(y), \qquad x = \chi_2(y), \qquad h \leq y \leq g, \ \chi_1 < \chi_2.$$

The problem of finding a solution of (1) in D, taking on preassigned continuous values on the boundary of D[1] has been treated by several authors. Holmgren[2] solved the problem, using integral equations, under the assumption that the $\chi_i(\eta)$ had continuous derivatives. Later papers have sought to weaken the hypothesis on the $\chi_i(\eta)$[3] or to use other methods for the solution.[4]

In the first part of this paper we treat a problem similar to that of a former paper,[5] where certain line integrals of solutions of (1) were made to take on boundary values of limited variation.[6] In Part 2 of the present paper,

---

* Received June 19, 1941; Revised September 18, 1942; Presented, in part, to the Society, April 8, 1939.

[1] No boundary values are assigned on the line $y = g$.

[2] E. Holmgren, "Sur une application de l'équation intégral de M. Volterra," *Arkiv för Matematik*, vol. 3 (1906-07), pp. 1-4.

[3] M. Gevrey, "Sur les équations aux dérivées partielles du type parabolique," *Journal de Mathématiques*, (6), vol. 9 (1913), pp. 305-471. Gevrey showed that, if the $\chi_i(\eta)$ satisfy a Lipschitz condition of order $\alpha > \frac{1}{2}$ the problem has a solution.

[4] W. Sternberg, "Über die Gleichung der Wärmeleitung," *Mathematische Annalen*, vol. 101 (1929), pp. 394-398. Here the problem was solved by Perron's method under the assumption that the derivatives of the $\chi_i(\eta)$ were bounded. I. Petrowsky ["Zur ersten Randwertaufgabe der Wärmeleitunggleichung," *Compositio Mathematica*, vol. 1 (1934-35), pp. 383-419] improved Sternberg's treatment and showed that a weaker condition than that of Gevrey was sufficient.

[5] F. G. Dressel, "A boundary value problem for the heat equation," *American Journal of Mathematics*, vol. 55 (1933), pp. 641-653. This will be referred to as paper (A).

[6] The results of Part 1 of this paper are taken from E. R. Elliott's doctoral dissertation presented while in the graduate school of Duke University. The investigation was suggested by Professor E. R. C. Miles, to whom Mr. Elliott is indebted for suggestions and encouragement.

we show how to construct a solution of equation (1) in D which takes on preassigned functions of limited variation on the boundary. However, the approaches to the boundary of D are in a restricted sense that we call parabolic approaches.

PART 1

In this part of the paper we shall show how to construct a solution $u(x, y)$ in the region D satisfying boundary conditions of the following type:

$$\lim_{\delta\to 0}\int_{x_1}^{x_2} u(t,\delta)\,dt = B(x_2) - B(x_1), \qquad \delta > 0, \tag{2}$$

$$\lim_{\lambda_i\to 0}\int_{y_1}^{y_2} u_x(\chi_i(t)+\lambda_i, t)\,dt = G_i(y_2) - G_i(y_1), \quad (-1)^{i+1}\lambda_i > 0, \; i = 1, 2, \tag{3}$$

where $B(x)$ and $G_i(y)$ are preassigned functions of limited variation. The problem differs from that in paper (A) in that here the line integral of u_x parallel to the curves χ_i is used in place of the line integral of the solution u itself. To carry through some of the proofs we have found it necessary to replace the Riemann-Stieltjes integral used in our former paper by the Young-Stieltjes integral.

Before stating our principal result we shall prove several introductory theorems about the class of functions we are using.

2. Lemmas. We first prove a lemma on the change of order of integration.

LEMMA 1. *Let* $G(x, y)$ *be continuous in* $m \leq x \leq n$, $p \leq y \leq q$, *with the possible exception of the point* (a, b), $m < a < n$, $p < b < q$. *Let*

$$\left| \int_m^n G(x,y)\,dx \right| < M, \qquad p \leq y \leq q.$$

If $F(y)$ *is of limited variation, then*

$$\int_m^n dx \cdot \int_p^q G(x,y)\,dF(y) = \int_p^q dF(y) \cdot \int_m^n G(x,y)\,dx,$$

when the above integrals are considered in the Young-Stieltjes sense.

From the hypothesis, it follows that $\int G(x,y)\,dx$ exists for each y, is bounded in y and continuous except for $y = b$; hence

$$\int_p^q dF(y) \cdot \int_m^n G(x,y)\,dx \tag{4}$$

exists in the Young-Stieltjes sense. Further by noting that $G(x, y)$ is continuous for $m \leq x \leq a - \delta$, $p \leq y \leq q$, $(\delta > 0)$, we have

$$\int_m^{a-\delta} dx \cdot \int_p^q G(x,y)\,dF(y) = \int_p^q dF(y) \cdot \int_m^{a-\delta} G(x,y)\,dx.$$

As $\delta \to 0$, the integral on the right is seen to have a bounded limit. A similar treatment of the interval $a \leq x \leq n$ shows that the integral

$$\int_m^n dx \cdot \int_p^q G(x,y)\,dF(y) \tag{5}$$

exists. Since the two integrals (4) and (5) exist, they are equal by a theorem of W. H. Young.[7]

The fundamental solution of (1) will be denoted by

$$U(x,y) = [4\pi y]^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4y}\right), \; y > 0; \qquad = 0, \; y \leq 0.$$

By $V(x,y)$ we shall mean the function

$$V(x,y) = -2U_x(x,y), \; y > 0; \qquad = 0, \; y \leq 0, \tag{6}$$

which is a solution of (1) for $y > 0$.

We shall now prove several theorems about solutions of the heat equation constructed from the functions U and V.

THEOREM 1. *Let $F(y)$ be of limited variation on $h \leq y \leq g$, and continuous at h. Let $\chi(y)$ be continuous on the same interval. Then*

$$I(\delta) \equiv \int_{x_1}^{x_2} dx \cdot \int_h^{h+\delta} U(x - \chi(\eta),\; h + \delta - \eta)\,df(\eta) \to 0, \; \textit{with} \; \delta \to 0.$$

$I(\delta)$ is the line integral of a solution of (1) along the line $y = h + \delta$. Applying our first lemma, we have

$$\begin{aligned} |\, I(\delta) \,| &= \left| \int_h^{h+\delta} dF(\eta) \cdot \int_{x_1}^{x_2} U(x - \chi(\eta),\; h + \delta - \eta)\,dx \right| \\ &\leq M \cdot T(h, h+\delta), \end{aligned}$$

where M is an upper bound for the integral of U with respect to x and $T(h, h+\delta)$ denotes the total variation of $F(y)$ over $h \leq y \leq y + \delta$. $T(h, h+\delta)$ approaches zero with δ, since $F(y)$ is continuous at $y = h$; hence

$$\lim_{\delta \to 0} I(\delta) = 0.$$

For completeness we state

---

[7] W. H. Young, "Integration with respect to a function of bounded variation," *Proceedings of the London Mathematical Society*, Series 2, vol. 13 (1914), pp. 109-150. See p. 148.

THEOREM 2. *Let $B(x)$ be of limited variation on $a \leq x \leq b$, continuous at a and b, and have regular discontinuities for $a < x < b$. Then*

$$\lim_{\delta\to 0} \int_{x_1}^{x_2} dx \cdot \int_a^b U(x-\xi, \delta)\, dB(\xi) = B(x_2) - B(x_1).$$

This follows from Theorem 2 of paper (A).

THEOREM 3. *Let $\chi'(y)$ be continuous on $h \leq y \leq g$, and let $B(x)$ be of limited variation on $a \leq x \leq b$ and continuous at $x = \chi(h)$ $(a \leq \chi(h) \leq b)$. We then have*

$$\lim_{\lambda\to 0} \int_h^y dt \cdot \int_a^b V(\chi(t)+\lambda-\xi, t-h)\, dB(\xi)$$
$$= \int_h^y dt \cdot \int_a^b V(\chi(t)-\xi, t-h)\, dB(\xi).$$

The limit function is absolutely continuous on $h \leq y \leq g$. (λ is assumed to approach zero either through positive or through negative values.)

We shall first show that $\int_h^y |V|\, dt$ is uniformly bounded. From the identity

$$(a-b)\exp[-(a-b)^2] = (a-c)\exp[-(a-b)^2] + (c-b)$$
$$\times \exp[-(c-b)^2]\exp[-2(c-b)(a-c)]\exp[-(a-c)^2]$$

and the fact that $(\chi(t)-\chi(h))/(t-h)$ is bounded, we see that

$$(7)\quad |V(\chi(t)+\lambda-\xi, t-h)| \leq M(t-h)^{-1/2} + M\,|V(\chi(h)+\lambda-\xi, t-h|,$$

where M is a constant depending on $\chi(t)$ and $\chi'(t)$. From (7) it follows that the integral

$$(8)\qquad \int_h^y |V(\chi(t)+\lambda-\xi, t-h)|\, dt$$

is uniformly bounded for a given $\chi(y)$.

From Lemma 1 and a property of Young-Stieltjes integrals we may write

$$\lim_{\lambda\to 0} \int_h^y dt \cdot \int_a^b V(\chi(t)+\lambda-\xi, t-h)\, dB(\xi)$$
$$= \int_a^b dB(\xi) \cdot \lim_{\lambda\to 0} \int_h^y V(\chi(t)+\lambda-\xi, t-h)\, dt,$$

provided the limit of the integrand exists. For $\xi \neq \chi(h)$ and λ sufficiently small, V is continuous; this gives us the first of the following results:

$$\lim_{\lambda\to 0\pm}\int_h^y V(\chi(t)+\lambda-\xi, t-h)\,dt = \int_h^y V(\chi(t)-\xi, t-h)\,dt,\ \xi\neq\chi(h),$$
$$= \pm 1 + \int_h^y V(\chi(t)-\xi, t-h)\,d\xi,\ \xi=\chi(h).$$

The second result, when $\xi=\chi(h)$, is well known.[8] Thus the limit of the Stieltjes integrand exists; in fact it is continuous in ξ except for $\xi=\chi(h)$. Since $B(\xi)$ is continuous at $\chi(h)$, the ± 1 contributes nothing, and we obtain

$$\lim_{\lambda\to 0}\int_h^y dt\cdot\int_a^b V(\chi(t)+\lambda-\xi, t-h)\,dB(\xi)$$
$$= \int_a^b dB(\xi)\cdot\int_h^y V(\chi(t)-\xi, t-h)\,dt \equiv \phi(y).$$

From (8) we may interchange the order of integration in $\phi(y)$. Again using (8), we then see that $\phi(y)$ is the indefinite integral of a summable function; hence $\phi(y)$ is an absolutely continuous function. This completes the proof of the theorem.

The function $H(\chi(y), y)$ appearing in Theorem 4 of paper (A) is a Stieltjes integral; we now proceed to change it into an ordinary integral.

$$H(x,y) = \int_h^y dG(\eta)\cdot\int_\eta^y 2\chi'(t)U(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt,$$
$$x=\lambda+\chi(y),\ \lambda\neq 0.$$

We may replace $G(\eta)$ by $\bar{G}(\eta)=G(\eta)-G(h)$ without changing the value of $H(x,y)$. We assume this done, and we also assume $\chi''(t)$ continuous. After integrating by parts, we can write H in the form

$$H(x,y) = -\int_h^y \bar{G}(\eta)\int_\eta^y \frac{\partial}{\partial\eta}[2\chi'(t)U]\,dt\cdot d\eta.$$

On adding and subtracting $\frac{\partial}{\partial t}[2\chi'(\eta)U]$ to the integrand of the inside integral, we obtain

$$(9)\quad H(x,y) = -\int_h^y \bar{G}(\eta)\int_\eta^y \left[\frac{\chi'(t)-\chi'(\eta)}{t-\eta}\right]\left[1-\frac{(\chi(t)+\lambda-\chi(\eta))^2}{2(t-\eta)}\right]$$
$$\cdot U(\chi(t)+\lambda-\chi(n), t-\eta)\,dt\,d\eta + \int_h^y 2\chi'(\eta)U(\chi(y)+\lambda-\chi(\eta), y-\eta)\bar{G}(\eta)\,d\eta.$$

Since χ'' is continuous and since there exists a constant K such that

$$|\alpha|\exp(-|\alpha|) < K\exp(-|\alpha|/2),$$

the $\lim_{\lambda\to 0} H(x,y)$ may be obtained by putting $\lambda=0$ in (9).

---

[8] E. Goursat, *Cours d'Analyse*, vol. 3, Paris (1927), p. 307.

To simplify the notation in the next theorem, define

$$(10)\quad R(\phi(y),\psi(\eta)) = V(\phi(y)-\psi(\eta), y-\eta)$$

$$-\int_{\eta}^{y}\left[\frac{\phi'(t)-\psi'(\eta)}{t-\eta}\right]\left[1-\frac{(\phi(t)-\psi(\eta))^2}{2(t-\eta)}\right]U(\phi(t)-\psi(\eta), t-\eta)\,dt.$$

THEOREM 4. *Let* χ'' *be continuous and* $G(y)$ *be of limited variation on* $h \leq y \leq g$. *Then, for each fixed* y *in* $h < y \leq g$, *we can represent the following limit in two forms:*

$$\lim_{\lambda\to 0\pm}\int_{h}^{y} dt \int_{h}^{t} V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dG(\eta)$$

is either

$$(11)\quad = \pm\,[G(y-0)-G(h)] + \int_{h}^{y} R(\chi(y),\chi(\eta))\,\bar{G}(\eta)\,d\eta,$$

or

$$(12)\quad = \pm\,[G(y-0)-G(h)] + \int_{h}^{y} dG(\eta)\cdot\int_{\eta}^{y} V(\chi(t)-\chi(\eta), t-\eta)\,dt.$$

The representation (11) follows from combining (9) for $\lambda = 0$ with the results of Theorem 4 of paper (A). We now proceed to prove (12). For $\lambda \neq 0$, we may change the order of integration and write

$$(13)\quad \lim_{\lambda\to 0\pm}\int_{h}^{y} dG(\eta)\cdot\int_{\eta}^{y} V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt$$

$$= \int_{h}^{y} dG(\eta)\lim_{\lambda\to 0\pm}\int_{\eta}^{y} V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt.$$

From a theorem similar to one of Goursat's [9]

$$\lim_{\lambda\to 0\pm}\int_{\eta}^{y} V(\chi(t)+\lambda-\chi(\eta), t-\eta)\,dt = \begin{cases} \pm 1 + \displaystyle\int_{\eta}^{y} V(\chi(t)-\chi(\eta), t-\eta)\,dt, & y \neq \eta, \\ = 0, & y = \eta. \end{cases}$$

Putting the above result in (13), we see that (12) is true.

3. A boundary value problem. Consider the domain D defined in the introduction. We shall refer to the segments of the line $y = h + \delta$, $\delta > 0$, with end points $(x_1, h+\delta)$ and $(x_2, h+\delta)$ as admissible δ-displacements, if for sufficiently small δ the following set of points (x, y) lies in D plus its boundary

$$x_1 \leq x \leq x_2, \qquad h \leq y \leq h+\delta.$$

---

[9] *Loc. cit.*, p. 307.

The curves

$$x = \chi_i(y) + \lambda_i, \qquad (-1)^{i+1}\lambda_i > 0, \qquad h \leq y \leq g,$$

will be called λ-displacements if they lie in D except for their end points, which lie on the boundary of D. We shall show how to find a solution $u(x,y)$ of the heat equation (1) in D which for admissible δ-displacements satisfies relation (2), where $B(x)$ is a given function of bounded variation with regular discontinuities and continuous at $x = a$ and $x = b$. Moreover, for λ-displacements, $u(x,y)$ satisfies (3), where the $G_i(y)$ are preassigned functions of limited variation continuous from the left and continuous at $\chi_i(h)$.

Set up the function

$$\text{(14)} \quad u(x,y) = \int_a^b U(x-\xi, y-\eta)\,dB(\xi) - \sum_{i=1}^{2} \int_h^y 2U(x-\chi_i(\eta), y-\eta)\,dF_i($$
$$= u_1(x,y) - u_2(x,y) - u_3(x,y).$$

We see that $u(x,y)$ satisfies (1) in D if the $F_i(\eta)$ are functions of limited variation. From Theorems 1 and 2, we have for admissible δ-displacements

$$\lim_{\delta \to 0} \int_{x_1}^{x_2} u(t, h+\delta)\,dt = B(x_2) - B(x_1),$$

if the $F_i(\eta)$ are continuous at $\eta = h$. For $u(x,y)$ to satisfy all our conditions, we have thus to show the existence of functions $F_i(\eta)$ of limited variation, continuous from the left and continuous at $\eta = h$, and such that

$$\text{(15)} \qquad \lim_{\lambda_i \to 0} \int_{y_1}^{y_2} u_x(\chi_i(t) + \lambda_i, t)\,dt = G_i(y_2) - G_i(y_1)$$

for λ-displacements.

We shall assume the $\chi''_i(y)$ continuous, and since $B(\xi)$ is continuous at $\xi = a$ and $\xi = b$, we have from Theorem 3

$$\lim_{\lambda_i \to 0} \int_b^y \frac{\partial}{\partial x}\, u_1(\chi_i(t) + \lambda_i, t)\,dt = \phi_i(y),$$

where the $\phi_i(y)$ are absolutely continuous functions of y.

Putting $y_1 = h$ and $y_2 = y$ and making use of result (11) in Theorem 4, the pair of equations (15) can be put in the form

$$\text{(16)} \qquad f_i(y) = g_i(y) + \sum_{j=1}^{2} \int_h^y K_{ij}(y,\eta) f_j(\eta)\,d\eta,$$

where the $g_i(y)$ are known functions of limited variation continuous from the left and continuous at $y = h$.

$$g_i(y) = (-1)^{i+1}[G_i(y) - G_i(h) - \phi_i(y)]$$

and

$$f_i(y) = F_i(y) - F_i(h).$$

$$K_{ij}(y,\eta) = (-1)^i R(\chi_i(y), \chi_j(\eta)), \tag{17}$$

the function R being given by (10).

Since the kernel $K_{ij}(y,\eta)$ is of the form of a continuous function of y and η divided by $(y-\eta)^{\frac{1}{2}}$, the system (16) will have a unique solution continuous from the left and continuous at $y = h$. It remains to show that the $f_i(y)$ are of limited variation. To show that the $f_i(y)$ are indeed of limited variation we proceed as we did in section 5 of paper (A). The only step we must supply here is to show that there exist functions $T_{ij}(y,\eta)$ such that

$$\int_\eta^y K_{ij}(y,\xi)\,d\xi = \int_\eta^y T_{ij}(\xi,\eta)\,d\xi.$$

Comparing the results (11) and (12) of Theorem 4, we see that we can take the

$$T_{ij}(y,\eta) = V(\chi_i(y) - \chi_j(\eta), y - \eta). \tag{18}$$

We collect the principal result of Part 1 into the following theorem:

THEOREM 5. *Let $B(x)$ be of limited variation for $\chi_1(h) \equiv a \leq x \leq b \equiv \chi_2(h)$, continuous at a and b, and having regular discontinuities for $a < x < b$. Let $G_1(y)$, $G_2(y)$ be of limited variation on $h \leq y \leq g$, continuous at h, and continuous from the left for $h < y \leq g$. Let χ_1 and χ_2 have continuous second derivatives. Then in the class of functions* (14), *there exists a unique function $u(x,y)$ satisfying the heat equation* (1) *in D and satisfying the boundary conditions* (2) *and* (3).

4. A note on paper (A). In our first paper the function $l(x,y)$ solving the boundary value problem was written

$$l(x,y) = \sum_{i=1}^{2} \int_h^y [V_i - 2\chi_i(\eta)U]\,dF_i(\eta) + \int_a^b U\,dB(\xi).$$

We wish to point out that by the use of Theorem 4 of this paper we could drop out the $\chi_i(\eta)U$ and write

$$l(x,y) = \sum_{i=1}^{2} \int_h^y V_i\,dF_i(\eta) + \int_a^b U\,dB(\xi). \tag{19}$$

This simplifies $l(x,y)$. The kernels of the integral equations that (19) leads to are the same as the functions K_{ij} of (17). These kernels are not simpler in form than the kernels of paper (A). However, the T_{ij} then become those defined in (18) and are simpler than those appearing in our first paper.

PART 2

In this part of the paper we return to the classical problem of finding a solution of the heat equation (1) in the region D which takes on preassigned values on the boundary of D. Here, however, our boundary values are not continuous but are functions of limited variation. The approaches to the boundary are in a restricted sense that we describe in the next section. The Riemann-Stieltjes integral will suffice for the work of this part of the paper.

5. Definitions and notation. A point $P(x,y)$ lying in the region D will be said to approach a point $M(x_0,h)$ on the characteristic $y=h$ in the parabolic sense if

$$(20) \qquad \lim_{P\to M} (x-x_0)/(y-h)^{\frac{1}{2}} = 0.$$

When a point P approaches a point $Q(\chi_i(y_0), y_0)$ through a set of points E of D, we divide E into two sets as follows: E_1 contains those points of E such that $y > y_0$, and $E_2 = E - E_1$. If Q is not a limit point of E_1, or if Q is a limit point of E_1 and in addition

$$(21) \qquad \lim_{\substack{P\to Q \\ P\subset E_1}} |\, x-\chi_i(y_0)\,|/(y-y_0)^{\frac{1}{2}} = \infty,$$

we shall say that P approaches the point Q in the parabolic sense.[10]

Let $W(x,y)$ denote the function

$$(22) \qquad W(x,y) = \frac{1}{\sqrt{\pi}} \int_{x/2\sqrt{y}}^{\infty} \exp(-u^2)\,du,\ y>0; \qquad =0,\ y\leq 0.$$

The following functions defined in terms of W are solutions of the heat equation (1) for points in D, and will play a fundamental rôle in the rest of the paper:

$$(23) \quad u(x,y) = \int_a^b [\tfrac{1}{2} - W(x-\xi, y-h)]\,dF(\xi),$$

$$(24) \quad v_i(x,y) = 2\int_h^y W((-1)^{i+1}(x-\chi_i(\eta)), y-\eta)\,dG(\eta),\ x\neq\chi_i(y),$$

$$= 2\int_h^y [W((-1)^{i+1}(x-\chi_i(\eta)), y-\eta) - W(0,1)]\,dG(\eta),\ x=\chi_i(y).$$

---

[10] The above definitions of parabolic approaches are radically different, and it is to be noted that a corner point $(\chi_i(h), h)$ can not be reached simultaneously under both definitions.

The functions $F(\xi)$ and $G(\eta)$ are functions of bounded variation and $a = \chi_1(h)$, $b = \chi_2(h)$.

6. Parabolic approaches.

In this section we shall be concerned with the limits of u and v_i under parabolic approaches to the boundary of D.

THEOREM 6. *If $P(x,y)$ approaches the point $M(x_0,h)$ $(a \leq x_0 \leq b)$ in the parabolic sense, then the function $u(P) \equiv u(x,y)$ defined by* (23) *has the following limit*

$$\lim_{P\to M} u(P) = \frac{F(x_0+0)+F(x_0-0)}{2} - \frac{F(b)+F(a)}{2},$$

where $F(b+0) = F(b)$ and $F(a-0) = F(a)$.

Suppose first that $a < x_0 < b$ and write $u(P)$ in the form

$$u(P) = \int_a^{x_0} [\tfrac{1}{2} - W(x-\xi, y-h)]dF(\xi) + \int_{x_0}^{b} [\tfrac{1}{2} - W(x-\xi, y-h)]dF(\xi)$$
$$= u_1(P) + u_2(P).$$

Integrating $u_1(P)$ by parts and replacing $F(\xi)d\xi$ by $d\int_a^{\xi} F(t)dt$, we may write

$$u_1(P) = [\tfrac{1}{2} - W(x-\xi, y-h)]F(\xi)\Big]_a^{x_0} + \int_a^{x_0} U(x-\xi, y-h)d\xi \int_a^{\xi} F(t)dt.$$

Under a parabolic approach the integrated part has the limit $-F(a)/2$. Since $\int_a^{\xi} F(t)dt$ has a left-sided derivative at x_0 equal to $F(x_0-0)$, Theorem 1 of paper (A) shows that the above integral in $u_1(P)$ has the limit $F(x_0-0)/2$. The limit of $u_1(P)$ is thus seen to be

$$\lim_{P\to M} u_1(P) = \frac{F(x_0-0)}{2} - \frac{F(a)}{2}.$$

Noting that $\int_a^{\xi} F(t)dt$ has a right-sided derivative at x_0 equal to $F(x+0)$, the same type of argument shows that

$$\lim_{P\to M} u_2(P) = \frac{F(x_0+0)}{2} - \frac{F(b)}{2}.$$

From the above limits the theorem readily follows for $a < x_0 < b$. If x_0 is a or b then $u_1(P)$ or $u_2(P)$ is zero and no new considerations are needed in these cases.

Turning now to the functions $v_i(P)$, we see that as the point P approaches the point $M(x_0,h)$ with $a < x_0 < b$ these functions each have the limit zero. Our next theorem is concerned with the limit of $v_i(P)$ as P approaches the point $Q(\chi_i(y_0), y_0)$.

THEOREM 7. *If P approaches Q in the parabolic sense, then the functions $v_i(P)$ defined by (24) have the limit*

$$\lim_{P\to Q} v_i(P) = [G(y_0 - 0) - G(h)] + v_i(\chi_i(y_0), y_0).$$

Let us consider the case of $v_1(P)$, and to simplify the notation write $v(P) \equiv v_1(P)$ and $\chi(\eta) \equiv \chi_i(\eta)$. It is then to be noted that $x - \chi(y) > 0$ for P in the region D.

For a given $\delta > 0$ we can select $\epsilon > 0$ so that the total variation of $G(\eta - 0)$ $[G(\eta + 0)]$ in $y_0 - \epsilon \leq \eta \leq y_0$ $[y_0 \leq \eta \leq y_0 + \epsilon]$ is less than δ. With ϵ fixed, since $y \to y_0$, we assume $y_0 - \epsilon/2 \leq y \leq y_0 + \epsilon/2$, and write $v(P)$ in the following form

$$\begin{aligned} v(P) &= \int_h^{y_0-\epsilon} 2W(x-\chi(\eta), y-\eta)\,dG(\eta-0) + \int_{y_0-\epsilon}^{y} 2W\,dG(\eta-0) \\ &= I_1(x,y) + I_2(x,y). \end{aligned}$$

Since $G(h - 0) = G(h)$ and the integrand W is continuous and equal to zero for $y = \eta$, the replacement of $G(\eta)$ by $G(\eta - 0)$ is permissible.

In I_1 the difference $y - \eta$ is bounded away from zero, hence

$$\lim_{P\to Q} I_1(x,y) = I_1(\chi(y_0), y_0).$$

For $y \leq y_0$, we have

$$| I_2(x,y) | < 2\delta.$$

If $y > y_0$, we write $I_2(x,y)$ as the sum of two integrals

$$I_2(x,y) = \left\{ \int_{y_0+\epsilon}^{y_0} + \int_{y_0}^{y} \right\} W(x-\chi(\eta), y-\eta)\,dG(\eta-0).$$

The first integral is seen to be less than 2δ, while the second can be put in the form

$$\int_{y_0}^{y} W(x-\chi(\eta), y-\eta)\,dG(\eta+0) + [G(y_0+0) - G(y_0-0)]\; W(x-\chi(y_0), y -$$

Here again the first term is less than 2δ, while the second has the limit zero when P approaches Q in the parabolic sense. Thus we see that

$$\lim_{P\to Q} | I_2 | < 4\delta.$$

Since δ is any positive number, and since

$$\left| \int_{y_0-\epsilon}^{y_0} 2W(\chi(y_0) - \chi(\eta), y_0 - \eta)\, dG(\eta - 0) \right| < 2\delta,$$

we have

$$\underset{\to Q}{\mathrm{m}}\, v(P) = \int_h^{y_0} 2W(\chi(y_0 - \chi(\eta), y_0 - \eta)\, dG(\eta - 0)$$

$$= \int_h^{y_0} 2W(0,1)\, dG(\eta - 0) + \int_h^{y_0} 2[W(\chi(y_0) - \chi(\eta), y_0 - \eta) - W(0,1)]\, dG(\eta - 0)$$

$$= G(y_0 - 0) - G(h) + v(\chi(y_0), y_0).$$

Theorem 7 is thus seen to hold for $v(P) = v_1(P)$ and a similar proof holds for $v_2(P)$.

7. The first boundary value problem. We shall prove the existence of a function $l(x, y)$ which is a solution of (1) in D, and whose limit as the point $P(x, y)$ approaches the point $M(x_0, h)$ in the parabolic sense is equal to a given function $B(x_0)$ of limited variation with regular discontinuities [here $a = \chi_1(h) < x_0 < \chi_2(h) = b$], also the limit of $l(P)$ as P approaches in the parabolic sense a point $Q(\chi_i(y_0), y_0)$ with $y_0 > h$ is equal to a given function $H_i(y_0)$ of limited variation continuous from the left.

Let $F(x) = B(x)$ for $a < x < b$ and let $F(a) = F(b) = 0$, and set up the function

$$(25) \qquad l(x, y) = \sum_{i=1}^{2} 2 \int_h^y W((-1)^{i+1}[x - \chi_i(\eta)], y - \eta)\, dG_i(\eta)$$

$$+ \int_a^b [\tfrac{1}{2} - W(x - \xi, y - h)]\, dF(\xi)$$

$$= v_1(x, y) + v_2(x, y) + u(x, y).$$

We see that in the domain D the function $l(x, y)$ satisfies equation (1) if the $G_i(\eta)$ are functions of limited variation. Since the $v_i(x, y)$ approach zero as P approaches $M(x_0, h)$ for $a < x_0 < b$, we see from Theorem 6 that under a parabolic approach

$$(26) \qquad \lim_{P \to M} l(x, y) = B(x_0).$$

As P approaches $Q(\chi_i(y_0), y_0)$ in the parabolic sense with $y_0 > h$, we have from Theorem 7

$$(27) \quad \lim_{P \to Q} l(x, y) = [G_i(y_0 - 0) - G_i(h)] + \sum_{j=1}^{2} v_j(\chi_i(y_0), y_0) + u(\chi_i(y_c), y),$$

and we must show the existence of functions of limited variation $G_i(y)$ so that the above limit is equal to the given functions:

$$(28) \qquad H_i(y_0).$$

Let us note that the $v_i(\chi_j(y), y)$ approach zero with y approaching h when $i = j$ and also when $i \neq j$. Also as $(\chi_i(y) - \chi_i(h))/(y - h)^{\frac{1}{2}}$ approaches zero with $y - h$, we see from Theorem 6 that $\lim u(\chi_i(y), y)$ as y approaches h exists and has a finite value. Taking this value for $u(\chi_i(h), h)$ the function $u(\chi_i(y), y)$ is a continuous function of y in $h \leq y \leq g$.

Set

$$(29)\quad f_i(y) = G_i(y-0) - G_i(h) + u(\chi_i(h), h) - H_i(h),\ y > h;\ = 0,\ y = h,$$

and on equating (28) to the right member of (27) with $y_0 = y$, we have the following system of integral equations

$$(30)\qquad f_i(y) = g_i(y) + \sum_{j=1}^{2} \int_h^y K_{ij}(y, \eta)\, df_j(\eta), \qquad (i = i, 2),$$

where [11]

$$g_i(y) = [H_i(y) - H_i(h)] - [u(\chi_i(y), y) - u(\chi_i(h), h)],$$

$$(31)\quad K_{ij}(y, \eta) = -2[W((-1)^{j+1}[\chi_i(y) - \chi_j(\eta)], y - \eta) - \delta_j^i W(0, 1)].$$

Here δ_j^i is 1 or 0 according as $i = j$ or $i \neq j$. The kernels K_{ij} can be put into the two integral forms

$$(32)\quad K_{ij}(y, \eta) = (-1)^j \int_\eta^y \left[\frac{\chi_i(y) - \chi_j(t)}{t - \eta} - 2\chi'_j(t)\right] U(\chi_i(y) - \chi_j(t), y - t)\, dt$$

$$(33)\qquad = (-1)^j \int_\eta^y \left[\frac{\chi_i(t) - \chi_j(\eta)}{t - \eta} - 2\chi'_i(t)\right] U(\chi_i(t) - \chi_j(\eta), t - \eta)\, dt.$$

We have seen [12] that the system of integral equations (30) having kernels with the representations (32)-(33) and with $g_i(h) = 0$, have unique solutions $f_i(y)$ of limited variation and continuous from the left provided that the $g_i(y)$ are such functions. The $H_i(y)$ have these properties and the $u(\chi_i(y), y)$ are continuous; it remains to show that the $u(\chi_i(y), y)$ are of limited variation. We shall do this in the last section. Thus we state the result:

LEMMA 2. *For a given set of functions $g_i(y)$ of limited variation, continuous from the left, and vanishing at h, there exists a unique set of functions $f_i(y)$ satisfying* (30) *and having these same properties.*

The equations (29) determine each $G_i(y - 0)$ to within an additive constant which does not affect the function $l(x, y)$; moreover, $l(x, y)$ is invariant under the replacement of $G_i(y)$ by $G_i(y-0)$. Considering then the $g_i(y)$, we see that if $u(x,y)$ is fixed the $g_i(y)$ are unique. If we put on the restriction

---

[11] The constant $u(\chi_i(h), h) - H_i(h)$ is put in to make $g_i(h) = 0$.

[12] Paper (A), pp. 651-653.

$$F(a)=F(b) \tag{34}$$

in forming the function $u(x,y)$ defined in equation (23), it then follows from Theorem 6 that the only $u(x,y)$ which, under a parabolic approach to $M(x_0,h)$ $(a<x_0<b)$, has the limiting value zero, is $u(x,y)$ identically zero. We shall say that $u(x,y)$ is regular if it is formed under the restriction (34).

We now state the main theorem of Part 2:

THEOREM 8. *In the class of functions* (25) *with* $u(x,y)$ *regular, there exists a unique function* $l(x,y)$ *such that* $l(x,y)$ *is a solution of* (1) *in* D *and, under parabolic approaches to the boundary, takes on preassigned functions of limited variation. On the characteristic* $y=h$ *the assigned function has regular discontinuities, elsewhere the assigned functions are continuous from the left.* [*Each* $\chi'_i(y)$ *is assumed continuous.*]

8. Functions of limited variation. In this section we shall show that the functions $u(\chi_i(y),y)$ appearing in the integral equations (30) are of bounded variation.

Let $\chi(y)$ have a continuous derivative and define

$$\begin{aligned} H(y)&=\int_a^b\int_\xi^{\chi(y)}U(t-\xi,y-h)\,dt\,dF(\xi), && h<y\leq g,\\ &=\lim_{y\to h}\int_a^b\int_\xi^{\chi(y)}U(t-\xi,y-h)\,dt\,dF(\xi), && y=h.\end{aligned} \tag{35}$$

$H(y)$ is seen to be equal to the function $u(\chi(y),y)$ and Theorem 1 insures the existence of the above limit.

LEMMA 3. *The function* $H(y)$ *defined by* (35) *is of limited variation in* $h\leq y\leq g$.

We give the proof for $\chi(h)=b$; the proof for other values of $\chi(h)$ can be handled in a similar manner.

Write $H(y)$ in the form

$$\begin{aligned} H(y)&=\int_a^b\int_\xi^b U\,dt\,dF(\xi)+\int_a^b\int_b^{\chi(y)}U\,dt\,dF(\xi), \qquad y>h\\ &=I_1(y)+I_2(y).\end{aligned}$$

Under the transformation $(t-\xi)/2(y-h)^{\frac{1}{2}}$ the function

$$\int_{\xi}^{b} U(t-\xi, y-h)\,dt$$

is seen to be monotonically decreasing with y for each ξ, and to have a total variation less than one on the set $y > h$. It then follows that the variation of $I_1(y)$ on this set is less than the total variation of $F(\xi)$.

With $y_2 \geq y_1 > h$, write

$$I_2(y_2) - I_2(y_1) = \int_a^b \int_{\chi(y_1)}^{\chi(y_2)} U(t-\xi, y_2-h)\,dt\,dF(\xi)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_a^b \int_b^{\chi(y_1)} \left[\frac{1}{\sqrt{y_2-h}} - \frac{1}{\sqrt{y_1-h}}\right] \exp \frac{(t-\xi)^2}{4(y_1-h)}\Big]\,dt\,dF(\xi)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_a^b \int_b^{\chi(y_1)} \frac{1}{\sqrt{y_2-h}} \left[\exp\left(-\frac{(t-\xi)^2}{4(y_2-h)}\right) - \exp\left(-\frac{(t-\xi)^2}{4(y_1-h)}\right)\right] dt\,dF($$

Remembering that $b = \chi(h)$ and that $\chi'(\eta)$ is bounded, the first two integrals on the right are easily shown to be less than $\sqrt{y_2-h} - \sqrt{y_1-h}$ times a constant C. The constant C depends only on the maximum of $\chi'(\eta)$ and the total variation of $F(\xi)$. We obtain for the third integral a similar bound, if, after using the theorem of the mean, the following property of $\exp(-\alpha)$ is used. If P, k are two positive constants with $0 < k < 1$ and α is a variable ≥ 0, then there exists a constant K such that

$$\alpha^P \exp(-\alpha) < K \exp(-k\alpha).$$

Since the function $\sqrt{y-h}$ is of limited variation, the function $I_2(y)$ is seen to have a variation on $h < y \leq g$ less than a constant times the variation of $\sqrt{y-h}$.

As the function $H(y)$ is bounded it then follows that it is of limited variation on the closed interval $h \leq y \leq g$.

DUKE UNIVERSITY AND
HAMPDEN SYDNEY COLLEGE.

ANALYTIC CONTINUATION OF FUNCTIONS DEFINED BY FACTORIAL SERIES.*

By GLEN T. MILLER and HOWARD K. HUGHES.

In this paper we shall be concerned with the determination of an analytic continuation of a function $\Omega(z)$ defined by a factorial series of the form:

$$(1) \qquad \Omega(z) = \sum_{m=0}^{\infty} \frac{g(m)(-1)^{m(k+1)}}{z(z+1)\cdots(z+m)},$$

where k is a positive integer, and z is a complex variable.

The results here obtained may be considered to be an extension of a theorem obtained by Hughes[1] for a function defined by a series of the type (1) when $k=0$. The methods used bear a close analogy to those used by Ford[2] and Newsom[3] in their work on functions defined by Maclaurin series.

It is well known that the region of convergence of a factorial series is a half-plane bounded on the left by a vertical line $R(z)=\lambda$, where $R(z)$ denotes the real part of z and $-\infty \leqq \lambda \leqq \infty$. Furthermore the convergence is uniform in the half-plane $R(z)=\lambda+\delta$ where δ is any positive number. Consequently the series (1) represents an analytic function in the region $R(z)>\lambda$.[4]

The results are given in the following

THEOREM. *Given the factorial series* (1), *whose abscissa of convergence is finite, and for which k is a positive integer; let it be assumed that the function $g(m)$ occurring in the coefficient of* (1), *when considered as a function $g(w)$ of the complex variable $w=x+iy$, satisfies the following conditions*:

(a) *it is single-valued and analytic throughout the half-plane* $x \geqq -\frac{1}{2}$;

---

* Received January 15, 1942.

[1] H. K. Hughes, "On the analytical extension of functions defined by factorial series," *American Journal of Mathematics*, vol. 53 (1931), pp. 771-776.

[2] W. B. Ford, "The asymptotic developments of functions defined by Maclaurin series," *University of Michigan Studies*, Scientific Series, Vol. 11.

[3] C. V. Newsom, "On the character of certain entire functions in distant portions of the plane," *American Journal of Mathematics*, vol. 60 (1938), pp. 561-572.

[4] For an account of the properties of factorial series, see *The Calculus of Finite Differences* by Milne-Thomson (1933), Ch. X.

(b) *for all* $x \geqq -\frac{1}{2}$, *and for all values of* $|y|$ *sufficiently large, we have:*

$$(2)\qquad \left| \frac{g(x+iy)}{g(x)} \right| < M \exp[\{\pi(k+\tfrac{1}{2}) - \epsilon\} |y|]$$

where ϵ *is an arbitrarily small positive number and* M *is a constant dependent on* ϵ.

Moreover, let there exist a positive constant q, *dependent on* ϵ, *but independent of* z, *such that when* z *is confined to any finite region, we have,*

$$(3)\qquad \left| \frac{g(x)}{\Gamma(z+x+1)} \right| < \frac{N}{x^{1+\epsilon}}$$

when $x > q$, *and where* N *is a positive constant.*

Then the function $\Omega(z)$, *defined by* (1) *may be continued analytically throughout the whole finite plane, except in the neighborhoods of the points* $z = 0, -1, -2, \cdots$, *and throughout this region will be defined by the equation*

$$(4)\qquad \Omega(z) = \Gamma(z) \int_{-\frac{1}{2}}^{\infty} \frac{\sin k\pi x}{\sin \pi x} \frac{g(x)}{\Gamma(z+x+1)}\, dx + \Gamma(z)(-\tfrac{1}{2}i)^k k\pi \int_{-\infty}^{+\infty} \frac{F(y,z)}{(\cosh \pi y)^{k+1}}\, dy$$

where

$$(5)\qquad F(y,z) = \int_0^y \frac{[\sinh \pi(s-y)]^{k-1} g(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\, ds.$$

The proof of this theorem is based on the following theorem in the Calculus of residues, due to Newsom:

"Let $P(w)$ be regular within a closed region bounded by the curve C in the w-complex plane, and enclosing the points $m = 0, 1, 2, \cdots, n$, but no other vanishing points of $\sin \pi w$. Furthermore, let $P(m) \neq 0$, $m = 0, 1, 2, \cdots, n$. Then

$$(6)\qquad \frac{1}{2\pi i} \int_C \frac{P(w)}{(\sin \pi w)^{k+1}}\, dw = \frac{(-1)^{m(k+1)}}{\pi^{k+1} k!} \sum_{t=0}^{k} \pi^t (\Sigma \alpha_1 \alpha_2 \cdots \alpha_t) P^{(k-t)}(m)$$

where $m = 0, 1, 2, \cdots, n$, and where $(\Sigma \alpha_1 \alpha_2 \cdots \alpha_t)$ denotes the sum of the $\binom{k}{t}$ products of t factors each formed by taking the possible combinations of the k quantities $\pm(k-1)i, \pm(k-3)i, \cdots \left\{ \begin{matrix} \pm i \\ 0 \end{matrix} \right\}$ t at a time. Here $\left\{ \begin{matrix} \pm i \\ 0 \end{matrix} \right\}$ is $\pm i$ or 0 according as k is even or odd. The integration is performed in the positive sense." [5]

---

[5] Newsom, *op. cit.*, pp. 562, 563.

Proof of the Theorem.

Let us suppose that z has any fixed value which is real and positive and greater than λ, the abscissa of convergence of series (1). We shall write the series (1) in the form:

$$\Omega(z) = \sum_{m=0}^{\infty} \frac{g(m)\Gamma(z)(-1)^{m(k+1)}}{\Gamma(z+m+1)} \tag{7}$$

and shall seek to determine $P(w)$ so that

$$\sum_{t=0}^{k} \pi^t (\Sigma\alpha_1\alpha_2\cdots\alpha_t) P^{(k-t)}(m) = \frac{g(m)}{\Gamma(z+m+1)}. \tag{8}$$

This is a differential equation in m with constant coefficients. The roots of the characteristic equation are distinct, and are $\pm(k-1)\pi i$, $\pm(k-3)\pi i, \cdots \left\{ \begin{matrix} \pm\pi i \\ 0 \end{matrix} \right\}$. If we denote these roots by $r_j\pi i$, where

$$r_j = k+1-2j, \qquad 1 \leqq j \leqq k, \tag{9}$$

then a particular solution of (8) is given by

$$P(m) = \sum_{j=1}^{k} b_j \exp r_j\pi i m \int_0^m \frac{\exp(-r_j\pi i m)\, g(m)}{\Gamma(z+m+1)}\, dm \tag{10}$$

where

$$b_j = \frac{1}{(2\pi i)^{k-1}} \prod_{a=1}^{k} \frac{1}{a-j}, \qquad a \neq j. \tag{11}$$

In some of our later considerations we shall find the following form for b_j useful:

$$b_j = \frac{1}{(2\pi i)^{k-1}(k-1)!} \binom{k-1}{j-1} (-1)^{j-1}. \tag{12}$$

If we replace m by the complex variable $w = x + iy$, then $P(w)$ is an analytic function of w in the finite-w-plane. Hence if $P(w)$ has the form indicated by (10), then the equation (6) can be written in the form:

$$\sum_{m=0}^{n} \frac{(-1)^{m(k+1)} g(m)\Gamma(z)}{\Gamma(z+m+1)} = \frac{\Gamma(z)\pi^k k!}{2i} \int_C \frac{P(w)}{(\sin \pi w)^{k+1}}\, dw \tag{13}$$

where the contour C is so chosen as to enclose the points $w = 0, 1, 2, \cdots, n$, but no other vanishing points of $\sin w$.

If we substitute the form for $P(w)$ from (10) in (13) we obtain a contour integral for the determination of the first $n+1$ terms of the series (7) as follows:

$$\sum_{m=0}^{n} \frac{(-1)^{m(k+1)} g(m)\Gamma(z)}{\Gamma(z+m+1)} = \frac{\Gamma(z)\pi^k k!}{2i} \sum_{j=1}^{k} b_j \int \frac{\exp r_j \pi i w \; I(w,z)}{(\sin \pi w)^{k+1}}\, dw \tag{14}$$

where

$$I(w,z) = \int_0^w \frac{\exp(-r_j \pi i w)\, g(w)}{\Gamma(z+w+1)}\, dw, \tag{15}$$

and where b_j is given by (11).

We shall choose as the path of integration the rectangle enclosed by the lines (1) $w = x + ip$, (2) $w = -\frac{1}{2} + iy$, (3) $w = x - ip$, and (4) $w = n + \frac{1}{2} + iy$, and shall denote the integrals along these sides by A, B, D, and E, respectively. Then in order to evaluate the contour integral

$$\int_C \frac{P(w)}{(\sin \pi w)^{k+1}}\, dw$$

we must consider k integrals of the form:

$$\int_C \frac{\exp r_j \pi i w \;\; I(w,z)}{(\sin \pi w)^{k+1}}\, dw \tag{16}$$

where $1 \leqq j \leqq k$, and $I(w,z)$ is given by (15).

In (15) let us take the path of integration along the x-axis from $w = 0$ to $w = x$, and then along a line parallel to the y-axis to the point $w = x + iy$. The function $I(w,z)$ then can be written in the form:

$$I(w,z) = R(x,z) + iS(x,y,z),$$

where

$$R(x,z) = \int_0^x \frac{\exp(-r_j \pi i x) \;\; g(x)}{\Gamma(z+x+1)}\, dx,$$

and

$$S(x,y,z) = \int_0^y \frac{\exp(-r_j \pi i(x+iy)) \;\; g(x+iy)}{\Gamma(z+x+1+iy)}\, dy.$$

With regard to $R(x,z)$ it is sufficient for our purpose to note that $|R(x,z)|$ is bounded when x is bounded.

It is known that for all $x > x_0$, where x_0 is arbitrary, and for all y sufficiently large, we have:[6]

$$\left| \frac{1}{\Gamma(x+iy)} \right| < K \exp[\tfrac{1}{2}(\pi + \epsilon)\,|\,y\,|]. \tag{17}$$

From the above inequality, together with (2) and (9), it follows that the absolute value of the integrand in (15) is less than

$$m_1 \exp[r_j \pi y + \{(k+1) - \tfrac{1}{2}\epsilon\}\,|\,y\,|],$$

---

[6] Ford, *op. cit.*, p. 61.

the constant m_1 being dependent only on ϵ. It follows that there exists a constant K, dependent only upon ϵ, such that

$$(18) \qquad |S(x, y, z)| < K \exp[r_j \pi y + \{(k+1) - \tfrac{1}{2}\epsilon\} |y|].$$

We are now ready to consider in detail the various contributions to the integral (16). We shall first consider the contribution A. Here we have $w = x + ip$, and

$$\sin \pi w = \sin \pi p (\sin \pi x \coth \pi p + i \cos \pi x).$$

Since $\lim\limits_{p \to \infty} (\sin \pi x \coth \pi p + i \cos \pi x) = 1$, $\sin \pi w$ is of the order of $\exp \pi p$ when p is large.

In order to evaluate A we must consider k integrals of the form:

$$A_j = \int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{\exp r_j \pi i (x + ip) [R(x, z) + iS(x, p, z)]}{\exp \pi p (k+1)} dx$$

where $1 \leqq j \leqq k$. It is evident that we may write

$$(19) \quad |A_j| \leqq \left| \int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{R(x, z)}{\exp \pi p (k + 1 + r_j)} dx \right| + \left| \int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{S(x, p, z)}{\exp \pi p (k + 1 + r_j)} dx \right| .$$

Since $R(x, z)$ is bounded if x is bounded, the first integral on the right of (19) approaches zero as p approaches infinity.

With regard to the second integral on the right of (19), we have from (18) that

$$|S(x, p, z)| < K \exp([\pi(r_j + k + 1) - \tfrac{1}{2}\epsilon] p).$$

The absolute value of the integrand in the integral in question is therefore of the order of $\exp(-\tfrac{1}{2}\epsilon p)$ for p large, and since ϵ is positive we conclude that $\lim\limits_{p \to \infty} A_j = 0$. It follows that $\lim\limits_{p \to \infty} A = 0$.

In a similar manner it may be shown that $\lim\limits_{p \to \infty} D = 0$.

We next consider the contribution E. We shall choose n to be an even integer so that $\sin \pi w = \cosh \pi y$. E consists of k integrals of the form:

$$(20) \quad E_j = i^{r_j} \Big[iR(n + \tfrac{1}{2}, z) \int_{-\infty}^{+\infty} \frac{\exp(-r_j \pi y)}{(\cosh \pi y)^{k+1}} dy - \int_{-\infty}^{+\infty} \frac{\exp(-r_j \pi y) S(n + \tfrac{1}{2}, y, z)}{(\cosh \pi y)^{k+1}} dy \Big] .$$

We shall set

$$\int_{-\infty}^{+\infty} \frac{\exp(-r_j \pi y)}{(\cosh \pi y)^{k+1}} dy = H_j.$$

Then by using the result of Newsom[7] and (9) we can write

---

[7] Newsom, *op. cit.*, pp. 567, 568.

$$H_j = \frac{(-1)^j 2^k}{-j\pi} \prod_{c=1}^{k} (1 - j/c) \qquad c \neq j.$$

Let us next consider the second integral on the right of (20). The integral can be written:

$$(21) \qquad \int_{-\infty}^{+\infty} \frac{\exp(-r_j\pi y) S(n+\frac{1}{2}, y, z)}{(\cosh \pi y)^{k+1}}\, dy$$

$$= \left\{ \frac{g(n)}{(z+n+1)} \right\} \int_{-\infty}^{+\infty} \frac{\exp(-r_j\pi y) S_1(n+\frac{1}{2}, y, z)}{(\cosh \pi y)^{k+1}}\, dy$$

where

$$S_1(n+\tfrac{1}{2}, y, z) = \int_0^y \frac{\Gamma(z+n+1) g(n+\frac{1}{2}+iy) \exp r_j\pi y}{\Gamma(z+n+3/2+iy) g(n)}\, dy.$$

We now make use of the following well known property of the gamma function:[8]

$$\Gamma(x+iy) = (2\pi)^{\frac{1}{2}} (x+iy)^{x-\frac{1}{2}} \exp(-x - y \operatorname{arc\,tan} y/x)(1+\delta)$$

where δ approaches zero as either x or y becomes infinite. Then we have

$$(22) \qquad \left| \frac{\Gamma(z+n+1)}{\Gamma(z+n+3/2+iy)} \right| < K \exp(\pi/2\, |y|).$$

By the use of (22) and (2), and taking into account the behavior of $(\cosh \pi y)$ for $|y|$ large, it follows that the integrand of the right member of (21) is of the order of $\exp(-\epsilon |y|)$, for $|y|$ large, and hence the integral converges. Furthermore, since the series (1) was assumed convergent, we have:

$$\lim_{n\to\infty} \frac{g(n)}{\Gamma(z+n+1)} = 0.$$

Consequently the integral on the left of (21) tends to zero as n approaches infinity. The total contribution E_j is given by

$$E_j = i^{r_j+1} H_j \lim_{n\to\infty} R(n+\tfrac{1}{2}, z).$$

We next consider the contribution B arising from the integration along the side where $w = -\frac{1}{2} + iy$. Here $\sin \pi w = -\cosh \pi y$ and we must consider k integrals of the form:

$$(23) \quad B_j = \int_{+\infty}^{-\infty} \frac{\exp r_j\pi i(-\frac{1}{2}+iy) R(-\frac{1}{2}, z)}{(-\cosh \pi y)^{k+1}}\, dy$$

$$+ i \int_{+\infty}^{-\infty} \frac{\exp r_j\pi i(-\frac{1}{2}+iy) S(-\frac{1}{2}, y, z)}{(-\cosh \pi y)^{k+1}}\, dy.$$

---

[8] Ford, *op. cit.*, p. 61.

The first of the integrals above, when combined with the contribution E_j, can be written in the form:

$$i^{r_j+1}H_j \lim_{n\to\infty} [R(n+\tfrac{1}{2}, z) - R(-\tfrac{1}{2}, z)] = i^{r_j+1}H_j \int_{-\frac{1}{2}}^{\infty} \frac{\exp(-r_j\pi i x)}{\Gamma(z+x+1)}\, dx.$$

If we multiply the above integral by b_j, we obtain as the sum of the j integrals in question when substituted in (14)

$$\Gamma(z) \int_{-\frac{1}{2}}^{\infty} \frac{\sin k\pi x}{\sin \pi x}\, \frac{g(x)}{\Gamma(z+x+1)}\, dx. \tag{24}$$

Finally let us investigate the second integral in (23). When substituted in (14), the sum of the k integrals in question may be written in the form:

$$\frac{\Gamma(z)\pi k}{(-2i)^k} \int_{+\infty}^{-\infty} \frac{\displaystyle\sum_{j=1}^{k} \int_0^y \binom{k-1}{j-1} \frac{(-1)^{j-1} \exp r_j\pi(s-y)\, g(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\, ds}{(\cosh \pi y)^{k+1}}\, dy. \tag{25}$$

But

$$\sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} \exp r_j\pi(s-y) = 2^{k-1}(\sinh \pi(s-y))^{k+1}. \tag{26}$$

Upon substituting (26) in (25), we find the value of this contribution to be:

$$\Gamma(z) k\pi(-i/2)^k \int_{+\infty}^{-\infty} \frac{\displaystyle\int_0^y \frac{[\sinh \pi(s-y)]^{k-1} q(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\, ds}{(\cosh \pi y)^{k+1}}\, dy. \tag{27}$$

The value of the right member of (14) is therefore the sum of the expressions (24) and (27). We thus obtain equation (4).

Up to this point we have restricted z to values which are real, positive and greater than λ, the abscissa of convergence. $\Omega(z)$ as defined by series (1) is a function which is analytic throughout the half-plane $R(z) > \lambda$ with the exception of the points $z = 0, -1, -2, \cdots$. The right member of (4) furnishes an analytic continuation of $\Omega(z)$ in any finite region which does not contain the points $z = 0, -1, -2, \cdots$, since throughout such a region the member under discussion is an analytic function. To prove this last statement, it is sufficient to show that the integrals on the right of (4) converge uniformly for all values of z in the proposed region. From (3), we see that the first integral in the right member of (4) converges uniformly, since it is dominated by $\int^{\infty} x^{-1-\epsilon} dx$. From (2), (5), and (17), it is evident that the second integral converges uniformly, since it is dominated by $\int^{\infty} \exp(-\frac{1}{2}\epsilon y)\, dy$. The theorem is therefore proved.

Extensions of the Theorem.

In the hypotheses of the theorem it was required that the function $g(w)$ be analytic for all values of $w=x+iy$ for which $x \geqq -\frac{1}{2}$. This condition is unnecessarily restrictive, as we shall now show. Let us assume that $g(w)$ has a singularity at a point $w=w_p$ in this region, and that w_p is not an integer. We shall now show that the theorem continues to hold provided that we subtract from the right member of (4) the loop integral

$$J_p=\frac{\Gamma(z)}{2i}\int_{L_p}\frac{\exp k\pi iw \sin \pi w\ \Gamma(z+w+1)}{g(w)}\,dw$$

wherein the contour L_p is described as follows: Draw a straight line from $w=w_p$ to infinity in a direction of the fourth quadrant, and let this line be regarded as a cut in the w-plane. Let the contour L_p consist of the two lines drawn on either side of and arbitrarily close to the cut, the ends near the point $w=w_p$ being joined by a circular arc of small radius about the same point. The integration is to be performed in the positive sense. We note that J_p is analytic for all finite values of z except $z=0,-1,-2,\cdots$.

An examination of the analysis of the proof of the theorem shows that if $g(w)$ has the singularity mentioned, then the theorem still holds if we subtract from the right member of (4) the loop integral

$$H_p=\frac{k!\,\pi^k\Gamma(z)}{2i}\int_{L_p}\frac{P(w)}{(\sin \pi w)^{k+1}}\,dw$$

where L_p denotes the contour just described. We shall show that H_p can be reduced to J_p.

Noting the value of $P(w)$ given by (10), it is evident that we may write $P(w)$ in the following form:

$$P(w)=\int_0^w \sum_{j=1}^{k} b_j \exp r_j\pi i(w-t)\,\frac{g(t)}{\Gamma(z+t+1)}\,dt.$$

By using the form of b_j given by (12) it is easily verified that

$$\sum_{j=1}^{k} b_j \exp r_j\pi i(w-t)=\frac{[\sin \pi(w-t)]^{k-1}}{\pi^{k-1}(k-1)!}.$$

Thus we have

$$H_p=\frac{k\pi\Gamma(z)}{2i}\int_{L_p}\frac{\displaystyle\int_0^w[\sin \pi(w-t)]^{k-1}\,\frac{g(t)}{\Gamma(z+t+1)}\,dt}{(\sin \pi w)^{k+1}}\,dw.$$

Let $\bar{w}$ be a point on the cut, and let L'_p be that part of L_p drawn from w_p to $\bar{w}$, on both sides of the cut. Let

$$H_p(\bar{w}) = \frac{k\pi\Gamma(z)}{2i} \int_{L'_p} \frac{\displaystyle\int_0^w [\sin\pi(w-t)]^{k-1} \frac{g(t)}{\Gamma(z+t+1)}\, dt}{(\sin\pi w)^{k+1}}\, dw.$$

If we use the multiple angle formula and expand $[\sin\pi(w-t)]^{k-1}$ by the binomial theorem, we obtain:

$$_p(\bar{w}) = \frac{k\pi\Gamma(z)}{2i} \sum_{j=1}^{k} \binom{k-1}{j-1} \int_{L'_p} (-\cot nw)^{j-1} \csc^2\pi w \int_0^w \frac{(\cos\pi t\ \sin nt)^{j-1} g(t)}{\Gamma(z+t+1)}\, dt\, dw.$$

After integrating by parts and making simple reductions, we obtain

$$H_p(\bar{w}) = \frac{k\Gamma(z)}{2i} \int_{L'_p} \frac{\displaystyle\sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{k-1}{j-1} [\cot n\bar{w}\ \sin\pi w]^j (\cos\pi w)^{k-j} - (\cos\pi w)^k}{\sin\pi w\Gamma(z+w+1)}\, dw.$$

If we carry out the summation under the integral sign and make simple transformations, we obtain

$$H_p(\bar{w}) = \frac{\Gamma(z)}{2i} \int_{L'_p} \frac{\left[\exp(-\pi i w) + \exp(-\pi i\bar{w}) \dfrac{\sin\pi w}{\sin\pi\bar{w}}\right]^k g(w)}{\sin\pi w\ \Gamma(z+w+1)}\, dw.$$

On setting

$$J_p(\bar{w}) = \frac{\Gamma(z)}{2i} \int_{L'_p} \frac{g(w)}{\exp k\pi i w \sin\pi w\ \Gamma(z+w+1)}\, dw$$

it is evident that $\lim\limits_{w\to\infty} J_p(\bar{w}) = J_p$.

We shall show that as $\bar{w} \to \infty$ along the cut, we have $\lim\limits_{\bar{w}\to\infty} [J_p(\bar{w}) - H_p(\bar{w})] = 0$, and it will follow that $\lim\limits_{\bar{w}\to\infty} H_p(\bar{w}) = J_p$. Now we have:

$$J_p(\bar{w}) - H_p(\bar{w}) = \Gamma(z) \exp(-\pi i w)$$
$$\times \int_{L'_p} \frac{\displaystyle\sum_{m=1}^{k} \left(\frac{k}{m}\right) \exp(-\pi i w)(k-m) \cdot \exp(-\pi i w)(m-1) \left(\frac{\sin\pi w}{\sin\pi\bar{w}}\right)^m g(w)}{\sin\pi w\ \Gamma(z+w+1)}\, dw.$$

But if $\bar{w} \to \infty$ in the fourth quadrant we have for $\bar{w}$ large and for all w on L'_p $\left|\dfrac{\sin\pi w}{\sin\pi\bar{w}}\right| < 1$ and $\exp(-\pi i\bar{w}) < \exp(-\pi i w)$ and hence

$$J_p(w) - H_p(w)| < \left|\frac{\Gamma(z)}{2} \exp(-\pi i\bar{w})\right| 2^k \int_{L_p} \frac{g(w)}{\exp(k-1)\pi w \sin\pi w\ \Gamma(z+w+1)}\, dw.$$

The integral exists because of the hypotheses of the theorem, and since the factor $\exp(-\pi i \bar{w})$ approaches zero, the above quantity can be made arbitrarily small. Hence if the function $g(w)$ has a finite number of singularities at the w_p, $p = 1, 2, \cdots, n$, the theorem continues to hold if we subtract from the right member of (4) the sum of the n integrals J_p. We note that if the singularity of $g(w)$ at W_p is a pole, then the integral J_p reduces to the residue of the function

$$\Gamma(z)^{k+1}\pi k! \, \frac{P(w)}{(\sin \pi w)^{k+1}}$$

where $w = w_p$ and $P(w)$ is given by (10).

PURDUE UNIVERSITY.

A KIND OF EVEN-DIMENSIONAL DIFFERENTIAL GEOMETRY AND ITS APPLICATION TO EXTERIOR CALCULUS.*

By Hwa-Chung Lee.[1]

Introduction. In this paper, we study the differential geometry of an even-dimensional space, into which a nonsingular skewsymmetric matrix is introduced as fundamental tensor. In this geometry we define a curvature tensor from which there arises the distinction between general, flat and conformally flat spaces. Necessary and sufficient conditions for such spaces to be flat or conformally flat are found in **1**, **2**. The results are applied, in **3**, to the exterior calculus to discover a necessary and sufficient condition that a nonsingular exterior differential form of degree two admit an integrating factor. The paper is concluded in **4** with a brief study of the infinitesimal automorphisms of the spaces.

1. The space L_{2n}. Consider a manifold of even dimensionality $2n$ whose points are described by a system of coördinates x^α $(\alpha, \beta, \gamma, \rho, \sigma, \tau = 1, \cdots, 2n)$, and let $a_{\alpha\beta}$ be a nonsingular skewsymmetric covariant tensor whose components are analytic functions of the x's. The manifold equipped with such a tensor is denoted by L_{2n}, and the tensor is called its *fundamental tensor*. It is at once verified that the skewsymmetric quantity

$$(1.1) \qquad K_{\alpha\beta\gamma} = \partial_\alpha a_{\beta\gamma} + \partial_\beta a_{\gamma\alpha} + \partial_\gamma a_{\alpha\beta} \qquad (\partial_\alpha = \partial/\partial x^\alpha)$$

is a covariant tensor, which we call the *curvature tensor* of L_{2n}. This is the simplest tensor involving the derivatives of the fundamental tensor. The curvature tensor evidently satisfies the identity

$$(1.2) \qquad \partial_\rho K_{\alpha\beta\gamma} - \partial_\alpha K_{\rho\beta\gamma} - \partial_\beta K_{\alpha\rho\gamma} - \partial_\gamma K_{\alpha\beta\rho} = 0$$

which is a tensorial equation, holding for all coördinate systems.

In terms of Cartan's well known notation in the exterior calculus,[2] there is associated with $a_{\alpha\beta}$ the *exterior differential form* of degree two

$$(1.3) \qquad \Omega = a_{\alpha\beta}[dx^\alpha dx^\beta]$$

which we call the *fundamental form* of L_{2n}. As $a_{\alpha\beta}$ is nonsingular, Ω is also called *nonsingular*. The *exterior derivative* of Ω is

---

* Received October 27, 1941.

[1] The author wishes to express his thanks to Professor J. M. Thomas for reading the manuscript and giving valuable criticisms.

[2] See e. g. E. Cartan, *Leçons sur les invariants intégraux*, chap. VII.

$$\Omega' = \tfrac{1}{3}K_{\alpha\beta\gamma}[dx^{\alpha}dx^{\beta}dx^{\gamma}] \tag{1.4}$$

and (1.2) is equivalent to the well known fact $\Omega'' = 0$.

An L_{2n} is called *flat* if its curvature tensor vanishes. Evidently, *every* L_2 *is flat.* For a flat L_{2n} we have $\Omega' = 0$ so that Ω itself is the exterior derivative of a pfaffian form.[3] Taking, by a change of coördinates, this pfaffian in its well known canonical form[4] we find that Ω may be written as

$$[dx^1dx^{2n}] + [dx^2dx^{2n-1}] + \cdots,$$

the last term being certainly $[dx^n dx^{n+1}]$ because of the nonsingularity of Ω. Hence in the new coördinate system the fundamental tensor assumes the form

$$a_{\alpha\beta} = \tfrac{1}{2}\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \tag{1.5}$$

where J denotes the matrix of order n with 1's on the secondary diagonal and 0's elsewhere and 0 denotes the zero matrix of order n.

It is also clear that there are other coördinate systems for which the components of $a_{\alpha\beta}$ are any constants. The converse is evident, and we have

THEOREM 1. *An L_{2n} is flat if and only if there exists a coördinate system for which the components of the fundamental tensor are constants.*

2. Conformal spaces. An L_{2n} and an $'L_{2n}$ over the same underlying manifold are said to be *conformal* to each other if their respective fundamental tensors $a_{\alpha\beta}(x)$ and $'a_{\alpha\beta}(x)$, referred to the same coördinate system, are connected by a relation of the form

$$'a_{\alpha\beta} = \phi a_{\alpha\beta}, \tag{2.1}$$

where ϕ is a nonvanishing scalar function of the coördinates in question. Evidently, *every L_2 is conformal to every other L_2.* We shall assume $2n > 2$ throughout this section. By (2.1) we find

$$'K_{\alpha\beta\gamma} = \phi(K_{\alpha\beta\gamma} + a_{\beta\gamma}\partial_{\alpha}\log\phi + a_{\gamma\alpha}\partial_{\beta}\log\phi + a_{\alpha\beta}\partial_{\gamma}\log\phi). \tag{2.2}$$

Since $a_{\alpha\beta}$ is nonsingular, it has an inverse $a^{\alpha\beta}$ (skewsymmetric in α, β) defined by $a^{\alpha\gamma}a_{\gamma\beta} = \delta_{\beta}^{\alpha}$ where δ_{β}^{α} is 1 or 0 according as α is or is not equal to β. Define the covariant vector

$$K_{\alpha} = K_{\alpha\beta\gamma}a^{\beta\gamma}$$

and call it the *curvature vector of L_{2n}.* Since by (2.1) we have $'a^{\alpha\beta} = \phi^{-1}a^{\alpha\beta}$, we find from (2.2)

$$'K_{\alpha} = K_{\alpha} - (2n-2)\partial_{\alpha}\log\phi. \tag{2.3}$$

---

[3] See e. g. E. Goursat, *Leçons sur le problème de Pfaff*, p. 105.

[4] Goursat, *loc. cit.*, p. 42.

Consequently the skewsymmetric covariant tensor

$$b_{\alpha\beta} = \partial_\alpha K_\beta - \partial_\beta K_\alpha \tag{2.4}$$

is an obsolute conformal invariant:

$$'b_{\alpha\beta} = b_{\alpha\beta}. \tag{2.5}$$

We call $b_{\alpha\beta}$ the *first conformal curvature tensor* of L_{2n}.

The result of eliminating the derivatives of $\log\phi$ in (2.2) by means of (2.3) may be written

$$'c_{\alpha\beta\gamma} = \phi\, c_{\alpha\beta\gamma} \tag{2.6}$$

where

$$c_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} + (K_\alpha a_{\beta\gamma} + K_\beta a_{\gamma\alpha} + K_\gamma a_{\alpha\beta})/(2n-2). \tag{2.7}$$

Thus the skewsymmetric covariant tensor $c_{\alpha\beta\gamma}$ is a relative conformal invariant which we call the *second conformal curvature tensor* of L_{2n}.

Solving (2.7) for $K_{\alpha\beta\gamma}$ and substituting into (1.2) give

$$a_{\rho\alpha} b_{\beta\gamma} + a_{\rho\beta} b_{\gamma\alpha} + a_{\rho\gamma} b_{\alpha\beta} + b_{\rho\alpha} a_{\beta\gamma} + b_{\rho\beta} a_{\gamma\alpha} + b_{\rho\gamma} a_{\alpha\beta} = c_{\rho\alpha\beta\gamma}, \tag{2.8}$$

where for the sake of brevity $c_{\rho\alpha\beta\gamma}$ indicates the aggregate of terms independent of the derivatives of K_α. Because of (1.1) and (2.7) this expression may be written

$$\begin{aligned} c_{\rho\alpha\beta\gamma} = {} & (2n-2)(\partial_\rho c_{\alpha\beta\gamma} - \partial_\alpha c_{\rho\beta\gamma} - \partial_\beta c_{\rho\gamma\alpha} - \partial_\gamma c_{\rho\alpha\beta}) \\ & + (K_\rho c_{\alpha\beta\gamma} - K_\alpha c_{\rho\beta\gamma} - K_\beta c_{\rho\gamma\alpha} - K_\gamma c_{\rho\alpha\beta}) \end{aligned}$$

(which is a skewsymmetric covariant tensor). Transvecting (2.8) by $a^{\rho\gamma}$ and writing $c_{\alpha\beta} = a^{\rho\gamma} c_{\rho\alpha\beta\gamma}$ we find

$$(2n-4) b_{\alpha\beta} = b a_{\alpha\beta} - c_{\alpha\beta}$$

where $b = a^{\alpha\beta} b_{\alpha\beta}$. Transvecting the above equation by $a^{\alpha\beta}$ and writing $c = a^{\alpha\beta} c_{\alpha\beta}$ we obtain $2(2n-2)b = -c$ so that

$$-2(2n-2)(2n-4) b_{\alpha\beta} = c a_{\alpha\beta} + 2(2n-2) c_{\alpha\beta}. \tag{2.9}$$

We observe that (2.4) and (2.7) immediately imply

THEOREM 2. *The first conformal curvature tensor vanishes if, and only if, the curvature vector is a gradient and the second conformal curvature tensor vanishes if, and only if, the curvature tensor is of the form*

$$K_{\alpha\beta\gamma} = g_\alpha a_{\beta\gamma} + g_\beta a_{\gamma\alpha} + g_\gamma a_{\alpha\beta} \tag{2.10}$$

where g_α is a covariant vector.

When $2n = 4$, (2.10) consists of four equations which are directly verified to be solvable for the four unknowns g_1, g_2, g_3, g_4. Hence

THEOREM 3. *The second conformal curvature tensor of every L_4 vanishes identically.*

An L_{2n} is said to be *conformally flat* if a scalar ϕ exists such that the $'L_{2n}$ defined by (2.1) is flat. Suppose that L_{2n} is conformally flat. If $'L_{2n}$ is the corresponding flat space, then $'K_{\alpha\beta\gamma}=0$ and therefore $'K_\alpha=0$, $'b_{\alpha\beta}=0$, $'c_{\alpha\beta\gamma}=0$ so that by (2.5) and (2.6) we have $b_{\alpha\beta}=0$, $c_{\alpha\beta\gamma}=0$. Suppose conversely that the latter hold. Then K_α is a gradient (by the first part of Theorem 2) and $'c_{\alpha\beta\gamma}=0$ (by (2.6)). The former fact shows, by (2.3), that we can make $'K_\alpha$ vanish by properly choosing ϕ. From $'c_{\alpha\beta\gamma}=0$ and $'K_\alpha=0$ we have $'K_{\alpha\beta\gamma}=0$ by (2.7), whence $'L_{2n}$ is flat and therefore L_{2n} is conformally flat. Thus we have proved

THEOREM 4. *A necessary and sufficient condition that an L_{2n} $(n>1)$ be conformally flat is that the first and second conformal curvature tensors vanish.*

When $2n>4$, the vanishing of the first conformal curvature tensor is a consequence of the vanishing of the second conformal curvature tensor, as is at once seen from (2.9). Hence, in view of Theorem 3 for $2n=4$, Theorem 4 may be refined into

THEOREM 5. *A necessary and sufficient condition that an L_{2n} be conformally flat is that the first conformal curvature tensor vanish for $2n=4$ and the second conformal curvature tensor vanish for $2n>4$.*

Given a conformally flat L_{2n}, the determination of the corresponding flat space is now immediate. For in this case we have, by Theorems 4, 2 equation (2.10) where $g_\alpha=-K_\alpha/(2n-2)$. The vector g_α is a gradient and so may be written in the form

$$g_\alpha=-\partial_\alpha \log\phi; \tag{2.11}$$

where ϕ is different from zero and is determined save for a constant factor. Because of (1.1) and (2.11), (2.10) may be written

$$\partial_\alpha(\phi a_{\beta\gamma})+\partial_\beta(\phi a_{\gamma\alpha})+\partial_\gamma(\phi a_{\alpha\beta})=0 \tag{2.12}$$

which shows that $\phi a_{\alpha\beta}$ is the required fundamental tensor of the corresponding flat space. (2.12) also implies the existence of a *covariant vector* a_α, determined save for an additive gradient, such that

$$\phi a_{\alpha\beta}=\partial_\alpha a_\beta-\partial_\beta a_\alpha. \tag{2.13}$$

3. Integrating factors of non-singular exterior differential forms of degree two. The theorems of the preceding section incidentally solve a particular problem in the exterior calculus. An exterior differential form (of any degree) is called an *exact differential* if its exterior derivative vanishes. An exterior differential form is said to admit an *integrating factor* if it can be made an exact differential by multiplication of a scalar factor.[5] Except for the case when the degree of the form is one (pfaffian form), a necessary and sufficient condition for a form to admit an integrating factor has not yet been found.

It is clear that the form Ω defined by (1.3) *admits an integrating factor if and only if the corresponding* L_{2n} *is conformally flat.* When $2n > 4$, Theorems 5 and 2 imply that L_{2n} is conformally flat if and only if (2.10) holds. Write the latter in the form

$$\Omega' = [\omega\Omega], \tag{3.1}$$

where ω denotes the pfaffian $g_\alpha dx^\alpha$. Thus Ω is a *factor* of Ω',[6] and we have

THEOREM 6. *A necessary and sufficient condition that a nonsingular exterior differential form of degree two in an even number* ($\geqq 4$) *of variables admit an integrating factor is that the form itself be a factor of its exterior derivative.*

Again by Theorems 5 and 2, an L_4 is conformally flat if and only if K_α is a gradient, i. e. the pfaffian form $\theta = K_\alpha dx^\alpha$ is an exact differential. Now

$$\theta = K_\alpha dx^\alpha = K_{\alpha\beta\gamma} a^{\beta\gamma} dx^\alpha = a^{\alpha\beta} K_{\alpha\beta\gamma} dx^\gamma = a^{\alpha\beta}\theta_{\alpha\beta}$$

where $\theta_{\alpha\beta} = K_{\alpha\beta\gamma}dx^\gamma$ is a system of (six) pfaffian forms, said to be *associated* with the form Ω'.[7] Calling $\theta = a^{\alpha\beta}\theta_{\alpha\beta}$ the *contraction*[8] of this associated system, we have

THEOREM 7. *A necessary and sufficient condition that a nonsingular exterior differential form of degree two in four variables admit an integrating factor is that the pfaffian form, which is the contraction of the system of pfaffian forms associated with the exterior derivative of the given form, be an exact differential.*

4. Infinitesimal automorphisms of L_{2n}.

We conclude this paper by giving a brief study of the infinitesimal automorphisms of an L_{2n}, which are, by definition, infinitesimal (point) transformations leaving invariant the fundamental form Ω. In the same way as Killing's equation in Riemannian geometry is obtained, we find that a contravariant vector v^α is the vector of an infinitesimal automorphism of L_{2n} if and only if it satisfies the equation

---

[5] Goursat, *loc. cit.*, p. 118.

[6] Goursat, *loc. cit.*, p. 97.

[7] Goursat, *loc. cit.*, p. 126.

[8] This operation is unfamiliar in the exterior calculus.

$$v^\rho \partial_\rho a_{\alpha\beta} + a_{\rho\beta}\partial_\alpha v^\rho + a_{\alpha\rho}\partial_\beta v^\rho = 0. \tag{4.1}$$

Introducing the *covariant associate* $v_\alpha = a_{\alpha\beta}v^\beta$ of v^α we may write this equation in the form

$$\partial_\alpha v_\beta - \partial_\beta v_\alpha = -v_\rho K^\rho{}_{.\alpha\beta}, \tag{4.2}$$

where $K^\rho{}_{.\alpha\beta} = a^{\rho\gamma}K_{\gamma\alpha\beta}$.

We shall only try to solve (4.2) in particular cases. If L_{2n} is flat, we have $K^\rho{}_{.\alpha\beta} = 0$ so that (4.2) implies that a covariant vector is the vector of an infinitesimal automorphism of a flat L_{2n} if and only if it is a gradient. More generally, if L_{2n} is conformally flat we may, using (2.10) and (2.11), write (4.2) in the form

$$\partial_\alpha(\phi v_\beta) - \partial_\beta(\phi v_\alpha) = v\phi a_{\alpha\beta}, \tag{4.3}$$

where $v = v^\rho g_\rho$, which can be shown to be a constant. In fact, if we operate on (4.3) with ∂_γ, perform a cyclic permutation of α, β, γ and then add the three equations thus obtained, we find, on account of (2.12), $a_{\beta\gamma}\partial_\alpha v + a_{\gamma\alpha}\partial_\beta v + a_{\alpha\beta}\partial_\gamma v = 0$, whence $\partial_\alpha v = 0$ and $v = \text{const.}$ Consequently, in view of (2.13), the general solution of (4.3) is of the form

$$\phi v_\alpha = v a_\alpha + \partial_\alpha w. \tag{4.4}$$

If we introduce the *contravariant associate* $g^\alpha = a^{\alpha\beta}g_\beta$ of g_α and transvect (4.4) by g^α, we find

$$g^\alpha \partial_\alpha w = -v(\phi + a) \tag{4.5}$$

where $a = g^\alpha a_\alpha$. Hence

THEOREM 8. *A covariant vector v_α is the vector of an infinitesimal automorphism of a conformally flat L_{2n} if and only if it is given by* (4.4) *where v is a constant and w is a solution of the linear equation* (4.5).

In this conformally flat case $(n > 1)$ it is easily seen that $v = 0$ if and only if v_α satisfies

$$v_\alpha(\partial_\beta v_\gamma - \partial_\gamma v_\beta) + v_\beta(\partial_\gamma v_\alpha - \partial_\alpha v_\gamma) + v_\gamma(\partial_\alpha v_\beta - \partial_\beta v_\alpha) = 0,$$

i.e., v_α is proportional to a gradient $\partial_\alpha \rho$, and this is the case if and only if the trajectories of the 1-parameter group of automorphisms generated by v^α are at the same time trajectories of other 1-parameter groups of automorphisms, the generators of the latter being then necessarily v^α multiplied by functions of ρ.

NATIONAL SZECHWAN UNIVERSITY, OMEI, CHINA.

OSCULATING CONICS OF THE PLANE SECTIONS THROUGH A POINT OF A SURFACE.*

By Buchin Su.

1. **Introduction.** According to the classical theorem of Moutard all the osculating conics of ∞^2 plane sections through a generic point O of an analytic surface immersed in a three-dimensional projective space are distributed on ∞^1 Moutard quadrics each of which corresponds to a non-asymptotic tangent of the surface at O. In connection with the definition of Darboux pencils of quadrics we have studied [1] Moutard pencils of quadrics (so called because each of these pencils contains a Moutard quadric). This notion has been generalized ingeniously by Louis Green [2] who shows, among other things, that two arbitrary non-asymptotic tangents at O suffice to determine a certain pencil of quadrics having a contact of the second order with the surface at O.

In studying the residual osculating conic on a given Moutard quadric Chenkuo Pa has recently proved that [3] if a quadric through the two asymptotic tangents at O possesses one and only one osculating conic of the surface at the same point, then it must necessarily belong to a Moutard pencil. Combining the investigation of L. Green with that of Pa we are naturally led to inquire whether the osculating conics belonging to *different* Moutard quadrics at O may be regulated in such a manner that they lie on a quadric of certain pencils.

The purpose of this paper is to solve the problem of determining a quadric by means of an osculating conic of a given direction and another osculating conic of another given direction with a suitably chosen plane, showing that on the quadric thus obtained there exists a third osculating conic. Thus all the osculating conics of the plane sections through O are newly distributed, three by three, on a quadric having a contact of the second order with the surface at O and meeting it in a curve containing these tangents as two of its triple tangents.

2. **Analytic basis.** If the four homogeneous projective coördinates x of a variable point on an analytic non-ruled surface S in ordinary space are given

---

* Received July 17, 1941.

[1] Buchin Su and Ichida, "On certain cones connected with a surface in the affine space," *Japanese Journal of Mathematics*, vol. 10 (1933), pp. 209-216.

[2] Louis Green, "Systems of quadrics associated with a point of a surface," *American Journal of Mathematics*, vol. 60 (1938), pp. 649-666.

[3] C. Pa, "On the quadrics of Moutard," not yet published.

as analytic functions of two independent variables u, v, and if the parametric net on S is the asymptotic net, then the functions x are solutions of a completely integrable system of differential equations in Fubini's canonical form

$$(1) \qquad \begin{cases} x_{uu} = \theta_u x_u + \beta x_v + px, \\ x_{vv} = \gamma x_u + \theta_v x_v + qx \end{cases} \qquad (\theta = \log \beta\gamma).$$

The coefficients of these equations are functions of u, v and satisfy certain conditions of integrability.

The coördinates of a point O' on S near O can be represented by the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$. For the non-homogeneous coördinates x, y, z of O' defined by putting

$$(2) \qquad x = x_2/x_1, \qquad y = x_3/x_1, \qquad z = x_4/x_1$$

we have the canonical expansion in the form

$$(3) \qquad z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3) + \phi_4(x, y) + \cdots,$$

where

$$(4) \qquad \phi_4(x, y) = \frac{1}{12}\{\beta\phi x^4 - 4\beta\psi x^3 y - 6\theta_{uv} x^2 y^2 - 4\gamma\phi x y^3 + \gamma\psi y^4\}$$

and

$$(5) \qquad \phi = \frac{\partial \log \beta\gamma^2}{\partial u}, \qquad \psi = \frac{\partial \log \beta^2\gamma}{\partial v}.[4]$$

Consider the section of the surface produced by a plane through a non-asymptotic tangent t_n whose equations are

$$(6) \qquad z = 0, \qquad y - nx = 0;$$

the equation of this plane is evidently of the form

$$(7) \qquad z = \rho(y - nx),$$

so that the osculating conic of the plane section at O is given by the equations (7) and

$$(8) \qquad \begin{aligned} -x^2 + Y[\{(\rho/3n^2)(\beta + \gamma n^3) - (1/n)\}x \\ - (\rho^2/n^2)\{(1/3n\rho)(\beta - 2\gamma n^3) + (1/n)\phi_4(1, n) \\ - (1/9n^2)(\beta + \gamma n^3)^2\}Y + (\rho/n)] = 0, \end{aligned}$$

where we have placed

$$(9) \qquad Y = y - nx.$$

---

[4] Cfr. E. P. Lane, "Power series expansions in the neighborhood of a point on a surface," *Proceedings of the National Academy of Sciences*, vol. 13 (1927), pp. 808-813.

Elimination of ρ from (7) and (8) gives the equation of the Moutard quadric belonging to the given tangent t_n, namely,

$$(10)\qquad 36n^3(z-xy)+12n^2(2\beta-\gamma n^3)xz-12n(\beta-2\gamma n^3)yz \\ +\{4(\beta+\gamma n^3)^2-36n\phi_4(1,n)\}z^2=0.$$

3. Quadrics having a contact of the second order with a surface. The equation of the most general quadric through the two asymptotic tangents of the surface S at the point O is

$$(11)\qquad xy+k_1z+k_2xz+k_3yz+k_4z^2=0,$$

where k_1, k_2, k_3, k_4 are arbitrary functions of u and v. The conic intersection of this quadric and the plane (7) is expressed by (7) and

$$(12)\qquad -x^2+Y[\{-(\rho/n)(k_2+k_3n)-1/n\}x \\ -(\rho^2/n)((1/\rho)k_3+k_4)Y-k_1(\rho/n)]=0.$$

In order that this conic be the osculating conic (8) of the plane section of the surface produced by the plane (7), it is necessary and sufficient that $k_2=-1$ and

$$(13)\qquad \begin{cases} k_2+k_3n=-(1/3n)(\beta+\gamma n^3), \\ k_3-(1/3n^2)(\beta-2\gamma n^3)=\rho\{(1/n^2)\phi_4(1,n) \\ \qquad\qquad -(1/9n^3)(\beta+\gamma n^3)^2-k_4\}. \end{cases}$$

Suppose that two arbitrary non-asymptotic tangents to S at O are given, respectively, by the equations

$$(14)\qquad z=0,\qquad y-n_ix=0 \qquad\qquad (i=1,2)$$

and that through each of them there passes a plane

$$(15)\qquad z=\rho_i(y-n_ix) \qquad\qquad (i=1,2)$$

on which lies the corresponding osculating conic K_i of the section at O. If the quadric (11) passes through both conics K_1 and K_2 the conditions (13) must hold for $n=n_i$, $\rho=\rho_i$ $(i=1,2)$ and the same set of k_2, k_3 and k_4; thus

$$(16)\qquad \begin{cases} k_2+k_3n_1=-(1/3n_1)(\beta+\gamma n_1^3), \\ k_2+k_3n_2=-(1/3n_2)(\beta+\gamma n_2^3); \end{cases}$$

$$(17)\qquad \begin{cases} k_3-(1/3n_1^2)(\beta-2\gamma n_1^3) \\ \qquad =\rho_1\{(1/n_1^2)\phi_4(1,n_1)-(1/9n_1^3)(\beta+\gamma n_1^3)^2-k_4\}, \\ k_3-(1/3n_2^2)(\beta-2\gamma n_2^3) \\ \qquad =\rho_2\{(1/n_2^2)\phi_4(1,n_2)-(1/9n_2^3)(\beta+\gamma n_2^3)^2-k_4\}. \end{cases}$$

Solving (16) for k_2 and k_3 we obtain

$$(18)\qquad \begin{cases} k_2 = -(1/3)\{(\beta/n_1n_2)(n_1+n_2)-\gamma n_1n_2\}, \\ k_3 = (1/3)\{\beta/n_1n_2-\gamma(n_1+n_2)\}. \end{cases}$$

The plane through one of the tangents t_1 and t_2 may be assigned arbitrarily and that through the other tangent is perfectly determined, since (17) gives k_4 for a given ρ_1 and therefore determines ρ_2, except when the quadric thus obtained coincides with a non-Moutard quadric of the Moutard pencil belonging to the tangent t_1 or t_2. In the exceptional case the plane (14) coincides with the tangent plane of the surface at the point O, so that no osculating conic can be obtained. That is to say, there is one and only one osculating conic on a non-Moutard quadric of any Moutard pencil, as was shown by Pa.

Let us consider the general case, confining ourselves to the determination of the other osculating conics on the quadric in consideration. Writing the equation of the plane of such a conic, if any, in the form

$$(19)\qquad z=\rho_3(y-n_3x),$$

we have

$$(20)\qquad \begin{cases} k_2+k_3n_3 = -(1/3n_3)(\beta+\gamma n_3^3), \\ k_3-(1/3n_3^2)(\beta-2\gamma n_3^2) \\ \qquad = \rho_3\{(1/n_3^2)\phi_4(1,n_3)-(1/9n_3)(\beta+\gamma n_3^3)^2-k_4\}. \end{cases}$$

The last condition evidently determines ρ_3 and consequently the plane (19) provided that the quadric is not the Moutard quadric belonging to the tangent t_3. Substituting the values of k_2 and k_3 given by (18) into the first equation of (20) and reducing, we are led to the relation

$$(21)\qquad n_1n_2n_3=-\beta/\gamma,$$

which shows that the third tangent t_3 is completely determinate when the first two t_1 and t_2 are known.

By virtue of (21) we can express (18) in the symmetrical form

$$(22)\qquad \begin{cases} k_2=\tfrac{1}{3}\gamma(n_1n_2+n_2n_3+n_3n_1), \\ k_3=-\tfrac{1}{3}\gamma(n_1+n_2+n_3). \end{cases}$$

Thus we arrive at the following conclusion:

Given two arbitrary non-asymptotic tangents t_1, t_2 to S at a generic point O and one plane π_1 through t_1 we can always determine a quadric Q through the two asymptotic tangents of S at O such that the osculating conic of the section of S produced by π_1 as well as that of the section produced by *the corresponding plane* π_2 *through* t_2 *lies on* Q. *This quadric, having a contact of the second order with* S *at* O, *contains a third osculating conic at* O *and forms a pencil when the plane* π_1 *turns about the tangent* t_1.

4. A covariant line associated with two non-asymptotic tangents. Let us consider now the correspondence between the planes π_1 and π_2 passing through the non-asymptotic tangents t_1 and t_2, respectively, such that the osculating conics of the sections produced by π_1, π_2 and the asymptotic tangents lie on the same quadric. By virtue of (17) it is easily seen that the planes (15) for $i = 1, 2$ are in the correspondence given by the equation

$$\begin{aligned}(23)\quad (1/\rho_1)\,\{k_3 - (1/3n_1^2)(\beta - 2\gamma n_1^3)\}& \\ - (1/\rho_2)\,\{k_3 - (1/3n_2^2)(\beta - 2\gamma n_2^3)\}& \\ = (1/n_1^2)\phi_4(1, n_1) - (1/n_2^2)\phi_4(1, n_2)& \\ -(1/9n_1^3)(\beta + \gamma n_1^3)^2 + (1/9n_2^3)(\beta + \gamma n_2^3)^2,&\end{aligned}$$

and are, therefore, in perspective position.

In order to derive the plane, denoted by π_{12}, on which lie all the lines of intersection of the corresponding pairs of planes π_1 and π_2, we have to substitute

$$1/\rho_i = (1/z)(y - n_i x) \qquad (i = 1, 2)$$

into (23) and then to reduce the resulting equation by means of the relations (21) and (22). A simple calculation suffices to obtain the equation of the plane π_{12}, namely,

$$\begin{aligned}(24)\quad \beta(2n_3 - p_1)x + \gamma n_3(p_2 - 3n_1n_2)y& \\ = [\tfrac{1}{4}\{ -\frac{\beta^2\phi}{\gamma}\,\frac{n_1+n_2}{n_1^2n_2^2} + 4\,\frac{\beta^2\psi}{\gamma}\,\frac{1}{n_1n_2} - 4\beta\phi + \beta\psi(n_1+n_2)\}& \\ + \tfrac{1}{3}\{\frac{\beta^3}{\gamma}\,\frac{n_1^2+n_1n_2+n_2^2}{n_1^3n_2^3} - \beta\gamma(n_1^2+n_1n_2+n_2^2)\}]z,&\end{aligned}$$

where we have placed

$$(26)\qquad p_1 = n_1 + n_2 + n_3, \qquad p_2 = n_1n_2 + n_2n_3 + n_3n_1.$$

If we take the three tangents t_i $(i = 1, 2, 3)$ at O given by

$$(27)\qquad z = y - n_i x = 0 \qquad (i = 1, 2, 3),$$

where n_1, n_2, n_3 satisfy (21); then there are three planes π_{12}, π_{23}, π_{31} which correspond to the pairs of tangents t_1, t_2; t_2, t_3 and t_3, t_1 respectively. These planes are evidently concurrent in a line l associated with the tangents t_1 and t_2 (and consequently t_3) of the surface S at O.

From the equation (24), and that obtained by interchanging n_1 and n_2, the equations of l follow immediately. The result of carrying out the computation is as follows:

$$(28)\quad \begin{aligned} x:y:z = {} & \tfrac{1}{4}(-15\beta\psi + 2\gamma p_2\phi - \gamma p_1 p_2 \psi) - \gamma^2 p_2{}^2 + \tfrac{1}{3}\gamma^2 p_1{}^2 p_2 \\ & : \tfrac{1}{4}(-15\beta\phi - 2\beta p_1 \psi - \gamma p_1 p_2 \phi) - \beta\gamma p_1{}^2 - \tfrac{1}{3}\gamma^2 p_1 p_2{}^2 \ : 9\beta + \gamma p_1 p \end{aligned}$$

Several particular cases of the above result are worth noting. If the three tangents are those of Darboux, then $p_1 = p_2 = 0$ and, consequently, the corresponding line l is the canonical line of the first kind for which $k = -5/12$, which we shall designate as *the first axis of Bompiani.*[5]

If two of the tangents lie in distinct Segre directions, the third is seen to lie in that Darboux direction which is conjugate to the residual Segre direction. Every Darboux direction thus determines a covariant line l. Putting

$$n_1 = (\beta/\gamma)^{1/3}\omega^{r+1}, \qquad n_2 = (\beta/\gamma)^{1/3}\omega^{r+2}, \qquad n_3 = (\beta/\gamma)^{1/3}\omega^{r},$$

where $\omega^3 = 1$, $\omega \neq 1$ and $r = 1, 2, 3$, so that

$$p_1 = -2(\beta/\gamma)^{1/3}\omega^r, \qquad p_2 = 2(\beta/\gamma)^{2/3}\omega^{2r},$$

we obtain the covariant line l_r

$$\begin{aligned} x:y:z = {} & -(11/20)\psi - (4/15)\beta^{1/3}\gamma^{2/3}\omega^r + (1/5)(\gamma/\beta)^{1/3}\phi\omega^{2r} \\ & : -(11/20)\phi + (1/5)(\beta/\gamma)^{1/3}\omega^r - (4/15)\beta^{2/3}\gamma^{1/3}\omega^{2r} : 1. \end{aligned}$$

The harmonic polar of the tangent plane $z = 0$ with respect to the trihedron formed by l_1, l_2, l_3 is easily found to be *the canonical line of the first kind for which* $k = -11/20$.

If two of the tangents are conjugate, namely, such that their directions are n, $-n$, the third must lie in the direction $\beta/\gamma n^2$, so that

$$p_1 = \beta/\gamma n^2, \qquad p_2 = -n^2$$

and the covariant line l_n corresponding to any non-asymptotic direction n (and its conjugate $-n$) is given by the equations

$$(29)\quad \begin{aligned} x:y:z = {} & \{-(7/16)\psi - (\gamma^2/8\beta)n^4 - (1/16)(\gamma/\beta)\phi n^2 - (1/24)\beta(1/n^2) \\ & : \{-(7/16)\phi - (\beta^2/8\gamma)(1/n^4) - (1/16)(\beta/\gamma)\psi(1/n^2) - (1/24)\gamma n^2\} : \end{aligned}$$

---

[5] Cf. L. Green, *loc. cit.*[2], p. 654.

In particular, for the Darboux directions

$$n = -(\beta/\gamma)^{1/3}\omega^r \qquad (r = 1, 2, 3)$$

(and the Segre directions) we have the lines:

$$\begin{aligned} x : y : z = {} & \{-(7/16)\psi - (1/6)\beta^{1/3}\gamma^{2/3}\omega^r - (1/16)(\gamma/\beta)^{1/3}\omega^{2r}\} \\ & : \{-(7/16)\phi - (1/16)(\beta/\gamma)^{1/3}\omega^r - (1/6)(\beta^{2/3}\gamma^{1/3}\omega^{2r}\} : 1, \end{aligned}$$

where $r = 1, 2, 3$. In consequence, the harmonic polar of the tangent plane $z = 0$ with respect to the trihedron formed the these lines is *the canonical line of the first kind for which* $k = -7/16$.

If two of the tangents coincide in a non-asymptotic direction n, the third must lie in the direction $-\beta/\gamma n^2$, so that

$$p_1 = -(1/\gamma n^2)(\beta - 2\gamma n^3), \qquad p_2 = -(1/\gamma n)(2\beta - \gamma n^3).$$

By means of these values there is no difficulty in determining the line (28) in question. Without expressing the equations explicitly we shall only consider two interesting cases, namely, the lines corresponding to a Segre direction and a Darboux direction. In the latter case we have

$$n = -(\beta/\gamma)^{1/3}\omega^r.$$

so that

$$p_1 = -3(\beta/\gamma)^{1/3}\omega^r, \qquad p_2 = -3(\beta/\gamma)^{2/3}\omega^{2r}$$

and the line becomes

$$\begin{aligned} x : y : z = {} & \{(1/3)\psi - (1/12)(\gamma/\beta)^{1/3}\phi\omega^{2r} - \gamma(\beta/\gamma)^{1/3}\omega^r\} \\ & : \{-(1/3)\phi + (1/12)(\beta/\gamma)^{1/3}\psi\omega^r\} : 1. \end{aligned}$$

These form a trihedron with respect to which the harmonic polar of the tangent plane $z = 0$ is obviously *the first axis of Čech.*

In the similar way we obtain a covariant line corresponding to every tangent of Segre; the harmonic polar in question is *the canonical line of the first kind for which* $k = -7/16$.

5. The quadrics of Darboux. Let us confine ourselves to an important case which can be derived from the general result stated in Section **3**, namely, the case where the tangents are in Darboux directions. The quadric in question must be a Darboux quadric, since (22) gives $k_2 = k_3 = 0$ for the roots n_1, n_2, n_3 of the equation

$$\beta + \gamma n^3 = 0,$$

so that the equation of the quadric becomes

$$xy - z + k_4 z^2 = 0. \tag{30}$$

The planes of osculating conics lying on this Darboux quadric are

$$z = \rho_i (y - n_i x) \qquad (i = 1, 2, 3), \tag{31}$$

where

$$1/\rho_i = -(1/\beta)[\phi_4(1, n_i) - n_i^2 k_4] \tag{32}$$

and

$$n_i = -(\beta/\gamma)^{1/3} \omega^i \qquad (i = 1, 2, 3) \tag{33}$$

In order that these planes be concurrent in a line it is necessary and sufficient that

$$\begin{vmatrix} n_1 & 1/\rho_1 & 1 \\ n_2 & 1/\rho_2 & 1 \\ n_3 & 1/\rho_3 & 1 \end{vmatrix} = 0$$

or

$$k_4 = -\tfrac{1}{2}\theta_{uv}. \tag{34}$$

Hence we have the following theorem:

Every plane through one tangent of Darboux uniquely determines a Darboux quadric Q such that the three osculating conics of sections produced by this plane and two others each passing through the remaining tangents of Darboux lie on Q. The planes of these conics are concurrent if, and only if, Q is the quadric of Wilczynski-Bompiani.

The common line of these planes, in the last case, can be found by substituting (34), (32) into (31), so that the equation of each plane is

$$n_i[x + (5/12)\psi z] - [y + (5/12)\phi z] = 0.$$

Hence *the planes of the three osculating conics on the quadric of Wilczynski-Bompiani are concurrent in the first axis of Bompiani.*

Since the expression (32) of $1/\rho_i$ for a given n_i is linear in k_4, we may conclude that if π_0, π_L, π_W be the tangent plane and the planes of osculating conics of sections through a non-asymptotic tangent and lying, respectively, on the quadric of Lie and that of Wilczynski-Bompiani and if π denotes the plane through the same tangent such that the double ratio

$$(\pi_0 \pi_L, \pi \pi_W) = h,$$

the quadric determined by π is the Darboux quadric

$$xy - z - \{\tfrac{1}{2}\theta_{uv} + (1-h)\beta\gamma\}z^2 = 0. \tag{35}$$

Moreover, by means of the pencil of Darboux quadrics, we can also determine the first and the second axes of Bompiani. In fact, the three planes of osculating conics lying on the quadric (30) form a trihedron of edges

$$n_i x - y + (1/\rho_i)z = 0, \qquad n_j x - y + (1/\rho_j)z = 0,$$

where $i \neq j$, $i, j = 1, 2, 3$ and ρ_i is given by (32). An easy calculation, which we shall omit here, gives the equations of the harmonic polar of the tangent plane with respect to the trihedron, namely,

$$x : y : z = -(5/12)\psi : -(5/12)\phi : 1,$$

which represent the first axis of Bompiani.

In the similar way the three conics in consideration intersect each other at three points distinct from O; when the quadric varies in the pencil of Darboux the plane of these points of intersection describes a flat pencil whose axis is given by

$$z = 0, \qquad (5/12)\phi x + (5/12)\psi y + 1 = 0,$$

that is, *the second axis of Bompiani.*

6. The curve of intersection of a surface and a quadric. As was shown in Section **3**, the planes

$$z = \rho_i(y - n_i x) \qquad (i = 1, 2, 3)$$

determine uniquely a quadric whose equation is

$$xy - z + k_2 xz + k_3 zy + k_4 z^2 = 0, \tag{36}$$

provided that the directions n_1, n_2, n_3 are connected by the relation (21). From (36) it follows that the quadric thus determined must have a contact of the second order with the surface S at O.

By means of the relation (22) we can easily demonstrate that the triple tangents of the curve of intersection of (36) and S at O coincide with the three tangents

$$(y - n_1 x)(y - n_2 x)(y - n_3 x) = 0. \tag{37}$$

Hence *the quadrics corresponding to the planes through a non-asymptotic tangent t_1 or t_2 form a pencil of quadrics having a contact of the second order with S at O and meeting S in a one-parameter family of curves each containing* *these tangents as two of its triple-point tangents. The residual tangent t_3 at O is the same for all the curves of the family and is the common tangent of all the osculating conics each of which lies on a quadric of the pencil.*

Conversely, suppose that a quadric Q has a contact of the second order with the surface S at O; the curve of intersection has O for its triple point. From (3) and (36) we have the equations of the triple-point tangents

$$z = y - n_i x = 0 \qquad (i = 1, 2, 3),$$

where n_1, n_2, n_3 are the roots of the equation

$$\tfrac{1}{3}\beta + k_2 n + k_3 n^2 + \tfrac{1}{3}\gamma n^3 = 0,$$

and therefore

$$n_1 n_2 n_3 = -\beta/\gamma, \quad k_2 = \tfrac{1}{3}\gamma(n_1 n_2 + n_2 n_3 + n_3 n_1), \quad k_3 = -\tfrac{1}{3}\gamma(n_1 + n_2 + n_3).$$

In order to obtain the osculating plane of each branch at O we have to assume that the expansions for the branch which touches the tangent t_i,

$$z = y - n_i x = 0,$$

are given by the equations

$$(38) \qquad y = n_i x + A x^2 + \cdots, \qquad z = n x^2 + \cdots.$$

and then to substitute (38) into the equation

$$\tfrac{1}{3}(\beta x^3 + \gamma y^3) + \phi_4(x, y) + k_2 x y + k_3 x z + k_4 z^2 = 0;$$

the left-hand side should be satisfied identically up to and including the term in x^4. Thus the coefficient A in (38) is given by

$$(39) \quad (1/n_i^2)\phi_4(1, n_i) - k_4 - (1/9n_i^3)(\beta + \gamma n_i^3)^2 \\ = (A/n_i)[k_3 - (1/3n_i^2)(\beta - 2\gamma n_i^3)].$$

Observing that the osculating plane of the branch (38) at O is

$$(40) \qquad A z = n_i(y - n_i x)$$

and comparing (39) with (17), we are led to the following result:

On every quadric Q having a contact of the second order with the surface S at O there are, in general, three osculating conics of the sections of S at O and their planes coincide with the osculating planes to three branches of the curve of intersection of S and Q.

From this we infer that the characterization of the quadric of Wilczynski-

Bompiani stated in Section 3 coincides with the one obtained by L. Green.[6] Similarly, we can put the interesting result of L. Green into a new form containing the osculating conics of sections of the surface.

Several exceptions to the last theorem shall be mentioned.

(i) If the quadric Q determined by an osculating conic C coincides with the Moutard quadric belonging to the tangent of C at O, then one of the two other osculating conics must necessarily coincide with C and the remaining one is *the residual conic of* Q.

(ii) If the quadric Q is a non-Moutard quadric in a Moutard pencil, then the curve of intersection of Q with S at O has two coincident triple-point tangents, the osculating plane to each being the tangent plane of S at O. Therefore there is *only one* osculating conic on a non-Moutard quadric in a Moutard pencil as Pa has shown.

(iii) If the quadric Q is the Moutard quadric belonging to one of the Darboux tangents at O, there is *no residual osculating conic.*

(iv) If Q is a non-Moutard quadric in the Moutard pencil belonging to one of Darboux tangents, it contains *no osculating conic of the surface at* O.

NATIONAL UNIVERSITY OF CHEKIANG,
TSUNYI, KWEICHOW, CHINA.

---

[6] L. Green, *loc. cit.*[2], p. 654.

THE ASYMPTOTIC BEHAVIOR OF A CLASS OF ENTIRE FUNCTIONS.*

By C. V. NEWSOM.

Introduction. This paper considers the behavior of the function,

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} g(n) z^{sn}, \qquad g(n) = 1/\prod_{j=1}^{s} \Gamma(n + \rho_j),$$

for large values of the complex variable z, if s is an integer > 1, and ρ_j is a complex constant. In particular, if $k \geqq [s/2] + 1$ is chosen as an odd integer, it is shown that

$$(2) \quad f(z) = (2\pi)^{(1-s)/2} s^{-\frac{1}{2}} \sum_{p=-(k-1)/2}^{(k-1)/2} e^{sz_p} z_p^{(s+1)/2-\rho} \left[\sum_{j=0}^{n} c_j/(sz_p)^j + \omega(z_p, n)/(sz_p)^n \right],$$

for all values of z satisfying the condition $-\pi/s < \arg z < \pi/s$; wherein $c_0 = 1$; $|z| = |z_p|$; $\arg z_p = \arg z - 2p\pi/s$; $\rho = \sum_{m=1}^{s} \rho_m$; and $\lim_{|z| \to \infty} \omega(z_p, n) = 0$. If k is even the same conclusion results except that the limits upon the first summation are $-k/2$ and $(k-2)/2$ when $-2\pi/s < \arg z < 0$, and are $-(k-2)/2$ and $k/2$ when $0 < \arg z < 2\pi/s$.

The asymptotic behavior of the same function was considered by Van Engen[1] in his dissertation, but the method employed here is sufficiently general that it appears to offer an approach to the study of a large class of entire functions related to (1).

1. A preliminary lemma. The present study employs an important lemma which is essentially a modification of a previous theorem due to the author.[2] The earlier theorem and the present lemma are both concerned with entire functions of the general type (1) in which the coefficient $g(n)$ is such that the function $g(w)$ of the complex variable $w = x + iy$ satisfies the following two conditions:

---

* Received January 11, 1942.

[1] Henry Van Engen, "On the asymptotic behavior of analytic functions," *Dissertation*, University of Michigan, 1934.

[2] C. V. Newsom, "On the character of certain entire functions in distant portions of the plane," *American Journal of Mathematics*, vol. 60 (1938), pp. 561-572. Note, also, C. V. Newsom and A. Franck, "Upon the asymptotic representation of functions of the Bessel type," *Bol. Mat.*, vol. 13 (1940), pp. 11-14.

(3) $g(w)$ is single valued and analytic throughout the finite w-plane; and

(4) $|g(w)| < K\,e^{k\pi y}$, where K is a constant independent of x and y, and k is a positive integer.

The theorem then states that

$$(5)\quad f(z) = \int_{-c-1/2}^{\infty} \{g(x)\,[\pm z^s]^x (\sin k\pi x)/(\sin \pi x)\}\,dx - \sum_{m=-c}^{-1} g(m) z^{sm} + \xi_k(c,z),$$

for all values of z satisfying the condition $-\pi < \arg(\pm z^s) < \pi$; wherein c is any arbitrary positive integer; k is the integer appearing in condition (4) above; the upper or the lower of the signs $\pm$ is to be taken according as k is odd or even; and $\lim\limits_{|z|\to\infty} z^c \xi_k(c,z) = 0$, irrespective of the value chosen for c.

To inaugurate the study of the representation (5), let k be odd. In view of the fact that $(\sin k\pi x)/(\sin \pi x) = 1 + 2\sum\limits_{p=1}^{(k-1)/2} \cos 2p\pi x$, it follows that the integral of (5) may be rewritten in the form,

$$(6)\qquad \int_{-c-1/2}^{\infty} \{g(x)\,[z^{sx} + 2\sum_{p=1}^{(k-1)/2} z_p^{sx} e^{2p\pi i x} \cos 2p\pi x]\}\,dx,$$

wherein $-\pi/s < \arg z < \pi/s$; $\arg z_p = \arg z - 2p\pi/s$; and $|z| = |z_p|$.

Since $2e^{2p\pi i x}\cos 2p\pi x = 1 + e^{4p\pi i x}$, the expression within the bracket of the integrand of (6) may be considerably simplified. In fact, under the conditions upon $g(w)$ previously given, we have the conclusion,

$$(7)\qquad f(z) = \int_{-c-1/2}^{\infty} [\sum_{p=-(k-1)/2}^{(k-1)/2} g(x) z_p^{sx}]\,dx - \sum_{m=-c}^{-1} g(m) z^{sm} + \xi_k(c,z),$$

in which $-\pi/s < \arg z < \pi/s$; $\arg z_p = \arg z - 2p\pi/s$; and $|z| = |z_p|$.

If k is even, a similar analysis is possible. The conclusion obtained differs from (7), however, in the fact that the limits upon the summation are $-k/2$ and $(k-2)/2$ when $-2\pi/s < \arg z < 0$, and are $-(k-2)/2$ and $k/2$ when $0 < \arg z < 2\pi/s$.

2. **Proof of the theorem.** To employ the result (7) in the study of the particular problem of this paper, it is merely necessary to specialize $g(x)$

as indicated in (1). As a consequence of a lemma demonstrated by Ford,[3] condition (4) is satisfied when $k \geqq [s/2] + 1$. Thus, it is apparent that a study should now be undertaken of integrals of the type,

$$(8) \qquad \int_{-c-1/2}^{\infty} [z^{sx}/\prod_{j=1}^{s}\Gamma(x+\rho_j)]dx,$$

subject to the restrictions upon z listed after (7).

If $x = (2y - 2\rho + s + 1)/2s$, where $\rho = \sum_{m=1}^{s}\rho_m$, the integral under consideration becomes

$$(9) \qquad z^{(s+1)/2-\rho/s}\int_{d}^{\infty}[F(y)(sz)^y/\Gamma(y+1)]dy,$$

in which, $d = \rho - sc - s - 1/2$, and $F(y)$ is the ratio of the functions, $\Gamma(y+1)$ and $s^y\prod_{j=1}^{s}\Gamma(y/s + 1/2 + 1/2s - \rho/s + \rho_j)$. The path of integration extends from the lower limit of the integral to positive infinity in the direction of the real y-axis.

The analysis of $\log F(y)$ follows immediately when it is recalled that

$$(10) \qquad \log\Gamma(y) = \tfrac{1}{2}\log 2\pi + (y - \tfrac{1}{2})\log y - y + \omega(y),$$

in which

$$\omega(y) = B_1/1\cdot 2y - B_2/3\cdot 4y^3 + \cdots + (-1)^{n-1}[B_n + \epsilon(y,n)]/(2n-1)(2n)y^{2n-1},$$

and where $\lim_{|y|\to\infty}\epsilon(y,n) = 0$, and $|\arg y| < \pi$. So, for sufficiently large values of y taken upon the path of integration, and after considerable algebraic simplification, it is determined that

$$(11) \quad \log F(y) = \tfrac{1}{2}(1-s)\log 2\pi + \tfrac{1}{2}\log s + a_1/y + a_2/y^2 + \cdots + [a_n + \alpha(y,n)]/y^n,$$

in which the a's are complex constants, and $\lim_{|y|\to\infty}\alpha(y,n) = 0$. Of course, then,

$$(12) \quad F(y) = s^{\frac{1}{2}}2\pi^{(1-s)/2}\exp\{a_1/y + a_2/y^2 + \cdots + [a_n + \alpha(y,n)]/y^n\}.$$

As a consequence of a well known principle in the theory of asymptotic series,[4] the exponential factor occurring in this equality may be represented asymptotically by a power series in $1/y$, in which the constant term is unity.

---

[3] W. B. Ford, *Asymptotic Developments*, Scientific Series, University of Michigan, XI (1936), p. 61.

[4] Ford, *op. cit.*, pp. 3-4.

This series, in turn, may be reformulated into a factorial series so that integral (9) now becomes:

$$(13)\qquad (2\pi)^{(1-s)/2}s^{-\frac{1}{2}}z^{(s+1)/2-\rho}\Big\{\int_d^\infty \sum_{j=0}^{n}[c_j(sz)^y/\Gamma(y+j+1)]dy + \int_d^\infty [\beta(y,n)(sz)^y/\Gamma(y+n+1)]dy\Big\};$$

wherein $c_0 = 1$, and the remaining c's are complex constants; and $\lim\limits_{|y|\to\infty}\beta(y,n) = 0$. In fact, each integral obtained from relation (7) will be of this type.

Integrals similar to those appearing in (13) were studied by Ford[5] in his consideration of functions of exponential type. In his study, however, it was necessary to make the restriction, $|\arg z| < \pi/2$. This condition is satisfied in the present analysis by all the z's except possibly $z_{(k-1)/2}$ and $z_{-(k-1)/2}$ when k is odd, and $z_{(k-2)/2}$ and $z_{-k/2}$ when k is even. Thus with the exception of these particular z's, typified temporarily by the variable u, the part of (13) inclosed within the brace may be written immediately in the following form as the result of the study of Ford:

$$(14)\qquad e^{sz}\Big[\sum_{j=0}^{n} c_j/(sz)^j + \omega(z,n)/(sz)^n\Big];$$

wherein $\lim\limits_{|z|\to\infty}\omega(z,n) = 0$.

For the case when the real part of $u \leqq 0$, a slightly different analysis of (13) is necessary. Referring again to the study of Ford, it will be observed that the first integral of (13) gives a result similar to (14) except that the remainder term does not involve the exponential factor, and is, in fact, in the form $\sum\limits_{j=0}^{n}\alpha_j(u,n)/(su)^j$, where $\lim\limits_{|u|\to\infty}\alpha_j(u,n) = 0$. Thus, for any z having the property that $|\arg z| < \pi/2$, it follows that the product of this remainder term and the factor $e^{-sz}(sz)^m$ produces a function which has the limit zero as $|z| = |u| \to \infty$, when m is any positive integer. Consequently it may be combined with any $\omega(z,n)$ occurring in (14). A similar method serves to dispose of that part of the right member of equality (7) which is outside the integral.

The second integral of (13), when the complex variable u is $z_{(k-1)/2}$, may be written thus:

---

[5] *Op. cit.*, pp. 65-67.

$$(15)\qquad 1/(su)^n \int_{d+n+1}^{\infty} [\beta(x-n-1,n)(su)^{x-1}/\Gamma(x)]\,dx$$

$$= 1/(sz_0)^n\{(z_0/u)^n \int_{d+n+1}^{\infty} [P(x)(sz_0)^{x-1}/\Gamma(x)]\,dx\},$$

in which $P(x) = \beta(x-n-1,n)e^{(x-1)(1-k)\pi i/s}$. Of course, a similar analysis is possible for any other value of u.

As the result of the application of another theorem given by Ford;[6] the function inclosed within the brace of (15), when multiplied by e^{-sz_0}, has the limit zero as $|z_0| \to \infty$. Thus $\omega(z_0, n)$ may be augmented by such a product. Consequently, result (14) is appropriate irrespective of the particular z selected. It is true, however, that the expansion ceases to be dominant for any z having the property that its real part $\leqq 0$.

In summary, the representation of $f(z)$ given in (2) follows immediately. In practice, k should be chosen as $[s/2]+1$, or perhaps, $[s/2]+2$, if it is desired to enlarge the sector for z over which the expansion is valid.

THE UNIVERSITY OF NEW MEXICO,
ALBUQUERQUE, NEW MEXICO.

---

[6] *Op. cit.*, p. 38.

CONFORMAL GEOMETRY OF HORN ANGLES OF HIGHER ORDER.*

By Mary Elizabeth Ladue.

Introduction. Under a conformal transformation a single curve has no invariant, but two intersecting curves have as an invariant the size of the angle between their tangents at their point of intersection. Since the order of contact is also a conformal invariant, two curves with first or higher order contact will be transformed into a pair of curves having the same order contact and the invariant mentioned above will have the trivial value zero. This configuration, an ordered pair of analytic curves with n-th order contact, is a *horn angle of the n-th order* with its *vertex* at their point of intersection. Kasner has shown that the lowest order non-trivial conformal invariant of a first order horn angle is a third order differential invariant. In fact he has shown that every horn angle has one and only one non-trivial conformal invariant which for an n-th order horn angle will be of differential order $2n+1$. This $2n+1$ order differential invariant of the n-th order horn angle he calls the *measure* of that horn angle, and he has found the expressions for the measures of first and second order horn angles (1). In this paper the measures of third, fourth and fifth order horn angles will be obtained.

The measure of an n-th order horn angle is determined by the values assumed by the first $2n+1$ derivatives of the two curves forming the sides of the angle at their point of intersection. Consequently its formula can be expressed in terms of the first $2n+2$ coefficients of the power series expansions of these curves (actually the first $n+1$ of these cancel out), or in terms of the curvatures and arc-length derivatives of curvature evaluated at the vertex, or finally, in simplest form, in terms of $n+1$ certain simple functions of these curvatures and their arc-length derivatives. For any particular curve the $n+1$ values assumed by these functions at any particular point can be regarded as the coördinates of a point in an $n+1$ dimensional space. If the set of all curves (analytic arcs) through a given point and having there n-th order contact with each other is called an *n-th order horn set,* then with such a horn set is associated the set of points corresponding to the curves of that set. This set of points is called the *K_{n+1}-space* associated with the given horn set. Since the distance between any two points of this space is given by the measure of the horn angle giving rise to these points, the study of the geometry

---

* Received June 6, 1942; Presented to the American Mathematical Society, February, 1943.

of this space is equivalent to the study of the conformal invariants of this set of curves. The properties of the geometries of the E-spaces associated with first and second order horn angles have been studied in considerable detail by Kasner, de Cicco, and Comenetz (2), and in this paper the geometry of a third order horn set is investigated in order that the similarities and differences in the geometries corresponding to various ordered horn sets may be considered.

Roughly dual to the group of conformal transformations is the group of equilong transformations of the plane which convert straight lines into straight lines in such a way that the distance between the points of contact of any two curves (defined as the envelope of an infinitude of lines) having a common tangent remains invariant (3). When this distance is zero and the two points of contact are the same, the configuration becomes a horn angle which Kasner has shown to have a non-trivial equilong invariant of higher order (1a). As in the conformal case, it is possible to associate with each horn set an equilong geometry which will have as its metric the equilong measure of horn angles of that order of contact. Kasner has investigated the equilong geometry associated with first order horn sets and found it to be identical with that of the conformal geometry of first order horn sets (4). In this paper the equilong geometries corresponding to horn sets of second and third order are investigated and, as in the earlier case, are found to be identical with the corresponding conformal geometries. The comparison of the conformal and equilong geometries of several higher orders is to be published soon.

The problem of a conformal measure of a horn angle is interesting not only as the study of the invariants of an important geometry—the geometry associated with the group of conformal transformations of the plane—but historically in that it answers by modern methods a question the ancient Greek mathematicians were unable to cope with satisfactorily. For the conformal measure of a horn angle provides a means of distinguishing according to size between angles which the Greeks were unwillingly forced to call the same, a measure furthermore which is significant in that it is invariant under a group of transformations which includes the Euclidean group of rigid motions of Greek geometry.

1. Conformal measure of Horn angles of higher order. In this section the measures of horn angles of several higher orders of contact will be developed. Since these invariants can be expressed in terms of the metric invariants of curvature and the arc-length derivatives of curvature of the curves forming the sides of the angle, they are obviously invariant under the group of rigid motions. Hence there is no loss in generality in considering the measure of the horn angle whose sides are the curves

$$(1) \quad y = a_2x^2 + a_3x^3 + \cdots$$

$$(1') \quad y = a'_2x^2 + a'_3x^3 + \cdots \quad (\text{with } a_2 = a'_2, \cdots, a_n = a'_n, \ a_{n+1} \neq a'_{n+1}),$$

which has the origin as its common point, the positive direction of the x-axis as its common direction, and has contact of order n at the origin. Furthermore, this study will be limited to the effect on this horn angle of those direct conformal transformations which carry the origin of the (x, y)-plane into the origin of the (X, Y)-plane and the positive direction on the x-axis into the positive direction on the X-axis. Such a transformation can, in general, be represented by a power series of the form

$$(2) \quad Z = c_1z + c_2z^2 + \cdots \quad (\text{with } c_1 \text{ real and positive}),$$

or if $z = x + iy$, $Z = X + iY$, and $c_n = \alpha_n + i\beta_n$ by

$$(2') \quad \begin{aligned} X &= \alpha_1x + \alpha_2(x^2 - y^2) - 2\beta_2xy + \cdots \\ Y &= \alpha_1y + \beta_2(x^2 - y^2) + 2\alpha_2xy + \cdots \end{aligned} \quad (\alpha_1 > 0).$$

Under the transformation $(2')$ the horn angle $(1, 1')$ is carried conformally into another horn angle of the same order of contact which may be given by the pair of curves

$$(3) \quad Y = A_2X^2 + A_3X^3 + \cdots$$

$$(3') \quad Y = A'^2X^2 + A'_3X^3 + \cdots \ (\text{with } A_2 = A'_2, \cdots, A_n = A'_n, A_{n+1} \neq A'_{n+1}).$$

The Measure of a Horn Angle of the Third Order. The measure of a horn angle of the third order will be derived by determining an expression in terms of the coefficients of the curves (1) and $(1')$ which is invariant under the transformation $(2')$. This will be done by first considering the expressions invariant under the modified form of the transformation which is obtained by setting $\alpha_1 = 1$,

$$(2'') \quad \begin{aligned} X &= x + \alpha_2(x^2 - y^2) - 2\beta_2xy + \cdots \\ Y &= y + \beta_2(x^2 - y^2) + 2\alpha_2xy + \cdots. \end{aligned}$$

This will cause a simplification in the calculations and a slight modification in the results which can easily be corrected later by following the transformation $(2'')$ by the simple magnification

$$(2''') \quad Z = \alpha_1z \quad (\alpha_1 > 0).$$

The condition that the horn angle $(1, 1')$ be equivalent to the horn angle $(3, 3')$ under the transformation $(2'')$ is expressed by the two identities in x

$$
\begin{aligned}
(4)\quad & (a_2x^2 + a_3x^3 + \cdots) \\
& + \beta_2[x^2 - (a_2x^2 + a_3x^3 + \cdots)^2] + 2\alpha_2 x(a_2x^2 + a_3x^3 + \cdots) + \cdots \\
& \equiv A_2\{x + \alpha_2[x^2 - (a_2x^2 + a_3x^3 + \cdots)^2] - 2\beta_2 x(a_2x^2 + a_3x^3 + \cdots) + \cdots\}^2 + \cdot
\end{aligned}
$$

$$
\begin{aligned}
(4')\quad & (a'_2x^2 + a'_3x^3 + \cdots) \\
& + \beta_2[x^2 - (a'_2x^2 + a'_3x^3 + \cdots)^2] + 2\alpha_2 x(a'_2x^2 + a'_3x^3 + \cdots) + \cdots \\
& \equiv A'_2\{x + \alpha_2[x^2 - (a'_2x^2 + a'_3x^3 + \cdots)^2] - 2\beta_2 x(a'_2x^2 + a'_3x^3 + \cdots) + \cdots\}^2 +
\end{aligned}
$$

obtained by substituting (2″) into (3) and (3′) and eliminating y by means of (1) and (1′). The following equations, representing the effect of the transformation (2″) on the coefficients of the curve (1), are obtained by equating to zero the coefficients of the various powers of x in the identity (4) and solving the resulting equations for $A_2, \cdots, A_7$:

$$(5_1)\quad A_2 = a_2 + \beta_2$$

$$(5_2)\quad A_3 = a_3 + \beta_3 - 2\alpha_2\beta_2$$

$$(5_3)\quad A_4 = a_4 - \alpha_2 a_3 + 3\beta_2 a_2^2 + (\alpha_3 - \alpha_2^2 + 4\beta_2^2)a_2 + \beta_4 - 2\alpha_3\beta_2 - 3\alpha_2\beta_3 + 5\alpha_2$$

$$
\begin{aligned}
(5_4)\quad A_5 = {} & a_5 - 2\alpha_2 a_4 + 8\beta_2 a_2 a_3 + (\alpha_2^2 + 4\beta_2^2)a_3 + 2\alpha_2 a_2^3 \\
& + (3\beta_3 - 6\alpha_2\beta_2)a_2^2 + (2\alpha_4 - 6\alpha_2\alpha_3 + 4\alpha_2^3 - 24\alpha_2\beta_2^2 + 12\beta_2\beta_3)a_2 \\
& + \beta_5 - 2\beta_2\alpha_4 - 4\alpha_2\beta_4 - 3\alpha_3\beta_3 + 12\alpha_2\beta_2\alpha_3 + 9\alpha_2^2\beta_3 - 14\alpha_2^3\beta_2
\end{aligned}
$$

$$(5_5)^1\quad A_6 = a_6 - 3\alpha_2 a_5 + \cdots + 42\alpha_2^4\beta_2$$

$$(5_6)^1\quad A_7 = a_7 - 4\alpha_2 a_6 + \cdots - 132\alpha_2^5\beta_2.$$

The corresponding relations $(5'_1), \cdots, (5'_6)$ for the second side of the horn angle are obtained from these by replacing $a_2, \cdots, a_7$, $A_2, \cdots, A_7$ by $a'_2, \cdots, a'_7$, $A'_2, \cdots, A'_7$ respectively in the above equations.

In order to find the invariants, it is necessary to eliminate the coefficients α_i, β_i of the transformation. Since $\Delta a_2 = \Delta a_3 = 0$,[2] the coefficients $\alpha_4, \alpha_5, \alpha_6$, $\beta_4, \beta_5, \beta_6, \beta_7$ can be eliminated by subtracting equations $(5_3), \cdots, (5_6)$ from the corresponding primed expressions. The resulting equations are

$$(6)\quad \Delta A_4 = \Delta a_4$$

$$(7)\quad \Delta A_5 = \Delta a_5 - 2\alpha_2\Delta a_4$$

$$(8)\quad \Delta A_6 = \Delta a_6 - 3\alpha_2\Delta a_5 + 10\beta_2 a_2\Delta a_4 - (\alpha_3 - 4\alpha_2^2 - 4\beta_2^2)\Delta a_4$$

---

[1] The complete expressions for A_6 and A_7 in terms of $a_2, \cdots, a_7$ and the coefficients of the transformation which involve, respectively, forty-five and eighty-three terms may be obtained from the author.

[2] Throughout this section of the paper the Δ notation will be used to indicate the difference between corresponding expressions connected with the two sides of a horn angle; e. g. $\Delta a_2 = a'_2 - a_2$, etc.

$$
\begin{aligned}
(9)\quad \Delta A_7 = \Delta a_7 &- 4\alpha_2\Delta a_6 + 12\beta_2 a_2\Delta a_5 - (2\alpha_3 - 8\alpha_2^2 - 4\beta_2^2)\Delta a_5 \\
&+ 12\beta_2 a_3\Delta a_4 + 8\alpha_2 a_2^2\Delta a_4 + (12\beta_3 - 48\alpha_2\beta_2)a_2\Delta a_4 \\
&+ (4\alpha_2\alpha_3 + 12\beta_2\beta_3 - 8\alpha_2^3 - 32\alpha_2\beta_2^2)\Delta a_4.
\end{aligned}
$$

Next β_2, β_3 are eliminated by substituting from equations (5_1) and (5_2) and α_2, α_3 by substituting from (7) and (8) into (9). Using (6) and rearranging, the expression

$$
\begin{aligned}
(10)\quad & \overline{\Delta A}_5^3 - 2\Delta A_6\Delta A_5\Delta A_4 + (\Delta A_7 + 4A_2^2\Delta A_5)\overline{\Delta A}_4^2 - 12A_2A_3\overline{\Delta A}_4^3 \\
&= \overline{\Delta a}_5^3 - 2\Delta a_5\Delta a_5\Delta a_4 + (\Delta a_7 + 4a_2^2\Delta a_5)\overline{\Delta a}_4^2 - 12a_2a_3\overline{\Delta a}_4^3.
\end{aligned}
$$

is obtained.

Equations (6) and (10) exhibit two quantities which are invariant under the special transformation (2″). Since the effect of the magnification (2‴) on the curve (1) is to divide the coefficient a_n by α_1^{n-1}, equations (6) and (10) still hold under the general transformation (2′) provided the factors $1/\alpha_1^3$ and $1/\alpha_1^{12}$ are inserted in the right members of (6) and (10) respectively. ($\alpha_1 \neq 0$ for α_1^2 is the Jacobian of the transformation (2′) at the origin.) Consequently, the quantities

$$(6')\quad \Delta a_4$$

$$(10')\quad \overline{\Delta a}_5^3 - 2\Delta a_6\Delta a_5\Delta a_4 + (\Delta a_7 + 4a_2^2\Delta a_5)\overline{\Delta a}_4^2 - 12a_2a_3\overline{\Delta a}_4^3$$

are relative invariants of the horn angle (1, 1′) under the conformal transformation (2′) of index 1 and 4 respectively, and

THEOREM I. *The quantity*

$$(11)\quad \frac{\overline{\Delta a}_4^4}{\overline{\Delta a}_5^3 - 2\Delta a_6\Delta a_5\Delta a_4 + (\Delta a_7 + 4a_2^2\Delta a_5)\overline{\Delta a}_4^2 - 12a_2a_3\overline{\Delta a}_4^3}$$

and the sign of Δa_4 *are the invariants of a horn angle of the third order under the conformal transformation* (2). *Conversely, if two such horn angles agree in the value of* (11) *and the sign of* Δa_4, *they can be made to coincide by a conformal transformation* (*at least to any finite order of contact*).

For the proof of the converse, see Kasner's Cambridge paper (1a). The quantity (11) will be called the *conformal measure of a third order horn angle* and will be denoted by M_{12}^4. It is a differential invariant of the seventh order.

The coefficients of the curve (1) are expressed in terms of the values, at the origin, of the curvature, γ_1, and its derivatives with respect to length of arc, $\gamma_1', \cdots, \gamma_1^{v}$ by the standard formulae

$$(12_1)\quad 2a_2 = \gamma_1$$

$$(12_2)\quad 6a_3 = \gamma_1'$$

$$(12_3)\quad 24a_4 = \gamma_1'' + 3\gamma_1^3$$

$(12_4)\quad 120a_5 = \gamma_1''' + 19\gamma_1^2\gamma_1'$

$(12_5)\quad 720a_6 = \gamma_1^{\mathrm{iv}} + 34\gamma_1^2\gamma_1'' + 48\gamma_1\gamma_1'^2 + 99\gamma_1^5$

$(12_6)\quad 5040a_7 = \gamma_1^{\mathrm{v}} + 55\gamma_1^2\gamma_1''' + 199\gamma_1\gamma_1'\gamma_1'' + 48\gamma_1'^3 + 1135\gamma_1^4\gamma_1'$

and the coefficients of the curve (1′) by similar formulae in $\gamma_2, \gamma_2', \cdots, \gamma_2^{\mathrm{v}}$. Consequently, the invariant measure (11) can be written as follows:

$$(13)\quad M_{12}^4 =$$

$$\frac{875}{4} \cdot \frac{\overline{\Delta\gamma''}^4}{42\overline{\Delta\gamma'''}^3 - 70\Delta\gamma''\Delta\gamma'''\Delta\gamma^{\mathrm{iv}} + (25\Delta\gamma^{\mathrm{v}} + 45\gamma^2\Delta\gamma''')\overline{\Delta\gamma''}^2 - 275\gamma\gamma'\overline{\Delta\gamma''}^3}$$

This might be regarded as the intrinsic form for the conformal measure of a third order horn angle since it is invariant under any conformal transformation, but it is an unwieldly expression and it will be convenient to use a simpler form.

Let the equations

$(14_1)\quad w = a_4$

$(14_2)\quad x = a_5$

$(14_3)\quad y = a_6$

$(14_4)\quad z = a_7 + 4a_2^2a_5 - 12a_2a_3a_4$

define a correspondence between the number quadruples (w, x, y, z) and the coefficients $a_4, \cdots, a_7$ of the curve (1). Then the conformal measure of the third order horn angle can be written in the form

$$(15)\quad M_{12}^4 = \frac{\overline{\Delta w}^4}{\overline{\Delta x}^3 - 2\Delta x\Delta y\Delta w + \Delta z\overline{\Delta w}^2}$$

where w, x, y, z and w', x', y', z' can be expressed in terms of the metric invariants $\gamma_1, \gamma_1', \cdots, \gamma_1^{\mathrm{v}}$ and $\gamma_2, \gamma_2', \cdots, \gamma_2^{\mathrm{v}}$ by combining the equations (12) and (14).

The measure of a fourth order horn angle. The derivation of the conformal invariant of a horn angle of the fourth order is similar to the procedure outlined above for the third order case, presenting no new difficulties other than the increasing mechanical difficulty in handling the computation. The work can be somewhat simplified, however, by omitting the determination of the set of equations analogous to the set (5) and, instead, forming immediately a set of difference equations obtained by equating to zero the coefficient of x^n in the identity (4) and in the identity (4′), then subtracting the first equation thus obtained from the second, and finally solving the resulting difference equation for ΔA_n $(n = 5, \cdots, 9)$.

> *Note*: These three operations, if done in this order, can be performed almost entirely by inspection of the two identities (4) and (4′) because the difference equations contain considerably fewer terms than the equations (5) due to the fact that in a fourth order horn angle $\Delta a_2 = \Delta a_3 = \Delta a_4 = 0$. However, the use of this method does entail the loss of the explicit statement of the effect of the transformation (2″) on the coefficients of the curve (1) which is needed in obtaining the group of induced transformations of a fourth order horn set and the expressions for β_8 and β_9 which would be needed in the elimination process in the derivation of the measures of horn angles of order greater than eight.

From these difference equations the coefficients α_i, β_i of the transformation can be eliminated as before. As a result of this elimination the quantities

(16) Δa_5

(17) $\overline{\Delta a}_6{}^4 - 3\Delta a_7\overline{\Delta a}_6{}^2\Delta a_5 + (\overline{\Delta a}_7{}^2 + 2\Delta a_8\Delta a_6 + 4a_2{}^2\overline{\Delta a}_6{}^2)\overline{\Delta a}_5{}^2$
$- (\Delta a_9 + 4a_2{}^2\Delta a_7 + 12a_2a_3\Delta a_6)\overline{\Delta a}_5{}^3 + (9a_3{}^2 + 16a_2a_4)\overline{\Delta a}_5{}^4$

are obtained as relative invariants of the fourth order horn angle formed by the curves (1, 1′) under the conformal transformation (2′). They are of index 1 and 5 respectively. Consequently,

THEOREM II. *The quantity*

(18) $$\frac{\overline{\Delta a}_5{}^5}{\overline{\Delta a}_6{}^4 - \cdots + (9a_3{}^2 + 16a_2a_4)\overline{\Delta a}_5{}^4}$$

and the sign of Δa_5 *are the invariants of a fourth order horn angle under the conformal transformation* (2). *Conversely, if two such horn angles agree in the value of* (18) *and in the sign of* Δa_5, *they can be made to coincide by a conformal transformations* (*at least to any finite order of contact*).

The quantity (18) will be called the *conformal measure of a fourth order horn angle* and denoted by $M_{12}{}^5$. It is a differential invariant of the ninth order.

Were it desirable, the quantities (16), (17), (18) could now be expressed in terms of the curvatures and arc-length derivatives of curvature of the curves of the horn angle as was done for the horn angle of third order contact in expression (13), but as this form is not needed, it will be omitted.

The measure, $M_{12}{}^5$, can be expressed in its simplified form analogous to (15) by means of the correspondence

(19_1) $v = a_5$.

(19_2) $w = a_6$

(19_3) $x = a_7$

$(19_4)\quad y = a_8 + 2a_2^2a_6$

$(19_5)\quad z = a_9 + 4a_2^2a_7 + 12a_2a_3a_6 - (9a_3^2 + 16a_2a_4)a_5.$

Then

$$(20)\qquad M_{12}^5 = \frac{\overline{\Delta v}^5}{\overline{\Delta w}^4 - 3\Delta v\overline{\Delta w}^2\Delta x + (\overline{\Delta x}^2 + 2\Delta w\Delta y)\overline{\Delta v}^2 - \overline{\Delta v}^3\Delta z}$$

is the measure of the fourth order horn angle in terms of certain simple functions of the coefficients of the curves of the horn angle and consequently in terms of simple functions of the curvatures and arc-length derivatives of curvature of these curves at the origin. Thus M_{12}^5 is a metric invariant also.

The measure of a fifth order horn angle. Analogous to the results in the two preceding sections, the quantities

$(21)\quad \Delta a_6$

$$(22)\quad \begin{aligned}&\overline{\Delta a_7}^5 - 4\overline{\Delta a_7}^3\Delta a_8\Delta a_6 + (3\overline{\Delta a_7}^2\Delta a_9 + 3\Delta a_7\overline{\Delta a_8}^2 + 4a_2^2\overline{\Delta a_7}^3)\overline{\Delta a_6}^2\\ &\quad - (2\Delta a_7\Delta a_{10} + 2\Delta a_8\Delta a_9 + 8a_2^2\Delta a_7\Delta a_8 + 12a_2a_3\overline{\Delta a_7}^2)\overline{\Delta a_6}^3\\ &\quad + [\Delta a_{11} + 4a_2^2\Delta a_9 + 12a_2a_3\Delta a_8 + (9a_3^2 + 16a_2a_4)\Delta a_7]\overline{\Delta a_6}^4\\ &\quad - (20a_2a_5 + 24a_3a_4)\overline{\Delta a_6}^5\end{aligned}$$

are relative invariants of the horn angle of fifth order contact formed by the curves $(1, 1')$ under the conformal transformation $(2')$. They are of index 1 and 6 respectively. Consequently,

THEOREM III. *The quantity*

$$(23)\qquad \frac{\overline{\Delta a_6}^6}{\overline{\Delta a_7}^5 - \cdot\cdot\cdot - \overline{\Delta a_6}^5(20a_2a_5 + 24a_3a_4)}$$

and the sign of Δa_6 *are the invariants of a fifth order horn angle under the conformal transformation* (2). *Conversely, if two such horn angles agree in the value of* (23) *and the sign of* Δa_6, *they can be made to coincide* (*at least to any finite order of contact*) *by a conformal transformation.*

The quantity (23) will be called the *conformal measure of the fifth order horn angle* and denoted by M_{12}^6. It is a differential invariant of the eleventh order.

Again the expression of these invariants in terms of the intrinsic metric invariants will be omitted. The measure, M_{12}^6, can be expressed in the simplified form

$$(24)\quad M_{12}^6 = \frac{\overline{\Delta u}^6}{\overline{\Delta v}^5 - 4\Delta u\overline{\Delta v}^3\Delta w + 3\overline{\Delta u}^2(\overline{\Delta v}^2\Delta x + \Delta v\overline{\Delta w}^2) - 2\overline{\Delta u}^3(\Delta v\Delta y + \Delta w\Delta x) + \overline{\Delta u}^4\Delta z}$$

by means of the correspondence

$$(25_1) \quad u = a_6$$

$$(25_2) \quad v = a_7$$

$$(25_3) \quad w = a_8$$

$$(25_4) \quad x = a_9 + \tfrac{4}{3}a_2^2 a_7$$

$$(25_5) \quad y = a_{10} + \tfrac{8}{3}a_2^2 a_8 + 6a_2 a_3 a_7$$

$$(25_6) \quad z = a_{11} + 4a_2^2 a_9 + 12a_2 a_3 a_8 + (9a_3^2 + 16a_2 a_4)a_7 - (20a_2 a_5 + 24a_3 a_4)a_6.$$

By combining equations (12_i) and (25_i), the functions u, v, w, x, y, z can be expressed in terms of the metric invariants, curvature and its arc-length derivatives, indicating that M_{12}^6 is a metric invariant also.

Since the existence of one and only one such conformal invariant has been established by Kasner for horn angles of any finite order of contact, and since the methods employed for each case are such simple extensions of the methods of the previous case, it seems probable that a general form for the invariant for the n-th order case exists. So far, however, it has not been possible to find a symbolism in which to express this conformal invariant of differential order $2n+1$ which is sufficiently compact to permit a proof of its existence to be carried through by producing the measure.

2. Conformal geometry of third order Horn angles. In an earlier section of this paper it has been shown that two given horn angles of n-th order contact can be transformed into one another by a direct conformal transformation if and only if the two horn angles agree in measure (and in the sign of Δa_{n+1}) and that the measure of a given horn angle is completely determined by the values of the first $2n+1$ derivatives of the sides at the vertex. Consequently, the study of the conformal geometry of horn angles may be reduced to the study of the conformal geometry of the differential elements of the $(2n+1)$-st order passing through a given point in a given direction. In the remainder of this section the term *curve* will mean the differential element defined by a point on a given curve and the derivatives of this curve at that point, and the term n-th *order horn set* will mean the totality of all curves [$(2n+1)$-st order elements] through a given point and each having at this point n-th order contact with every curve of the set. Again the n-th order horn set for which the common point and common direction are the origin and horizontal axis may be considered without loss of generality.

Associated with each third order horn set (γ, γ') (i. e., the horn set of curves through the origin having there slope zero, curvature γ, and arc-length derivative of curvature γ') is a four dimensional space, $K_4(\gamma, \gamma')$, which consists of the totality of number quadruples (w, x, y, z) where w, x, y, z are the simple functions of the coefficients of the curves of the horn set (γ, γ') given by the equations $(14_i; 1)$. The conformal transformation $(2; 1)$ induces a transformation between the curves of the third order horn set (γ, γ') and those of another third order horn set (Γ, Γ') and, consequently, between the elements (w, x, y, z) of the space $K_4(\gamma, \gamma')$ and the elements (W, X, Y, Z) of the space $K_4(\Gamma, \Gamma')$. The geometry of the K_4-space associated with a given horn set (i. e., the invariants of this space under the transformation thus induced) will be the subject of this section of the paper.

Definitions of the elements of a k_4-space. A *point* of the K_4-space is a number quadruple (w, x, y, z) as defined by the equations $(14_i; 1)$.

A *series* consists of any single infinitude of points of the space.

A *line* (linear series) consists of the single infinitude of points of the space which satisfy three linear equations of the form

$$(1) \qquad x = p_1 w + r_1, \qquad y = p_2 w + r_2, \qquad z = p_3 w + r_3$$

where p_i, r_i $(i = 1, 2, 3)$ are constants. When two lines are considered, double subscripts will be used, the second subscript indicating to which line the coefficient belongs.

A *plane* (flat congruence) consists of the double infinitude of points of the space which satisfy two linear equations of the form

$$(2) \qquad y = a_1 w + b_1 x + c_1, \qquad z = a_2 w + b_2 x + c_2$$

where a_i, b_i, c_i $(i = 1, 2)$ are constants. Second subscripts will be used to distinguish between the elements when more than one is considered.

A *flat* consists of the triple infinitude of points which satisfy the single linear equation

$$(3) \qquad z = dw + fx + gy + k$$

where d, f, g, k are constants, subscripts indicating the flat to which they belong when more than one flat is considered.

Derivation of the fundamental group, G_7. The transformation induced by the conformal transformation $(2; 1)$ and carrying the point (w, x, y, z) of the space $K_4(\gamma, \gamma')$ into the point (W, X, Y, Z) of the space $K_4(\Gamma, \Gamma')$ is given by the equations

$$(4) \qquad \begin{aligned} W &= m_1^3 w + h_1 \\ X &= m_1^4 x + 2m_1^2 m_2 w + h_2 \\ Y &= m_1^5 y + 3m_1^3 m_2 x + m_1 m_3 w + h_3 \\ Z &= m_1^6 z + 4m_1^4 m_2 y + 2m_1^2 m_3 x + 4m_2(m_3 - 2m_2^2) w + h_4 \end{aligned}$$

where $m_1 \neq 0$, m_2, m_3, h_1, h_2, h_3, h_4 are constants. The verification of this set of equations lies in expressing the functions W, X, Y, Z, defined as functions of the coefficients of the transformed curve $(3;\mathbf{1})$, as functions of the coefficients of the curve $(1;\mathbf{1})$ and then expressing this last set of coefficients as functions of w, x, y, z. The effect of the transformation $(2'';\mathbf{1})$ on the coefficients of the curve $(1;\mathbf{1})$ is given by the equations $(5;\mathbf{1})$ from which β_2 and β_3 can be eliminated as before by the use of $(5_1;\mathbf{1})$ and $(5_2;\mathbf{1})$ leaving

$$
(5)\qquad
\begin{aligned}
A_4 &= a_4 + f(a_2, a_3, \alpha_i, \beta_i)\\
A_5 &= a_5 - 2\alpha_2 a_4 + g(a_2, a_3, \alpha_i, \beta_i)\\
A_6 &= a_6 - 3\alpha_2 a_5 + (-\alpha_3 + 4\alpha_2^2 + 4A_2^2 + 2A_2 a_2 - 6a_2^2)a_4\\
&\quad + h(a_2, a_3, \alpha_i, \beta_i)\\
A_7 &= a_7 - 4\alpha_2 a_6 + 2(-\alpha_3 + 4\alpha_2^2 + 2A_2^2 + 2A_2 a_2 - 4a_2^2)a_5\\
&\quad + [-4\alpha_2(-\alpha_3 + 2\alpha_2^2 + 2A_2^2 + 2A_2 a_2 - 6a_2^2)\\
&\quad + 12(A_2 A_3 - a_2 a_3)]a_4 + k(a_2, a_3, \alpha_i, \beta_i)
\end{aligned}
$$

where f, g, h, k are functions of a_2, a_3, α_i, β_i $(i = 2, \cdots, 7)$ and do not involve $a_4, \cdots, a_7$. The effect of using the modified transformation $(2'';\mathbf{1})$ in place of the transformation $(2;\mathbf{1})$ is eliminated by dividing each of the coefficients a_n of the right members of equations (5) by α_1^{n-1} as before and the resulting equations,

$$
(6)\qquad
\begin{aligned}
A_4 &= (1/\alpha_1^3)a_4 + f'\\
A_5 &= (1/\alpha_1^4)a_5 - 2(\alpha_2/\alpha_1^3)a_4 + g'\\
A_6 &= (1/\alpha_1^5)a_6 - 3(\alpha_2/\alpha_1^4)a_5 + (1/\alpha_1^5)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)a_4 + h'\\
A_7 &= (1/\alpha_1^6)a_7 - 4(\alpha_2/\alpha_1^5)a_6 + 2(1/\alpha_1^6)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 4a_2^2)a_5\\
&\quad + [-4(\alpha_2/\alpha_1^5)(-\alpha_1^2\alpha_3 + 2\alpha_1^2\alpha_2^2 + 2\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)\\
&\quad + 12(1/\alpha_1^6)(\alpha_1^3 A_2 A_3 - a_2 a_3)]a_4 + k',
\end{aligned}
$$

express the effect of a more general transformation like $(2;\mathbf{1})$ on the coefficients of the curve $(1;\mathbf{1})$. From these equations (6), the equations $(14_i;\mathbf{1})$ and the corresponding equations expressing W, X, Y, Z as functions of $A_4, \cdots, A_7$, are obtained the equations

$$
(7)\qquad
\begin{aligned}
W &= (1/\alpha_1^3)w + f'\\
X &= (1/\alpha_1^4)x - 2(\alpha_2/\alpha_1^3)w + g'\\
Y &= (1/\alpha_1^5)y - 3(\alpha_2/\alpha_1^4)x + (1/\alpha_1^5)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)w + h'\\
Z &= (1/\alpha_1^6)z - 4(\alpha_2/\alpha_1^5)y + 2(1/\alpha_1^6)(-\alpha_1^2\alpha_3 + 4\alpha_1^2\alpha_2^2\\
&\quad + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)x\\
&\quad - 4(\alpha_2/\alpha_1^5)(-\alpha_1^2\alpha_3 + 2\alpha_1^2\alpha_2^2 + 4\alpha_1^2 A_2^2 + 2\alpha_1 A_2 a_2 - 6a_2^2)w\\
&\quad + k' + 4A_2^2 g' - 12A_2 A_3 f'
\end{aligned}
$$

which become the equations of the induced transformation (4) when the substitutions

$$
(8)\quad \begin{aligned} &m_1 = 1/\alpha_1,\ m_2 = -\alpha_2/\alpha_1,\ m_3 = (1/\alpha_1{}^4)(-\alpha_1{}^2\alpha_3 + 4\alpha_1{}^2\alpha_2{}^2 + 4\alpha_1{}^2A_2{}^2 + 2\alpha_1A_2a_2 - \\ &h_1 = f',\ h_2 = g',\ h_3 = h',\ h_4 = k' + 4A_2{}^2g' - 12A_2A_3f' \end{aligned}
$$

are made. Consequently,

THEOREM IV. *The group of conformal transformations of the plane induces a seven parameter group, G_7, of transformations between the points of the two K_4-spaces associated with the given and transformed horn sets.*

Furthermore, Theorem I may be restated as follows

THEOREM V. *An ordered pair of points of a space $K_4(\gamma, \gamma')$ has the unique absolute conformal invariant, the measure or distance between the points, $M_{12}{}^4$* (15; **1**).

Thus the differential of arc in this space is

$$
(9)\qquad ds = \frac{dw^4}{dw^2dz - 2dwdxdy + dx^3}\,.
$$

The general line. A *null series* shall mean any series of the K_4-space for which the measure of any pair of distinct points of the series is either zero or infinity. Thus from (9) it is seen that the null series are solutions of the differential equation (prime indicating differentiation with respect to w)

$$
(10)\qquad z' - 2x'y' + x'^3 = 0
$$

and the special lines for which

$$
(11)\quad w = \text{constant}.
$$

The line (1) is said to be a *general line* if

$$
(12)\quad p_3 - 2p_1p_2 + p_1{}^3 \neq 0.
$$

Geometrically, a general line is a line which is not tangent to a null series (i. e., the line and series do not have two consecutive points in common).

Any general line (1) of a space $K_4(\gamma, \gamma')$ is converted into a general line

$$
(1')\qquad X = P_1W + R_1,\qquad Y = P_2W + R_2,\qquad Z = P_3W + R_3
$$

of a space $K_4(\Gamma, \Gamma')$ under transformations of the fundamental group. This statement is proved by obtaining the effect of the transformation (4) on the coefficients of the line (1) which is given by the equations

$$
(13)\quad
\begin{aligned}
P_1 &= m_1p_1 + 2m_2/m_1 \\
P_2 &= m_1^2p_2 + 3m_2p_1 + m_3/m_1^2 \\
P_3 &= m_1^3p_3 + 4m_1m_2p_2 + 2m_5p_1/m_1 + (4m_2/m_1^3)(m_3 - 2m_2^2) \\
R_1 &= m_1^4r_1 - m_1h_1p_1 + h_2 - 2m_2h_1/m_1 \\
R_2 &= m_1^5r_2 + 3m_1^3m_2r_1 - m_1^2h_1p_2 - 3m_2h_1p_1 + h_3 - m_3h_1/m_1^2 \\
R_3 &= m_1^6r_3 + 4m_1^4m_2r_2 + 2m_1^2m_3r_1 - m_1^3h_1p_3 - 4m_1m_2h_1p_2 \\
&\qquad - 2m_3h_1p_1/m_1 + h_4 - (4m_2/m_1^3)(m_3 - 2m_2^2)h_1.
\end{aligned}
$$

Then it can be seen that

$$
(14)\quad P_3 - 2P_1P_2 + P_1^3 = m_1^3(p_3 - 2p_1p_2 + p_1^3),
$$

and consequently the line (1′) is a general line if the line (1) is.

Given two general lines, l_1 and l_2, it is possible to obtain from the equations (13) the following expressions from which the coefficients m_2, m_3, $h_1, \cdots, h_4$ of the transformation (4) have been eliminated[3]

$$
(15)\quad
\begin{aligned}
&\Delta P_1 = m_1\Delta p_1 \\
&\Delta P_2\Delta R_1 - \Delta P_1\Delta R_2 = m_1^6(\Delta p_2\Delta r_1 - \Delta p_1\Delta r_2) \\
&3\Delta P_1(\Delta P_3\Delta R_1 - \Delta P_1\Delta R_3) - 4\Delta P_2(\Delta P_2\Delta R_1 - \Delta P_1\Delta R_2) \\
&\qquad = m_1^8[3\Delta p_1(\Delta p_3\Delta r_1 - \Delta p_1\Delta r_3) - 4\Delta p_2(\Delta p_2\Delta r_1 - \Delta p_1\Delta r_2)].
\end{aligned}
$$

The left sides of these equations (15) and of the two expressions (14) stated once for each line are then relative invariants, consequently,

THEOREM VI. *Two*[4] *general lines possess the four independent absolute invariants*

$$
(16_j)\qquad \frac{\overline{\Delta p_1}^3}{p_{3j} - 2p_{2j}p_{1j} + p_{1j}^3} \qquad (j = 1, 2)
$$

$$
(16_3)\qquad \frac{\Delta p_2\Delta r_1 - \Delta p_1\Delta r_2}{(p_{31} - 2p_{11}p_{21} + p_{11}^3)^2}
$$

$$
(16_4)\qquad \frac{3\Delta p_1(\Delta p_3\Delta r_1 - \Delta p_1\Delta r_3) - 4\Delta p_2(\Delta p_2\Delta r_1 - \Delta p_1\Delta r_2)}{(p_{31} - 2p_{11}p_{21}p_{11}^3)^{8/3}}.
$$

Note: Two intersecting general lines possess only the two independent invariants (16_j).

The *direct dihorn angle*, α_{ij}, between the lines l_i and l_j is defined to be the quantity

---

[3] Throughout this section of the paper the second set of subscripts will be used to indicate to which of two elements the coefficient belongs and the Δ notation to indicate the difference between corresponding coefficients for the two elements.

[4] A single line possesses no absolute invariants but has the expression (12) as a relative invariant.

$$(17)\qquad \frac{(p_{1j}-p_{1i})^3}{p_{3j}-2p_{1j}p_{2j}+p_{1j}^3};$$

and the *reverse dihorn angle*, $\varkappa_{ji}$, for this ordered pair of lines, is defined by formula (17) with the i and the j interchanged. From the definition of these two angles and definition (15; I) the trihornometry (2b) of the third order can be developed.

Similar to the proof of Theorem VI is the proof of

THEOREM VII. *A point* (w_0, x_0, y_0, z_0) *and the general line* (1), *not containing the general point, have the three independent absolute invariants*

$$(18_1)\quad I_{Pl_1}=\frac{x_0-p_1w_0-r_1}{(p_3-2p_1p_2+p_1^3)^{4/3}}$$

$$(18_2)\quad I_{Pl_2}=\frac{2(y_0-p_2w_0-r_2)-3p_1(x_0-p_1w_0-r_1)}{(p_3-2p_1p_2+p_1^3)^{5/3}}$$

$$(18_3)\quad I_{Pl_3}=\frac{(z_0-p_3w_0-r_3)-2p_1(y_0-p_2w_0-r_2)+(3p_1^2-2p_2)(x_0-p_1w_0-}{(p_3-2p_1p_2+p_1^3)^2}$$

The extremum distance between the point and the line is a function of these invariants

$$(18_4)\qquad \text{Extremum } D=\frac{W^4}{W^3-I_{Pl_3}W^2-I_{Pl_1}I_{Pl_2}W-I_{Pl_1}^3}$$

where W is that function of $I_{Pl_1}, I_{Pl_2}, I_{Pl_3}$ obtained by solving the cubic equation

$$(19)\qquad W^3-2I_{Pl_3}W^2-3I_{Pl_1}I_{Pl_2}W-4I_{Pl_1}^3=0.$$

The general plane. The plane (2) is a *general plane* if

$$(20)\quad [27a_2-18b_1(2a_1-b_2)-16b_1^3]^2-4[3(2a_1-b_2)+4b_1^2]^3\neq 0.$$

Geometrically a general plane is a plane which is not the osculating plane of any null series (i. e., the plane and series do not have three consecutive points in common).

From equations (4) it is possible to show that under the fundamental group, G_7, any general plane (2) of a space $K_4(\gamma,\gamma')$ is converted into a general plane of a space $K_4(\Gamma,\Gamma')$

$$(2')\qquad Y=A_1W+B_1X+C_1,\qquad Z=A_2W+B_2X+C_2.$$

The transformation between the planes of the two K_4-spaces is given by the equations

$$\begin{aligned}
A_1 &= m_1^2 a_1 - 2m_2 b_1 + (m_3 - 6m_2^2)/m_1^2 \\
B_1 &= m_1 b_1 + 3m_2/m_1 \\
C_1 &= m_1^5 c_1 + (2m_2 h_1 - m_1 h_2) b_1 - m_1^2 h_1 a_1 + h_3 - 3m_2 h_2/m_1 - (m_3 - 6m_2^2) h_1/m_1^2 \\
A_2 &= m_1^3 a_2 - 2m_1 m_2 b_2 + 4m_1 m_2 a_1 - 8m_2^2 b_1/m_1 - 8m_2^3/m_1^3 \\
B_2 &= m_1^2 b_2 + 4m_2 b_1 + 2m_3/m_1^2 \\
C_2 &= m_1^6 c_2 + m_1(2m_2 h_1 - m_1 h_2) b_2 - m_1^3 h_1 a_2 + 4m_1^4 m_2 c_1 \\
&\quad + (4m_2/m_1)(2m_2 h_1 - m_1 h_2) b_1 - 4m_1 m_2 h_1 a_1 + h_4 - 2m_3 h_2/m_1^2 + 8m_2^3 h_1/m_1^3
\end{aligned} \tag{21}$$

From equations (21) it is possible to obtain the following invariants:

THEOREM VIII. *A general plane* (2) *possesses the absolute invariant*

$$\frac{[27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2}{4[3(2a_1 - b_2) + 4b_1^2]^3} \neq 1. \tag{22}$$

THEOREM IX. *Two general planes possess, in addition to the invariant* (22) *stated for each plane, the four independent absolute invariants*

$$\frac{[3(2a_{12} - b_{22}) + 4b_{12}^2]^3}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3} \tag{23_1}$$

$$\frac{\overline{\Delta b}_1^6}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3} \tag{23_2}$$

$$\frac{(2\Delta a_1 + \Delta b_2)^3}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3} \tag{23_3}$$

$$\frac{(\Delta a_2 \Delta b_1 - \Delta a_1 \Delta b_2)^{3/2}}{[27a_{21} - 18b_{11}(2a_{11} - b_{21}) - 16b_{11}^3]^2 - 4[3(2a_{11} - b_{21}) + 4b_{11}^2]^3}. \tag{23_4}$$

Note: In general two planes have only a single point in common, but if they have a line in common, the invariant (23_4) vanishes.

THEOREM X. *A point* (w_0, x_0, y_0, z_0) *and a general plane* (2) *not containing the given point possess, in addition to the invariant* (22) *of the plane, the two independent absolute invariants*

$$\frac{(y_0 - a_1 w_0 - b_1 x_0 - c_1)^{6/5}}{[27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2 - 4[3(2a_1 - b_2) + 4b_1^2]^3}. \tag{24_1}$$

$$\frac{3(z_0 - a_2 w_0 - b_2 x_0 - c_2) - 4b_1(y_0 - a_1 w_0 - b_1 x_0 - c_1)}{[27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3]^2 - 4[3(2a_1 - b_2) + 4b_1^2]^3}. \tag{24_2}$$

THEOREM XI. *A general line* (1) *and a general plane* (2) *possess, in addition to the invariant* (22) *of the plane, the four independent absolute invariants*

$$\frac{27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3}{p_3 - 2p_1 p_2 + p_1^3} \tag{25_1}$$

$$(25_2)\qquad \frac{(2b_1-3p_1)^3}{p_3-2p_1p_2+p_1^3}$$

$$(25_3)\qquad \frac{(a_1+p_1b_1-p_2)^3}{(p_3-2p_1p_2+p_1^3)^2}$$

$$(25_4)\qquad \frac{[(a_2+p_1b_2-p_3)(c_1+r_1b_1-r_2)-(a_1+p_1b_1-p_2)(c_2+r_1b_2-r_3)]^3}{(p_3-2p_1p_2+p_1^3)^8}$$

Note: If the line (1) lies in the plane (2), the invariants (25_3) and (25_4) vanish.

The general flat. The flat (3) is said to be a *general flat* if

$$(26)\qquad 8d+4fg+g^3\neq 0.$$

Such a flat is not the osculating flat of a null series (i. e., the flat and the series do not have four consecutive points in common) at a point for which $x'=g/2$.

From equations (4) it is possible to show that under the fundamental group, G_7, any general flat (3) of a space $K_4(\gamma,\gamma')$ is converted into a general flat of a space $K_4(\Gamma,\Gamma')$

$$(3')\qquad Z=DW+FX+GY+K.$$

The transformation between the flats of the two K-spaces is given by the equations

$$(27)\qquad \begin{aligned} D&=m_1^3d-2m_1m_2f-(m_3-6m_2^2)g/m_1-(4m_2/m_1^3)(m_3-4m_2^2)\\ F&=m_1^2f-3m_2g+(2/m_1^2)(m_3-6m_2^2)\\ G&=m_1g+4m_2/m_1\\ K&=m_1^6k-m_1^3h_1d+m_1(2m_2h_1-m_1h_2)f\\ &\quad+[(m_3-6m_2^2)h_1/m_1+3m_2h_2-m_1h_3]g+h_4-4m_2h_3/m_1\\ &\quad-(2/m_1^2)(m_3-6m_2^2)h_2+(4m_2/m_1^3)(m_3-4m_2^2)h_1. \end{aligned}$$

From equations (27) it is possible to obtain the invariants listed in the following theorems.

THEOREM XII. *Two*[5] *general flats possess the two independent absolute invariants*

$$(28_j)\qquad \frac{\overline{\Delta g^3}}{8d_j+4f_jg_j+g_j^3}\qquad (j=1,2).$$

THEOREM XIII. *A point* (w_0,x_0,y_0,z_0) *and a general flat* (3) *not containing the given point possess the absolute invariant*

---

[5] A single flat possesses no absolute invariant but has the expression (26) as a relative invariant.

$$(29)\qquad \frac{256(z_0 - dw_0 - fx_0 - gy_0 - k)}{(8d + 4fg + g^3)^2}$$

which is the extremum distance between the point and the flat.

THEOREM XIV. *A general line* (1) *and a general flat* (3) *possess the three independent absolute invariants*

$$(30_1)\qquad \frac{(2p_1 - g)^3}{8d + 4fg + g^3}$$

$$(30_2)\qquad \frac{(2p_1 - g)^3}{p_3 - 2p_1p_2 + p_1^3}$$

$$(30_3)\qquad \frac{p_3 - d - fp_1 - gp_2}{8d + 4fg + g^3}.$$

Note: In general a line and a flat have a single point in common, in which case

$$(31)\qquad p_3 - d - fp_1 - gp_2 \neq 0$$

and no extremum distance can be computed. If in particular the line and the flat do not intersect (i. e., the line is "parallel" to the flat), the expression (31) vanishes but

$$(32)\qquad r_3 - k - fr_1 - gr_2 \neq 0$$

and the extremum distance is given by the expression

$$(33)\qquad \text{Extremum } D = \frac{256(r_3 - k - fr_1 - gr_2)}{(8d + 4fg + g^3)^2}$$

which in this case *only* is an absolute invariant. If on the other hand the line lies entirely within the flat, both (31) and (32) vanish and the expression (33) gives the extremum distance, which is then zero.

THEOREM XV. *A general plane* (2) *and a general flat* (3) *possess, in addition to the invariant* (22) *of the plane, the three independent absolute invariants*

$$(34_1)\qquad \frac{27a_2 - 18b_1(2a_1 - b_2) - 16b_1^3}{8d + 4fg + g^3}$$

$$(34_2)\qquad \frac{(4b_1 - 3g)^3}{8d + 4fg + g^3}$$

$$(34_3)\qquad \frac{[3(2a_1 - f) + b_1(4b_1 - 3g)]^3}{(8d + 4fg + g^3)^2}.$$

Note: As in Theorem XIV, no extremum distance can be computed if the plane and flat intersect either in a line or in a point; but if they do not intersect

$$(35) \qquad a_2 - d - a_1 g = b_2 - f - b_1 g = 0$$

and the extremum distance is given by the expression

$$(36) \qquad \text{Extremum } D = \frac{256(c_2 - c_1 g - k)}{(8d + 4fg + g^3)^2}$$

which in this case *only* is an absolute invariant, or if the flat contains the plane both (35) and (36) vanish, the extremum distance being zero.

3. Equilong geometry of second and third order Horn angles.

The equilong measure of second and third order horn angles. In this section of this paper the equilong measure and the equilong geometry of second and third order horn angles will be developed in order that the rough duality existing between the conformal and the equilong theories may be studied further.

An equilong transformation of the plane is any contact transformation of the plane which converts straight lines into straight lines in such a way that the distance, δ, between the points of contact of any two curves having a common tangent remains invariant. Since the element is the straight line, it will be convenient to use Hessian line coördinates (u, v) to indicate that directed line having a normal through the origin which is v units long and makes an angle of u radians with the prime direction. In this coördinate system a curve is given as the envelope of a single infinitude of lines and therefore by an equation of the form

$$v = a_0 + a_1 u + a_2 u^2 + \cdots.$$

Two such curves will have contact of exactly n-th order at the line $v = a_0$ if the first $n + 1$ coefficients in their power series expansions are the same and the $(n + 2)$-nd are distinct. If these two curves have contact of first or higher order, the invariant distance, δ, has the trivial value zero and the pair of curves form a horn angle of the given order of contact. This horn angle will have a unique equilong invariant. The form of this invariant for second and third order horn angles is obtained below. As in the conformal case there is no loss in generality in limiting the study to the invariants of the horn angle

$$(1) \qquad \begin{aligned} v &= a_2 u^2 + a_3 u^3 + \cdots \\ v &= a'_2 u^2 + a'_3 u^3 + \cdots \quad (\text{with } a_2 = a'_2, \cdots, a_n = a'_n,\ a_{n+1} \neq a'_{n+1}). \end{aligned}$$

Since the group of equilong transformations of the plane can be shown to consist of the set of all line transformations of the form

$$U = \phi(u)$$

$$V = \phi_u(u) v + \psi(u)$$

where ϕ and ψ are arbitrary analytic functions (3), the equations

$$(2)\qquad \begin{aligned} U &= \alpha_1 u + \alpha_2 u^2 + \cdots \\ V &= (\alpha_1 + 2\alpha_2 u + \cdots)v + \beta_1 u + \beta_2 u^2 + \cdots \qquad (\alpha_1 \neq 0) \end{aligned}$$

represent an adequately general equilong transformation under which the horn angle (1) is equivalent to another horn angle of the same order of contact given by the equations

$$(3)\qquad \begin{aligned} V &= A_2 U^2 + A_3 U^3 + \cdots \\ V &= A'_2 U^2 + A'_3 U^3 + \cdots \ \text{(with } A_2 = A'_2, \cdots, A_n = A'_n, A_{n+1} \neq A'_{n+1}\text{)}. \end{aligned}$$

Again the two identities

$$(4)\qquad \begin{aligned} &(\alpha_1 + 2\alpha_2 u + \cdots)(a_2 u^2 + a_3 u^3 + \cdots) + \beta_1 u + \beta_2 u^2 + \cdots \\ &\equiv A_2(\alpha_1 u + \alpha_2 u^2 + \cdots)^2 + A_3(\alpha_1 u + \alpha_2 u^2 + \cdots)^3 + \cdots \end{aligned}$$

$$(4')\qquad \begin{aligned} &(\alpha_1 + 2\alpha_2 u + \cdots)(a'_2 u^2 + a'_3 u^3 + \cdots) + \beta_1 u + \beta_2 u^2 + \cdots \\ &\equiv A'_2(\alpha_1 u + \alpha_2 u^2 + \cdots)^2 + A'_3(\alpha_1 u + \alpha_2 u^2 + \cdots)^3 + \cdots \end{aligned}$$

are the conditions for the equivalence of the horn angles (1) and (3) under the transformation (2). As before, when for a horn angle of a given order the coefficients of like powers of u are equated to zero in each identity, difference equations formed from the corresponding pairs of equations, and the constants α_i and β_i of the transformation (2) eliminated, a pair of relative invariants are obtained for this horn angle. For the second order horn angle the relative invariants are

$$(5)\qquad \begin{aligned} &\Delta a_3 && \text{of index } 2 \\ &\overline{\Delta a_4}^{2} - \Delta a_3 \Delta a_5 && \text{of index } 6 \end{aligned}$$

and for the third order horn angle

$$(6)\qquad \begin{aligned} &\Delta a_4 && \text{of index } 3 \\ &\overline{\Delta a_5}^{3} - 2\Delta a_4 \Delta a_5 \Delta a_0 + \overline{\Delta a_4}^{2} \Delta a_7 && \text{of index } 12 \end{aligned}$$

Consequently,

THEOREM XVI. *A horn angle of second order has the unique absolute equilong invariant*

$$(5')\qquad \frac{\overline{\Delta a_3}^{3}}{\overline{\Delta a_4}^{2} - 2\Delta a_3 \Delta_5},$$

and a horn angle of third order has the unique absolute equilong invariant

$$(6')\qquad \frac{\overline{\Delta a_4}^{4}}{\overline{\Delta a_5}^{3} - 2\Delta a_4 \Delta a_5 \Delta a_6 + \overline{\Delta a_4}^{2} \Delta a_7}.$$

These quantities (5′) and (6′) are called the *equilong measures of second and third order horn angles* respectively and are denoted by $\mu_{12}{}^3$ and $\mu_{12}{}^4$.

As in the conformal theory, the invariants can be expressed in terms of intrinsic metric invariants, in this case in terms of the radii of curvature and their derivatives with respect to the inclinations of the curves.[6] However, it is more desirable for comparing the equilong and conformal measures to leave the equilong measures expressed in terms of the power series coefficients, regarding these as functions of the metric invariants of radius of curvature and its inclination-derivatives. Then the comparison of the equilong measures $\mu_{12}{}^3$ (5′) and $\mu_{12}{}^4$ (6′) with the corresponding conformal measures $M_{12}{}^3$ [6] and $M_{12}{}^4$ (15; 1) reveals that the two measures for second and third order horn angles are identical in algebraic structure.

The "duality" of the conformal and equilong geometries of second and third order horn angles. In section **2** of this paper the transformation induced in the space of elements (w, x, y, z) associated with a third order horn set by the conformal transformation (2; 1) has been discussed in detail. The conformal geometries associated with first and second order horn sets has also been discussed thoroughly by Kasner and de Cicco (2). Furthermore Kasner has shown that the equilong geometry associated with a first order horn set is identical with the conformal geometry associated with that horn set (4b). In this section of this paper the extension of this "duality" to the geometries of second and third order horn angles is established.

The conformal K_3-space associated with a given second order horn set may be defined as the space of elements (x, y, z) given by the equations (2c, d)

$$(7)\qquad \begin{aligned} x &= a_3 \\ y &= a_4 \\ z &= a_5 - 4a_2{}^2a_3 \end{aligned}$$

where the a_i are the coefficients in the power series expansion (1; 1) of the curves of the second order horn set. For this space the fundamental transformation induced by the conformal transformation (2; 1) of the Cartesian plane is given by the equations[7]

---

[6] $$\mu_{12}{}^3 = \frac{40}{3}\,\frac{\overline{\Delta r'}{}^3}{5\overline{\Delta r''}{}^2 - 4\Delta r'\Delta r''' + 4\overline{\Delta r'}{}^2};$$

$$\mu_{12}{}^4 = \frac{875}{4}\,\frac{\overline{\Delta r''}{}^4}{42\overline{\Delta r'''}{}^3 - 70\Delta r''\Delta r'''\Delta r^{\mathrm{iv}} + (25\Delta r^{\mathrm{v}} + 45\Delta r''')\overline{\Delta r''}{}^2}; \qquad M_{12}{}^3 = \frac{\overline{\Delta x}{}^3}{\overline{\Delta y}{}^2 - \Delta x\Delta z}$$

[7] Equations (7) and (8) differ a little from the definition and transformation of the K-space given by Kasner and de Cicco because the author found this definition more convenient. The effect on the geometry is unimportant.

$$
\begin{aligned}
&X = m_1{}^2x + h_1\\
(8)\quad &Y = m_1{}^3y + m_1m_2x + h_2\\
&Z = m_1{}^4z + 2m_1{}^2m_2y + m_2{}^2x + h_3
\end{aligned}
$$

and the fundamental invariant distance under this transformation (i. e., $M_{12}{}^3$, the conformal measure of a second order horn angle) by

$$
(9)\qquad \frac{\overline{\Delta x}^3}{\overline{\Delta y}^2 - \Delta x\Delta z}.
$$

If the equilong K'_4-space associated with the second order horn set is defined as the space of elements (x, y, z) given by the equations

$$
(10)\qquad x = a_3,\quad y = a_4,\quad z = a_5,
$$

where the a_i are the coefficients in the power series expansion (1) of the curves of the horn set in Hessian line coördinates, then the transformation induced in this equilong K'_4-space by the equilong transformation (2) of the Hessian plane is also given by the equations (8) and the fundamental distance invariant, the equilong measure of a second order horn angle, by equation (9). Thus the "duality" extends to second order horn angles.

In section **2** of this paper the conformal K_4-space associated with a third order horn set was defined as the space of elements (w, x, y, z) given by the equations (14_i; **1**) with an induced transformation given by the equations (4; **2**) and a metric given by the expression (15; **1**). In this paragraph the corresponding equilong K'_4-space will be defined and shown to be identical with this conformal K_4-space. The *equilong K'_4-space* associated with a given third order horn set is defined to be the space of elements (w, x, y, z) given by the equations

$$
(11)\qquad w = a_4,\quad x = a_5,\quad y = a_6,\quad z = a_7,
$$

where the a_i are the coefficients in the power series expansion (1) of the curves of the horn set in Hessian line coördinates. The development of the equations of the transformation induced in this space by an equilong transformation (2) of the Hessian plane follows the same procedure as that outlined on pages 18 and 19 for the conformal case. Given a third order horn angle expressed in Hessian line coördinates in the form (1), it is possible to obtain from the identity (4) the set of equations

$$
\begin{aligned}
&A_4 = (1/\alpha_1{}^3)a_4 + h_1(a_2, a_3, \alpha_i)\\
&A_5 = (1/\alpha_1{}^4)a_5 - 2(\alpha_2/\alpha_1{}^5)a_4 + h_2(a_2, a_3, \alpha_i)\\
(12)\quad &A_6 = (1/\alpha_1{}^5)a_6 - 3(\alpha_2/\alpha_1{}^6)a_5 - (1/\alpha_1)(\alpha_3/\alpha_1{}^5 - 4\alpha_2{}^2/\alpha_1{}^6)a_4 + h_3(a_2, a_3, \alpha_i)\\
&A_7 = (1/\alpha_1{}^6)a_7 - 4(\alpha_2/\alpha_1{}^7)a_6 - 2(1/\alpha_1{}^2)(\alpha_3/\alpha_1{}^5 - 4\alpha_2{}^2/\alpha_1{}^6)a_5\\
&\qquad + 4(\alpha_2/\alpha_1{}^3)(\alpha_3/\alpha_1{}^5 - 2\alpha_2{}^2/\alpha_1{}^6)a_4 + h_4(a_2, a_3, \alpha_i)
\end{aligned}
$$

(where $h_1, \cdots, h_4$ are functions of a_2, a_3, α_1 and do not involve $a_4, \cdots, a_7$). These equations, which express the effect of the equilong transformation (2) on the coefficients of the curve (1), become the equations of the desired transformation by means of (11) when the substitutions

$$(13) \qquad m_1 = 1/\alpha_1, \quad m_2 = -\alpha_2/\alpha_1^3, \quad m_3 = -[\alpha_3/\alpha_1^5 - 4(\alpha_2^2/\alpha_1^6)]$$

are made. It is only necessary to write out this transformation in order to see that it is given by the equations (4; 2) and consequently has the expression (15; 1) as its fundamental invariant. Consequently,

THEOREM XVII. *The equilong geometry of a second or third order horn set is identical with the conformal geometry of that horn set.*

BARNARD COLLEGE,
COLUMBIA UNIVERSITY.

REFERENCES.

(1a) E. Kasner, "Conformal geometry," *Proceedings of the Fifth International Congress of Mathematicians,* Cambridge, vol. 2 (1912), p. 81.

b) E. Kasner, "Two conformal invariants of fifth order," *Transactions of the American Mathematical Society,* vol. 44 (1938), p. 25.

(2a) E. Kasner and G. Comenetz, "Conformal geometry of Horn angles," *Proceedings of the National Academy of Sciences,* vol. 22 (1936), p. 303.

b) E. Kasner, "Trihornometry, a new chapter of conformal geometry," *Proceedings of the National Academy of Sciences,* vol. 23 (1937), p. 337.

c) E. Kasner and J. de Cicco, "Conformal geometry of Horn angles of second order," *Proceedings of the National Academy of Sciences,* vol. 24 (1938), p. 393.

d) E. Kasner and J. de Cicco, "General trihornometry of second order," *Proceedings of the National Academy of Sciences,* vol. 25 (1939), p. 479.

(3) G. Scheffers, "Isogonalkurven, Äquitangentialkurven und komplexe Zahlen," *Mathematische Annalen,* vol. 60 (1905), p. 491.

(4a) E. Kasner, "Equilong invariants and convergence proofs," *Bulletin of the American Mathematical Society,* vol. 23 (1917), p. 341.

b) E. Kasner, "Characterization of the conformal group and the equilong group by Horn angles," *Duke Mathematical Journal,* vol. 4 (1938), p. 95.

ON THE SINGULARITIES AND THE QUALITATIVE BEHAVIOR OF THE SOLUTIONS OF THE PROBLEM OF CENTRAL FORCES.*

By E. K. Haviland.

Introduction. The purpose of the present note is to investigate the motion of two bodies under the influence of their mutual attractions, when these latter satisfy certain specific conditions. By choosing the origin of the coördinate system so that it coincides with one of the particles, the motion may be reduced essentially to that of a single particle under a static central force. The force function, U, will in addition be subjected to the requirement that it be homogeneous of degree λ $(\neq 0)$ in the Cartesian coördinates x and y. In the particular case where $\lambda = 0$ the latter requirement is replaced by the condition that the first partial derivatives U_x, U_y be homogeneous in x, y of degree $\lambda - 1 = -1$. This leads in the former case to a force function U proportional to r^λ and in the latter to one proportional to $\log r$, where the constant of proportionality is chosen so as to correspond to an attractive force.[1]

While a number of the results concerning the nature of the paths in various particular cases have long been known, the present paper is concerned primarily with an aspect of the subject which appears not to have been treated in detail hitherto, viz., the analytic character of the collisions, a subject which appears elementary at first glance, but which turns out to be complicated enough to involve a rather deep analytical apparatus; in particular, the proof of the non-existence of an analytic continuation in certain cases will be made to depend on Bohr's theory of almost-periodic functions of a complex variable.

At the same time, it has seemed desirable to complete and systematize the description of all the various cases arising from force functions of the above type, and in this connection the most striking result is that in the range $-2 < \lambda < 0$ the situation is the same as for the Newtonian exponent $\lambda = -1$; in the sense that the non-vanishing of the constant angular momentum, c, remains necessary and sufficient for the non-existence of collisions

---

* Received March 25, 1942.

[1] Cf. W. Whewell, *On the Free Motion of Points*, Cambridge (1836), especially pp. 39-79. E. T. Whittaker, *Analytical Dynamics*, Cambridge (1917), pp. 77-86. P. Stäckel, *Encyclopädie der mathematischen Wissenschaften*, vol. 4, pp. 494-496.

and singularities, and that in addition the three subcases belonging to a fixed $c \neq 0$ are qualitatively the same as in the classical case.

1. Classification of cases. Turning now to the detailed treatment of the problem, we observe that, if polar coördinates be introduced, the equations of motion possess the integrals

$$\tfrac{1}{2}(r'^2 + r^2\phi'^2) - U(r) = h \tag{1}$$

and

$$r^2\phi' = c \qquad (' = d/dt), \tag{2}$$

the former representing the conservation of energy and the latter the conservation of angular momentum. Since ϕ is an ignorable coördinate, (1) may be replaced by

$$\tfrac{1}{2}r'^2 - U(r) + \tfrac{1}{2}c^2/r^2 = h \tag{1 bis}$$

i. e., by $\tfrac{1}{2}r'^2 - U^* = h$, where $U^* = U^*(r; c) = U(r) - \tfrac{1}{2}c^2/r^2$, so that

$$\tfrac{1}{2}r'^2 = U(r) - \tfrac{1}{2}c^2r^2 + h = F(r), \text{ say.}$$

We proceed to deduce from (1 bis) and (2) the nature of the motions corresponding to various forms of $U(r)$. Since the number of possibilities is rather large, it has been thought desirable to summarize the various cases and subcases in the following table:

I. $c = 0$. (A) $U = r^{-\lambda}$, $0 < \lambda < +\infty$
    (a) $h > 0$.
        1. λ irrational. 2. λ rational $= p/q$.
            1^*. p odd, 2^*. p even $= 2\kappa$.
                2_1^*. $\kappa + q$ odd, 2_2^*. $\kappa + q$ e
    (b) $h = 0$.
    (c) $h < 0$.
  (B) $U = -\log r$
  (C) $U = -r^{\lambda}$
    (α) $0 < \lambda < 1$. (β) $1 \leqq \lambda < +\infty$.
        (a) $h > 0$. (a) $h > 0$.
        (b) $h \leqq 0$. (b) $h \leqq 0$.

II. $c \neq 0$. $(c > 0)$.

(A) $U = r^{-\lambda}$.

(α) $0 < \lambda < 2$.

(a) $h > 0$. (b) $h = 0$. (c) $h < 0$.

(i) $F(r_0) > 0$, (ii) $F(r_0) = 0$, (iii) $F(r_0) < 0$.

(β) $\lambda = 2$. $1 - \frac{1}{2}c^2 = \zeta$.

(aa) $\zeta > 0, h > 0$; (ab) $\zeta > 0, h < 0$; (ac) $\zeta > 0, h = 0$.

(ba) $\zeta = 0, h > 0$; (bb) $\zeta = 0, h = 0$, (bc) $\zeta = 0, h < 0$.

(c) $\zeta < 0$.

(γ) $2 < \lambda < +\infty$.

(a) $h > 0$.

$$\left.\begin{array}{l}\text{(i) } F(r_0) < 0. \quad (\text{i}_1)\\ \text{(ii) } F(r_0) = 0.\\ \text{(iii) } F(r_0) > 0.\end{array}\right\} \begin{array}{l}(\text{i}_2)\\ 1.\ \lambda \text{ irrational, } 2.\ \lambda\\ \text{rational} = p/q.\end{array}$$

1*. p odd.
2*. p even $= 2\kappa$.
  2_1*. $\kappa + q$ odd.
  2_2*. $\kappa + q$ even.

(b) $h \leqq 0$.

(B) $U = -\log r$.

(i) $h < \log c + \frac{1}{2}$, (ii) $h = \log c + \frac{1}{2}$, (iii) $h > \log c + \frac{1}{2}$.

(C) $U = -r^{\lambda}$, $0 < \lambda < +\infty$.

(i) $F(r_0) < 0$, (ii) $F(r_0) = 0$, (iii) $F(r_0) > 0$.

2. The case of vanishing angular momentum. For convenience in treating the individual cases, we have written (1 bis) in the form

$$\frac{1}{2}r'^2 = U(r) - \frac{1}{2}c^2/r^2 + h = F(r, h) = F(r) \tag{3}$$

whence

$$2^{\frac{1}{2}}(t - t_0) = \int_{r_0}^{r} \pm \,|\, F(r, h) \,|^{-\frac{1}{2}} dr, \qquad (t_0 = \text{arb. const.}). \tag{3 bis}$$

We shall first suppose that

(I) $c = 0$; (A) $U = r^{-\lambda}$, $(0 < \lambda < +\infty)$; and (a) $h > 0$.

Under the assumption (I), the motion is rectilinear, so that we may write $x\ (> 0)$ for r, and (3) then becomes

(3 bis) $$\tfrac{1}{2}x'^2 = x^{-\lambda} + h = F(x),$$
whence

$$2^{\frac{1}{2}}t = \int_0^x (x^{-\lambda} + h)^{-\frac{1}{2}}dx = \int_0^x x^{\lambda/2}(1 + hx^{\lambda})^{-\frac{1}{2}}dx.$$

As $x \to 0$, $t \to 0$, so that a collision occurs, corresponding to $t = 0$. To investigate its nature, we consider only values of x for which $0 < hx^{\lambda} < \rho$, say, $(\rho < 1)$. Then the parenthesis in the preceding integral may be expanded by the binomial theorem and the resulting series integrated term by term, leading to the equation

$$\begin{aligned} 2^{\frac{1}{2}}t = x^{\frac{1}{2}(\lambda+2)}\{&2/(\lambda+2) - h/(3\lambda+2)x^{\lambda} \\ &+ 3h^2/[4(5\lambda+2)]x^{2\lambda} - 5h^3/[8(7\lambda+2)]x^{3\lambda} + \cdots\}. \end{aligned}$$

Regarding now t and x as complex variables, we set $x^{\lambda} = y$. Then

$$\begin{aligned} z = t^{2\lambda/(\lambda+2)} = 2^{-\lambda/(\lambda+2)}y\{&2/(\lambda+2) - h/(3\lambda+2)y \\ &+ 3h^2/[4(5\lambda+2)]y^2 - 5h^3/[8(7\lambda+2)]y^3 + \cdots\}^{2\lambda/(\lambda+2)}, \end{aligned}$$

$x^{\lambda}, t^{2\lambda/(\lambda+2)}, \cdots$ denoting those branches which are real and positive for real positive values of the argument. Thus $z = y\mathfrak{A}(y)$, where $\mathfrak{A}(y)$ is a regular analytic function of y in some neighborhood of $y = 0$ and $\mathfrak{A}(0) \neq 0$. Consequently, $z'(0) \neq 0$, so that y is a regular analytic function of z in some neighborhood of $z = 0$ and may be developed into a power series there, and since the coefficients of the series for z are real, the same is true of the series for y. On expressing y, z in terms of x, t respectively, this leads to

(4) $x = \beta_1 t^{2/(\lambda+2)}(1 + \beta_2 t^{\nu} + \beta_3 t^{2\nu} + \cdots)$, where $\beta_1 > 0$ and $\nu = 2\lambda(\lambda+2)$,

the Puiseux series in parenthesis converging uniformly for all t such that $|t| < \rho_1$, say, $\rho_1 > 0$.

Two cases arise:

1. λ is irrational. Then $2/(\lambda+2)$ is also irrational, and x has a logarithmic branch point at $t = 0$, so that $x(t)$ possesses no analytic continuation through $t = 0$.

To prove this, we observe that, if we start with that branch of x which is real and positive for real positive values of t (which is seen to be possible from the fact that the coefficients in the expression for x are real and $\beta_1 > 0$), then since $2/(\lambda+2)$ is irrational, it is clear that no number of circuits about $t=0$ will restore the first factor on the right of (4) to a real value. If, then, x is to assume its original value, it must, in particular, be real and this is possible only if, after n circuits, the second factor is proportional to the conjugate of the first, with a real proportionality factor k. Put $t=\rho\exp(i\vartheta)$. Then after n circuits

$$t^{2/(\lambda+2)}=\rho^{2/(\lambda+2)}\exp(i4\pi n/(\lambda+2))=\rho^{2/(\lambda+2)}\exp(i2\pi\alpha),$$

where $\alpha=(2n/(\lambda+2))=2n/(\lambda+2)-[2n/(\lambda+2)]$.

The second factor on the right-hand side of (4) would then have to be of the form $k\rho^{2/(\lambda+2)}\exp(-i2\pi\alpha)$ for all $\rho<\rho_1$. But this is impossible, since inspection of the factor in question shows that, for ρ sufficiently small, it lies in an arbitrarily small neighborhood of $t=1$ and hence in the interior of the angle bounded by the half-rays $\vartheta=\pm 2\pi\alpha$. From this contradiction, it follows that x has a logarithmic branch point at $t=0$.

2. λ is rational, say of form p/q, where p and q are relatively prime. Reasoning similar to that of case 1 shows that x is real if and only if $t^{2/(\lambda+2)}$ is real. Therefore, x will be continuable through $t=0$ under and only under conditions such that $t^{2/(\lambda+2)}$ is real for negative t.

This last factor may be written as $t^{2q/(p+2q)}$, where $2q/(p+2q)$ is to be interpreted as reduced to its lowest terms, viz., $2q/(p+2q)$, if p is odd; $q/(\frac{1}{2}p+q)$, if p is even. Then

(1*) If p is odd, $p+2q$ is odd and real analytic continuation is always possible, q arbitrary. This includes the Newtonian case $p=q=1$.

(2*) If p is even, put $p=2\kappa$. Then $2/(\lambda+2)$ becomes $q/(\kappa+q)$, and
  (2_1*) If $\kappa+q$ is odd, real analytic continuation is again possible;
  (2_2*) If $\kappa+q$ is even, q must be odd, since otherwise we should not have p and q relatively prime. Then κ is necessarily odd. In this case, real analytic continuation is impossible, e. g. $p=2$, $q=1$.

If the positive square root be chosen in (3 bis), it is seen that x increases steadily with t. As $x\to\infty$, $t\to\infty$ and $x\sim(2h)^{\frac{1}{2}}t$. Hence the solution is not bounded. If there are in general to be equilibrium solutions or asymptotic solutions, we must have [2] for some r_0

$$F(r_0)=F_r(r_0)=0, \tag{5}$$

---

[2] Cf. equation (3) above and A. Wintner, *Analytical Foundations of Celestial Mechanics*, p. 132 and p. 153.

or in the present case, $F = F_x = 0$. But from (3 bis) it is seen that neither F nor F_x vanishes for $0 < x < + \infty$, so that under the present assumptions neither equilibrium solutions nor asymptotic solutions exist.

Suppose next that

(b) $h = 0$. Here the motion is the same as in (a), with respect both to boundedness and to the nature of the collision and to the non-existence of equilibrium or asymptotic solutions, but the results are more easily obtained, since

$$t = 2^{-\frac{1}{2}} \int_0^x x^{\frac{1}{2}\lambda} dx = 2^{\frac{1}{2}} x^{\frac{1}{2}(\lambda+2)}/(\lambda + 2) \text{ or } x = \text{const. } t^{2/(\lambda+2)}.$$

Again, suppose that

(c) $h < 0$. Then the nature of the collision is as before and there are no equilibrium solutions or asymptotic paths, since $F_x \neq 0$, $(0 < x < + \infty)$, but $x \sim (2h)^{\frac{1}{2}} t$ implies that the solution is bounded.

Let assumption (I) be retained, but (A) be replaced by

$$\text{(B)} \qquad U = - \log r, \qquad (r = x),$$

so that $\frac{1}{2} x'^2 = F(x) = - \log x + h$; $F_x = - x^{-1}$; $F_{xx} = x^{-2}$. Thus there are no finite positive zeros of either F_x or F_{xx}. For all $h \lesseqgtr 0$, $F(0+) = + \infty$, $F(+ \infty) = - \infty$. The path is rectilinear and has a maximum at $x = e^h$, corresponding to a cusp.[3]

Since

$$(6) \qquad dt = 2^{-\frac{1}{2}} (h - \log x)^{-\frac{1}{2}} dx,$$

it follows that, if t^* be the time arbitrarily corresponding to the position x^*, $(0 < x^* < e^h)$, then

$$(7) \qquad t - t^* = 2^{-\frac{1}{2}} \int_{x^*}^{x} (h - \log x)^{-\frac{1}{2}} dx,$$

from which it is seen that the solution path, starting from x^*, reaches both $x = 0$ and $x = e^h$ in finite time. To $x = 0$ may be made to correspond $t = 0$, these representing the place and the time of collision.

A necessary condition that $x(t)$ be continuable analytically through $t = 0$ is that it should have at most an algebraic singularity there, whereupon the same would be true of the inverse function $t = t(x)$ and hence of the derivative of the latter, since differentiation cannot introduce transcendental singularities. But (6) shows that this derivative has a transcendental (logarithmic) singularity at $x=0$, from which it follows that the analytic continuation of $x=x(t)$ through $t=0$ is impossible.[4]

---

[3] Cf. A. Wintner, *op. cit.*, §§ 185, 186.

As in the preceding case, there are no equilibrium solutions and no asymptotic paths.

Continuing the assumption $c=0$, we now suppose

$$\text{(C)} \qquad U=U(x)=-x^{\lambda},$$

$$(\alpha) \qquad 0<\lambda<1,$$

then $F(x)=-x^{\lambda}+h$; $F_x=-\lambda x^{\lambda-1}$; $F_{xx}=-\lambda(\lambda-1)x^{\lambda-2}$, so that F_x and F_{xx} are $\neq 0$ for all x, $0<x<+\infty$. Suppose further, that

$$\text{(a)} \qquad h>0.$$

Then $F(0)=h>0$, $F(+\infty)=-\infty$, so that there exists a bounded collision path, the maximum value of x corresponding to a cusp. As before, there are no equilibrium solutions or asymptotic solutions. To examine the nature of the collision, we consider

$$(8) \qquad x'^2=2(h-x^{\lambda}).$$

Taking, as we may, the positive square root $x'=(2h)^{\frac{1}{2}}(1-x^{\lambda}/h)^{\frac{1}{2}}$, and setting $x/h^{1/\lambda}=r$, we may expand the parenthesis by the binomial theorem and the resulting series converges uniformly for $|r^{\lambda}|<\rho$, say, $(\rho<1)$, and may be integrated there term by term, leading to the expression

$$t=kr\left(1+\frac{1}{2(\lambda+1)}r^{\lambda}+\frac{3}{8(2\lambda+1)}r^{2\lambda}+\frac{5}{16(3\lambda+1)}r^{3\lambda}+\cdots\right),$$
$$(k=\text{const.}\neq 0).$$

Regarding t and r as complex variables, we put $r^{\lambda}=y$, so that

$$z=t^{\lambda}=k^{\lambda}y\left(1+\frac{1}{2(\lambda+1)}y+\frac{3}{8(2\lambda+1)}y^2+\cdots\right)^{\lambda},$$

$z=t^{\lambda}$ and $y=r^{\lambda}$ denoting those branches which are real for real values of the argument. Then, as in case (Aa), $z=y\mathfrak{A}(y)$, where $\mathfrak{A}(y)$ is a regular analytic function of y in some neighborhood of $y = 0$ and $\mathfrak{A}(0) \neq 0$, so that y may be expressed as a regular analytic function of z in some neighborhood of $z = 0$, and this leads to an expression of the form

---

[4] This fact may also be verified by making the substitution $\log x=y$, followed by $y=h-z^2$, whereupon (7) becomes $t-t^*=-\sqrt{2}\exp(h)\int_{z^*}^{z}\exp(-z^2)\,dz$. If $z=z(t)$ represents the inverse of the foregoing, then x is given as a function of t by $x=\exp(h-z^2(t))$. Since the nature of the transcendental singularity of $t=t(z)$ at $z=\infty$ is quite different from that of $y=\log x$ at $x=0$ (or $x=\infty$), it follows that x as a function of t has a transcendental singularity at $t=0$. For the nature of $t(z)$ at $z=\infty$, cf. Polya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin (1925), vol. II, p. 33, no. 189.

$$x = \gamma_1 t[1 + \gamma_k t^{(k-1)\lambda}(1 + \gamma_{k+1}t^{\lambda} + \gamma_{k+2}t^{2\lambda} + \cdots)],$$

where $\gamma_1 > 0$ and $\gamma_k \neq 0$, $(k \geqq 2)$, whereupon reasoning similar to that of case (Aa), shows that

1. If λ is irrational, no analytic continuation is possible.
2. If λ is rational, $= p/q$ say, where p and q are relatively prime, we write $(k - 1)p/q = m/s$, where it is understood that m and s are relatively prime. Here real analytic continuation is possible, if s is odd, but not if s is even.

If, on the other hand,

(b) $$h \leqq 0,$$

it is clear from (8) and from the expression for F_x that there is no motion at all.

Finally, if

(β) $$1 \leqq \lambda < +\infty$$

and (a) : $h > 0$, there exists, as before, a bounded collision path, the situation with respect to the behavior at the maximum value of x and with respect to the analytic nature of the collision being precisely as in case (α). In the cases in which analytic continuation through the origin is possible and $\lambda > 1$, the motion is similar to the case $\lambda = 2$ of the harmonic oscillator. Since F and F_x do not vanish simultaneously, there are no equilibrium or asymptotic solutions.

If, however, (b) : $h \leqq 0$, then there is no motion except, in the case $h = 0$, for the trivial equilibrium solution $x \equiv 0$.

3. The case of non-vanishing angular momentum. Having considered linear motions, we now turn to the case

(II) $$c \neq 0,$$

where we may suppose without restriction that $c > 0$. As our first sub-case, we shall consider that in which

(A) $$U(r) = r^{-\lambda} \text{ and } (\alpha)\ 0 < \lambda < 2.$$

Here (3) becomes

(9) $$\tfrac{1}{2}r'^2 = r^{-\lambda} - \tfrac{1}{2}c^2 r^{-2} + h = F(r),$$

whence

(10) $F_r(r) = -\lambda r^{-\lambda-1} + c^2 r^{-3}$ and (11) $F_{rr}(r) = \lambda(\lambda+1) r^{-\lambda-2} - 3c^2 r^{-4}$.

As $r \to 0+$, $r'^2 \sim -c^2/r^2$, which is impossible for real r, so that there is no collision in any of the three cases $h \lesseqgtr 0$.

As $r \to +\infty$, $r'^2 \sim 2h$, so that if (a) : $h > 0$, then $r \sim kt$, k a constant, which may be supposed positive. In this case, a solution exists and is not bounded. dr/dt has just one zero, since F_r has but one positive real zero and F crosses the r-axis an odd number of times, therefore once. The zero of dr/dt corresponds to a minimum value of r, which we may suppose to occur at $t = 0$. For the nature of the path at this point, cf. the reference under (i) below. By (2), $\phi' = c/r^2$, $(c > 0)$, so that ϕ increases steadily with t. Moreover, as $r \sim kt$, integration of (2) shows that ϕ approaches a finite limit as $t \to +\infty$, and a similar result holds if $t \to -\infty$.

In this case, $F(r_0) = F_r(r_0) = 0$ would imply $F_{rr}(r_0) = 0$ also. But the latter two equations imply $\lambda = 2$, contrary to hypothesis, so that no equilibrium or asymptotic solutions exist.

Again, if (b) : $h = 0$, $2(dt/dr)^2 \sim r^\lambda$, so that $r \sim \text{const.}\, t^{2/(\lambda+2)}$. Here r becomes infinite more slowly than in case (a), but the motion is otherwise the same.

Thirdly, if (c) : $h < 0$, any possible solution will be bounded. To determine whether or not a solution exists at all, we observe from (11) that

$$F_{rr}(r) \lesseqgtr 0, \text{ according as } r \lesseqgtr r_1 = \left(\frac{3c^2}{\lambda(\lambda+1)}\right)^{1/(2-\lambda)}.$$

Consequently, $F(r)$ is first concave and then convex from below. The extremum of $F(r)$ is located at $r_0 = (c^2/\lambda)^{1/(2-\lambda)}$, where $F_r(r_0) = 0$. For $0 < \lambda < 2$, it is seen that $r_0 < r_1$, so that the extremum is a maximum. $F(0+) = -\infty$, $F(+\infty) < 0$. Therefore,

(i) if $F(r_0) > 0$, there will be a bounded (periodic) non-circular motion, for a more detailed discussion of which the reader is referred to A. Wintner, *The Analytical Foundations of Celestial Mechanics*, § 215, p. 153;

(ii) if $F(r_0) = 0$, there will be a circular motion and nothing else;

(iii) if $F(r_0) < 0$, no solution will exist.

In case (i), $0 < R_0 \leqq r \leqq R_1$, say, so that from (2) $|\phi - \phi_0| \geqq cR_1^{-2}|t|$, and ϕ becomes positively or negatively infinite with t, and the same is true in case (ii).

F and F_r vanish simultaneously in, and only in, case (ii). The circular path then represents an equilibrium solution, but there are no asymptotic paths.

Under the assumptions (II) : $c > 0$ and (A) : $U = r^{-\lambda}$, we now consider case

(β) $\lambda = 2.$

We set $1 - \frac{1}{2}c^2 = \zeta$, and examine first the subcase corresponding to (aa) : $\zeta > 0$; $h > 0$. Then

(12) $r'^2 = 2(h + \zeta/r^2)$, whence (13) $r = 2^{\frac{1}{2}}t^{\frac{1}{2}}((2\zeta)^{\frac{1}{2}} + ht)^{\frac{1}{2}}$;

if $t = 0$ corresponds to $r = 0$, and conversely. In this case, there is a collision, but no real analytic continuation of r through $t = 0$ is possible. Substitution of (13) into (2) and subsequent integration leads to

$$\phi - \phi_0 = c[2(2\zeta)^{\frac{1}{2}}]^{-1}\log[t/((2\zeta)^{\frac{1}{2}} + ht)],$$

so that ϕ becomes logarithmically infinite as $t \to 0 +$. As t increases and becomes positively infinite, the same is true of r. ϕ also increases steadily, but approaches a finite limit as $t \to + \infty$.

If, on the other hand, (ab) : $h < 0$, the situation with respect to the collision as $t \to 0 +$ is the same as before, but as t increases, r remains bounded. ϕ increases steadily with t and becomes infinite as $t \to -(2\zeta)^{\frac{1}{2}}/h > 0$, while as $t \to 0 +$, ϕ becomes negatively infinite.

In both the foregoing cases, $x = r \cos \phi$ becomes

$$x = 2^{\frac{1}{2}}t^{\frac{1}{2}}((2\zeta)^{\frac{1}{2}} + ht)^{\frac{1}{2}}\cos\{c(8\zeta)^{-\frac{1}{2}}\log[t/((2\zeta)^{\frac{1}{2}} + ht)]\},$$

and the expression for $y = r \sin \phi$ is similar.

Thirdly, if (ac) : $h = 0$, we have $r'^2 = 2\zeta/r^2$, so $r = (8\zeta)^{\frac{1}{4}}t^{\frac{1}{2}}$. Here, as before, there is a collision, and real analytic continuation of r through $t = 0$ is impossible. $\phi' = c/r^2 = c/((8\zeta)^{\frac{1}{2}}t)$; $\phi = c(8\zeta)^{-\frac{1}{2}}\log t + \text{const.}$, so that $\phi \to -\infty$ as $t \to 0 +$; $\phi \to + \infty$ as $t \to + \infty$, in which latter case $r \to + \infty$ also.

Again, if (ba) : $\zeta = 0$, $h > 0$, (12) reduces to $r'^2 = 2h$, so $r = (2h)^{\frac{1}{2}}t$. In this case, r can be continued analytically through $t = 0$, $\phi' = c/2ht^2$; $\phi = -c(2h)^{-1}t^{-1}$, so that as $t \to 0 +$, ϕ becomes negatively infinite (assuming $c > 0$), but more strongly than in the preceding three cases. As $t \to + \infty$, $r \to + \infty$ and $\phi \to 0 -$. $x = (2h)^{\frac{1}{2}}t \cos (c/2ht)$, $y = -(2h)^{\frac{1}{2}}t \sin (c/2ht)$.

If (bb) : $\zeta = 0$, $h = 0$, (12) reduces to $r' = 0$, $r = \rho = \text{const.}$ and $\phi = c\rho^{-2}t + \text{const.}$, where the latter constant may be taken to be zero, and we have circular motion.

If (bc) : $\zeta = 0$, $h < 0$, there is no real solution.

Finally if (c) : $\zeta < 0$, there is no collision for any value of h. If $h \leqq 0$, there is no real solution at all, just as in case (bc). If, on the other hand, $h > 0$,

$$t = 2^{-\frac{1}{2}} \int_R^r r(hr^2 + \zeta)^{-\frac{1}{2}} dr = (hr^2 + \zeta)^{\frac{1}{2}}/h2^{\frac{1}{2}},$$

where R is the minimum value of r, corresponding to $t = 0$. For the behavior of the path there, cf. the reference under case (II A α c i). $r^2 = h^{-1}(2h^2t^2 - \zeta)$, so that upon substitution in (2) and integration, $\phi = c\eta/(2h)$ arctan (ηt), where $\eta^2 = 2h^2/(-\zeta)$, from which it appears that ϕ approaches a finite limit as $t \rightarrow 0+$ and as $t \rightarrow +\infty$, while in the latter case, $r \rightarrow +\infty$.

The usual tests applied to the case $\lambda = 2$ show that there are no equilibrium solutions except in the case (bb), where $F \equiv 0$ and $F_r \equiv 0$, in which case all circular motions are equilibrium solutions. In no case do there exist paths asymptotic as $t \rightarrow \infty$.

We next assume (II) : $c > 0$; (A) : $U = r^{-\lambda}$;

$$(\gamma) \qquad 2 < \lambda < +\infty.$$

In this case, $r_1 < r_0$ (cf. p. 485 for the notation), so that the extremum of $F(r)$ is a minimum.

We first suppose (a) : $h > 0$, so that it is seen from (9) that $F(r)$ tends to $+\infty$ as $r \rightarrow 0+$ and is positive as $r \rightarrow +\infty$. Then according as

$$\text{(i)} \quad F(r_0) < 0, \qquad \text{(ii)} \quad F(r_0) = 0, \qquad \text{(iii)} \quad F(r_0) > 0,$$

there will be

(i_1) a bounded collision path *and* (i_2) an unbounded path without collision;

(ii) a circular equilibrium path and asymptotic paths corresponding to (i_1) and (i_2);

(iii) an unbounded collision path.

We examine first case (i_1). Here

$$\tfrac{1}{2}r'^2 = r^{-\lambda} - \tfrac{1}{2}c^2r^{-2} + h, \qquad (2 < \lambda < +\infty),$$

so that

$$(14) \quad t = 2^{-\frac{1}{2}} \int_0^r r^{\lambda/2}\{1 + (hr^\lambda - \tfrac{1}{2}c^2r^{\lambda-2})\}^{-\frac{1}{2}} dr,$$

$$(15) \quad t = 2^{-\frac{1}{2}} \int_0^r r^{\lambda/2}\{1 - \tfrac{1}{2}(hr^\lambda - \tfrac{1}{2}c^2r^{\lambda-2}) + 3/8\,(hr^\lambda - \tfrac{1}{2}c^2r^{\lambda-2})^2 - \cdots\} dr.$$

The series in brackets converges uniformly and absolutely for $|r^{\lambda-2}|(\tfrac{1}{2}c^2 + |hr^2|) < \rho$, say, $(0 < \rho < 1)$, i. e. for $|r| < r^{**}$, for a suitable $r^{**} > 0$

Consequently, (15) may be integrated term by term there and the resulting series will converge uniformly for r in the above range. Before carrying out this integration, it is convenient to write (15) as

$$t = 2^{-\frac{1}{2}} \int_0^r r^{\lambda/2}\{1 - \tfrac{1}{2}r^{\lambda-2}(hr^2 - \tfrac{1}{2}c^2) + \tfrac{3}{8}r^{2(\lambda-2)}(hr^2 - \tfrac{1}{2}c^2)^2 + \cdots\}dr$$

$$= 2^{-\frac{1}{2}} \int_0^r \{r^{\lambda/2} - \tfrac{1}{2}hr^{3\lambda/2} + \tfrac{1}{4}c^2r^{3\lambda/2-2} + \tfrac{3}{8}h^2r^{5\lambda/2} - \tfrac{3}{8}hc^2r^{5\lambda/2-2} + \cdots\}dr,$$

so that

$$(16)\quad t = r^{\frac{1}{2}(\lambda+2)} \left\{ \frac{2^{\frac{1}{2}}}{\lambda+2} - \frac{h}{2^{\frac{1}{2}}(3\lambda+2)}r^{\lambda} + \frac{c^2}{2^{3/2}(3\lambda-2)}r^{\lambda-2} + \frac{3h^2}{2^{5/2}(5\lambda+2)}r^{2\lambda} - \frac{3hc^2}{2^{5/2}(5\lambda-2)}r^{2\lambda-2} + \cdots \right\};$$

i. e., t is of the form

$$(17)\qquad t = r^{\frac{1}{2}(\lambda+2)}\{a_1 + r^{\lambda-2}[a_2 + a_3r^{\mu_3} + a_4r^{\mu_4} + \cdots]\}$$

or

$$(18)\qquad t = r^{\frac{1}{2}(\lambda+2)}\{a_1 + a_2r^{\nu_2} + a_3r^{\nu_3} + a_4r^{\nu_4} + \cdots\},$$

where t and r are henceforth to be regarded as complex variables, the coefficients and the exponents, however, being real and r^{ν} for any exponent ν referring to that branch of the function which is real and positive for positive real r.

From (16) and the remarks preceding it, it is seen that t is a regular analytic function of r in the domain $0 < |r| < r^{**}$. Moreover, from the uniform convergence of the integrated series, it follows by the approximation theorem for almost-periodic functions of a complex variable[5] that t is an almost-periodic function of r in that domain. Furthermore, if λ is irrational, t has an almost-periodic singularity (corresponding to a logarithmic branch point) at $r = 0$, as is seen by applying to (17), where $a_1 > 0$, the reasoning used in case (I A a 1). Also, since $\lambda > 2$, (16) shows that the exponents of the series for t are all positive and that amongst them there is a smallest, viz., $\frac{1}{2}(\lambda + 2)$. Consequently, $t = f(r)$ is an analytic almost-periodic function in the domain $0 < |r| < r^{**}$ and has a normal almost-periodic singularity at the point $r = 0$. Then, by a theorem of Bohr,[6] for a sufficiently small $r^{*} > 0$ the function $t = f(r)$ will represent the neighborhood E_{r^*} of $r = 0$ on a simply connected domain D of the Riemann surface S_t, which domain D

---

[5] Cf. H. Bohr, "On the inverse function of an analytic almost periodic function," *Annals of Mathematics*, 2nd Ser., vol. 32 (1931), p. 249.

[6] Cf. H. Bohr, *ibid.*, p. 251.

will contain a complete neighborhood E_{ρ^*} of $t = 0$. Hence in E_{ρ^*} there exists an inverse function $r = g(t)$ and this function will (for sufficiently small ρ^*) be analytic and almost-periodic in $0 < |t| < \rho^*$ and have a normal almost-periodic singularity at $t = 0$.

Furthermore, if $\Lambda = \frac{1}{2}(\lambda + 2) > 0$ denotes the smallest exponent of $f(r)$ and $M > 0$ the smallest exponent of $g(t)$, then

$$\mathrm{M} = 1/\Lambda = 2/(\lambda + 2),$$

and if we write $f(r) = r^{\Lambda} f_1(r)$ and $g(t) = t^{\mathrm{M}} g_1(t)$, the modules of $f_1(r)$ and of $g_1(t)$ will be connected by the equation

$$\mathfrak{M} g_1 = \Lambda^{-1} \mathfrak{M} f_1,$$

(where it is seen from (16) that $\mathfrak{M} f_1$ is generated by λ and 1). Consequently, the reasoning of case (I A a 1), applied above to (17), now shows that r as a function of t possesses, in the case λ irrational, a logarithmic branch point at $t = 0$, so that in this case analytic continuation of the motion through the instant of collision is impossible.

If, on the other hand, λ is rational, the right-hand member of (16) becomes a Puiseux series which may be inverted just as in case (I A a). As there, we conclude that if $\lambda = p/q$, p and q relatively prime, then

1* If p is odd, q arbitrary, real analytic continuation is possible;

2* If p is even, $= 2\kappa$, then
  (2_1*) If $\kappa + q$ is odd, real analytic continuation is again possible.
  (2_2*) If $\kappa + q$ is even, no real analytic continuation is possible.

The smaller zero of $F(r)$ is simple and corresponds to a maximum R_1 of $r = r(t)$, which, by (3 bis), is reached in finite time.[3] r decreases steadily with t from $r = R_1$ to $r = 0$, the point of collision. From (2) it follows that ϕ increases steadily with t as r increases from 0 to R_1 and then decreases to 0. If $0 < r^* < R_1$ and if t^*, t_1 be the times corresponding to r^*, R_1 respectively, then the finite value of $\phi - \phi^* = \int_{t^*}^{t} (c/r^2)\,dt$ for all t, $t^* \leqq t \leqq t_1$ shows that ϕ approaches a finite limit as $r \to R_1$. If $t \to 0$, then $r \to 0$ and

$$\phi - \phi_0 \sim k \int_0^t t^{-4/(\lambda+2)}\,dt, \qquad k = \text{const.}, \ \lambda > 2,$$

so that ϕ approaches a finite limit in this case also.

With respect to the case (II A γ ai$_2$), it is seen that R_2, the larger zero of $F(r)$, is also simple and corresponds to a minimum of $r = r(t)$, which is reached in a finite time from a point on the path for which $r = r^{**} > R_2$. As t varies in any finite range, the same is true of ϕ, so that ϕ approaches a finite limit as $t \rightarrow 0$, if $t = 0$ is made to correspond to $r = R_2$. As $r \rightarrow +\infty$, $r' \sim (2h)^{\frac{1}{2}}$, so that $r \sim (2h)^{\frac{1}{2}} t$ and by (2) $\phi \sim k_1 t^{-1} + k_2$, ($k_1$ and k_2 constants), so that ϕ approaches a finite limit as $t \rightarrow +\infty$ also. The motion is similar to that in cases (II A α a) and (II A α b). This completes the discussion of the motion under the assumptions (II): $c > 0$; (A): $U = r^{-\lambda}$; (γ): $2 < \lambda < +\infty$; (a) $h > 0$; (i) $F(r_0) < 0$.

If (ii): $F(r_0) = 0$, then, as already pointed out, there is a circular motion, corresponding to an equilibrium solution and asymptotic paths one lying inside the equilibrium solution and corresponding to case (i_1) above, while the other lies outside and corresponds to (i_2). The former is a collision path behaving at the moment of collision as the path in case (i_1) above. r increases steadily with t and approaches r_0 as $t \rightarrow +\infty$. ϕ likewise increases steadily with t and becomes infinite as $r \rightarrow r_0$ (i. e., as $t \rightarrow \infty$) as may readily be seen from the integration of (2) in view of the fact that r remains bounded. On the outer asymptotic path, t becomes infinite and ϕ approaches a finite limit as r, starting from some $r^{**} > r_0$, becomes infinite, just as in case (i_2), but as r decreases toward r_0, t becomes negatively infinite and ϕ decreases indefinitely with t.

If, finally (iii): $F(r_0) > 0$, then $F(r)$ has no zeros in $0 < r < +\infty$, so that there exists an unbounded collision path. The nature of the collision, corresponding to $t = 0$, is exactly the same as in subcase (i_1) above. As t increases from 0 to $+\infty$, r increases steadily and $\sim$ const. t, while ϕ approaches a finite limit, this portion of the path being similar to that of case (i_2).

Let assumptions (II), (A), (γ) be retained, but let (a) be replaced by (b): $h \leqq 0$. Here $F(0+) = +\infty$, but $F(r) < 0$ for all sufficiently large r. $F(r)$ crosses the r-axis an odd number of times, hence just once, as $F_r(r)$ has but one positive real zero. The motion corresponds to a bounded collision path precisely similar to that of subcase (i_1) under (a). This concludes the treatment of case (II A). The only equilibrium solutions or asymptotic paths are those treated under subcase (aii).

Retaining the assumption (II): $c > 0$, we now suppose that

$$\text{(B)} \qquad U = -\log r.$$

Then

$$F(r) = -\log r - \tfrac{1}{2}c^2 r^{-2} + h; \; F_r(r) = -r^{-1} + c^2 r^{-3}; \; F_{rr}(r) = r^{-2} - 3c^2 r^{-4}.$$

$F_r = 0$ implies $r = c\ (> 0)$, while $F_{rr} \lesseqqgtr 0$ according as $r^2 \lesseqqgtr 3c^2$. For all $h \lesseqqgtr 0$, we have $F(0+) = F(+\infty) = -\infty$. $F(r)$ has one maximum, viz., at $r = c$, and $F(c) \lesseqqgtr 0$ according as $h \lesseqqgtr \log c + \frac{1}{2}$.

Consequently, (i) if $h < \log c + \frac{1}{2}$, there is no solution;

(ii) if $h = \log c + \frac{1}{2}$, there is a circular motion and nothing else;

(iii) if $h > \log c + \frac{1}{2}$, there is no collision, but a bounded periodic motion like that of case (II A α ci).

Finally let assumption (II) be retained but (B) be replaced by

$$\text{(C)} \qquad U = -r^{\lambda}, \qquad 0 < \lambda < +\infty$$

Then

$$F(r) = -r^{\lambda} - \tfrac{1}{2}c^2r^{-2} + h;\ F_r(r) = -\lambda r^{\lambda-1} + c^2r^{-3};\ F_{rr} = -\lambda(\lambda - 1)r^{\lambda-2} - 3c^2r^{-4}.$$

As in the previous case, for all h, $F(0+) = F(+\infty) = -\infty$. F_r has just one zero, viz., at $r_0 = (c^2/r)^{1/(\lambda+2)}$. Then

(α): if $0 < \lambda < 1$, it follows that $F_{rr} \lesseqqgtr 0$ according as $r \lesseqqgtr r_1 = (3c^2\lambda^{-1}(1-\lambda)^{-1})^{1/(\lambda+2)}$, respectively, and $r_0 < r_1$, since $1 < 3/(1-\lambda)$.

(β): if $\lambda = 1$, direct calculation shows that F_r vanishes at $r_0 = c^{2/3}$, while $F_{rr} = -3c^2r^4 < 0$.

(γ) if $\lambda > 1$, F_r vanishes at $r_0 = (c^2/\lambda)^{1/(\lambda+2)}$ and $F_{rr} < 0$ for all $r > 0$.

Hence the extremum is a maximum, and according as $F(r_0) \lesseqqgtr 0$, there exists (i) no solution; (ii) a circular motion and nothing else; (iii) a bounded periodic motion like that of case (II A α ci).

THE LINCOLN UNIVERSITY,
CHESTER COUNTY, PENNSYLVANIA.

RAMANUJAN IDENTITIES INVOLVING THE PARTITION FUNCTION FOR THE MODULI 11^α.*

By JOSEPH LEHNER.

1. Introduction.[1] Ramanujan conjectured that

$$p(n) \equiv 0 \pmod{q} \quad \text{if} \quad 24n \equiv 1 \pmod{q}, \tag{1.1}$$

where $p(n)$, with $n > 0$, is the number of unrestricted partitions of n, and q is one of the primes 5, 7, 11. Direct proofs of each of these three congruences were given by Ramanujan.[2]

He also indicated an indirect method of proof; namely, as immediate consequences of certain *identities*, which he stated without proof, and of which the following is an example:

$$\sum_{n=0}^{\infty} p(7n+5)x^n = 7\,\frac{\prod_1^\infty (1-x^{7m})^3}{\prod_1^\infty (1-x^m)^4} + 49x\,\frac{\prod_1^\infty (1-x^{7m})^7}{\prod_1^\infty (1-x^m)^8}. \tag{1.2}$$

The identities for the moduli 5 and 7 have since been proved by several writers.[3] Ramanujan did not give an identity for $q = 11$.

Ramanujan stated more than (1.1). His most general conjecture—that

$$p(n) \equiv 0 \pmod{q^\alpha} \text{ if } 24n \equiv 1 \pmod{q^\alpha}, \quad \alpha = \text{integer} > 0 \tag{1.3}$$

—is known to be false for $q = 7$, $\alpha > 2$, but G. N. Watson[4] [15] has proved the correctness of (1.3) for $q = 5$, $\alpha \geqq 1$, and has proposed and proved a substitute congruence for $q = 7$ and all α. In the same paper he promised a discussion of the moduli 11^α, but to date it has not appeared. A proof of (1.3) for $q^\alpha = 11^2$ is contained in one of Ramanujan's notebooks but has not been published.[5] For higher powers of 11, nothing is known concerning the

---

* Received March 16, 1942.

[1] The author wishes to express his thanks to Professor H. Rademacher, who suggested to him the problem treated in this paper, and to Professor J. B. Rosser for helpful advice in connection with the computations of 9.

[2] [14], pp. 210-213, 232-238. (Square brackets refer to the bibliography at the end of this paper.)

[3] For references see footnotes 3, 4, 4a in [11].

[4] See also [16] and the reference in footnote 3 of that paper.

[5] [4], p. 100.

correctness of (1.3), except for an affirmative test by D. H. Lehmer[6] of an isolated value.

The procedure of Watson and Zuckerman was to establish identities of type (1.2) containing a factor q^α in the right member. Recently Rademacher [12] developed a systematic method for the investigation of these identities, and applied it to the moduli 5^α, 7, 7^2, 13^α. Rademacher's method displays very clearly the underlying connection between the Ramanujan identities and the theory of modular functions. But it is not so powerful as Watson's process, which employs modular equations: it does not establish, except for the first few values of α where direct computation is possible, the *integrality* of the coefficients of the modular functions (which themselves have integral coefficient expansions) in the identity, and so the *congruence* (1.3) does not follow.

In the present paper we use Rademacher's method to establish identities for the moduli 11^α. For $\alpha = 1$, the coefficients in the identity are computed—**6**—while in the case $\alpha = 2$—**9**—they are determined sufficiently to prove the congruence (1.3). For higher values of α we cannot prove the integrality of the coefficients in the identities and so cannot prove (1.3). Thus, our goal, the proof or disproof of the congruences (1.3) in the unknown cases (i.e., $\alpha > 2$), is not reached.

It is felt that the identities possess some interest in themselves, particularly since they highlight the arithmetic character inherent in the larger part of the theory of modular functions. From this point of view the Ramanujan congruences appear to be an arithmetic expression of certain facts pertaining to special subgroups of the modular group and the functions belonging to them.

2. Outline of the method. We shall make use of a simple modification of an operator studied by Hecke.[7] Let $F(\tau)$ be an entire modular function belonging to the subgroup $\Gamma_0(q)$ of the full modular group. $\Gamma_0(q)$ is defined by the condition $c \equiv 0 \pmod{q}$ in the substitution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where q is a prime > 3. We define the linear operator

$$U_qF(\tau) = q^{-1} \sum_{\lambda=0}^{q-1} F\left(\frac{\tau+\lambda}{q}\right). \tag{2.1}$$

In general, we cannot prove that U_qF is entire modular on $\Gamma_0(q)$. However, if F exhibits a simple behavior under the (non-modular) substitution $\tau' = -1/q\tau$, such as

$$F(-1/q\tau) = cF(\tau), \text{ or } F(-1/q\tau) = cF^{-1}(\tau),\ c = \text{const.}, \tag{2.2}$$

---

[6] See [9], p. 90.

[7] See [5] for a readable sketch of this theory and [6] for a complete account.

then U_qF will belong to $\Gamma_0(q)$. Moreover, in certain cases F may be a modular function of $\Gamma_0(q^2)$, and not of $\Gamma_0(q)$, but still the same conclusion will hold. This is true of the function

$$\Phi(\tau; q^2) = \frac{\eta(q^2\tau)}{\eta(\tau)}, \tag{2.3}$$

where $\eta(\tau)$ is the well-known Dedekind function

$$\eta(\tau) = \exp(\pi i\tau/12) \cdot \prod_1^\infty (1 - x^m), \quad x = \exp 2\pi i\tau. \tag{2.4}$$

$\Phi(\tau; q^2)$ is an entire modular function on $\Gamma_0(q^2)$, but $U_q\Phi(\tau; q^2)$ belongs to the larger group $\Gamma_0(q)$.

We now take $q = 11$ and show in **5** that every entire modular function of $\Gamma_0(11)$ is a rational function of certain basic functions $A_{11}(\tau)$, $C_{11}(\tau)$. The latter have power series expansions in $x = \exp 2\pi i\tau$ with integral coefficients. We find in **6** that

$$\begin{gathered} U_{11}\Phi(\tau; 11^2) = 11(11AC^2 - 11^2C + 2AC - 32C - 2), \\ A = A_{11}(\tau), \quad C = C_{11}(\tau). \end{gathered} \tag{2.5}$$

On the other hand, we have as a special case ($q = 11$, $\alpha = 1$) of Theorem 7,

$$U_{11}\Phi(\tau; 121) = x\prod_1^\infty (1 - x^m) \sum_{l=0}^\infty p(11l + 6)x^l, \quad x = \exp 2\pi i\tau. \tag{2.6}$$

This, together with (2.5), gives

$$\sum_{l=0}^\infty p(11l + 6)x^{l+1} = 11\prod_1^\infty (1 - x^{11m})^{-1}\{11AC^2 - 11^2C + 2AC - 32C - 2\}. \tag{2.7}$$

From (2.7) we deduce, by comparison of coefficients in the two members, the correctness of Ramanujan's conjecture (1.1) for $q = 11$.

The identity (2.5) is equivalent to (2.7) and is far more convenient, since it involves only modular functions. We shall obtain all identities in the form (2.5), the transition to the partition function then being made by analogues of (2.6).

I. The Identities.

3. Preliminaries. By the statement "$F(\tau)$ is an entire modular function on a subgroup Γ of the full modular group" we mean that F satisfies the following conditions:

(A) $F(\tau)$ is an analytic function of τ which is regular for $\Im\tau > 0$.

(B) $F(\tau)$ has at most polar singularities, measured in the uniformizing variable, at the parabolic vertices of Γ.

(C) For every substitution

$$V\tau = \frac{a\tau + b}{c\tau + d} \tag{3.1}$$

belonging to Γ, we have

$$F(V\tau) = F(\tau). \tag{3.2}$$

The subgroup of greatest interest to us is $\Gamma_0(q)$, defined by $c \equiv 0 \pmod q$ in (3.1), where q is a prime > 3. Because of the primality of q, the fundamental region of $\Gamma_0(q)$ has only two parabolic vertices, which we may take to be $\tau' = 0$, $\tau = i\infty$.

The uniformizing variable for $\tau = i\infty$ is evidently $x = \exp 2\pi i\tau$, since $\Gamma_0(q)$ contains the substitution $S\tau = \tau + 1$. For $\tau' = 0$, we use the substitution $\tau' = -1/q\tau$ and employ the transformation equation of (2.2). This will give us an expansion in x, so that x is a uniformizing variable also for $\tau' = 0$.[8]

Notation. τ is a complex variable with $\mathfrak{J}\tau > 0$. The point $\tau = i\infty$ is the infinite point of the τ-sphere considered as the limit of iy, $y \to +\infty$, $y > 0$. $x = \exp 2\pi i\tau$. q is a prime > 3. $S, T, V, W, \cdots$, are linear substitutions, $\tau' = (a\tau + b)/(c\tau + d)$, with integral a, b, c, d and determinant $ad - bc = 1$. $\Gamma(1)$ is the full modular group, consisting of all such substitutions. $\Gamma_0(n)$ is the subgroup of $\Gamma(1)$ defined by $c \equiv 0 \pmod n$. $O(x^s)$ denotes a power series in x with *integral* coefficients beginning with a term of order $\geq s$.

Formulae.

$$\prod_1^\infty (1 - x^m) = \sum_{-\infty}^{+\infty} (-1)^n x^{n(3n+1)/2} = 1 - x - x^2 + x^5 + x^7 - - + + \cdots. \tag{3.3}$$

$$\prod_1^\infty (1 - x^m)^{-1} = \sum_{n=0}^\infty p(n) x^n, \quad p(0) = 1. \tag{3.4}$$

$$\eta(\tau + 1) = \exp(\pi i/12) \cdot \eta(\tau), \qquad \eta(\tau + 24) = \eta(\tau). \tag{3.5}$$

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}} \eta(\tau), \tag{3.6}$$

where the square-root is positive when τ is purely imaginary.

For every modular substitution $V\tau = (a\tau + b)/(c\tau + d) \in \Gamma(1)$, with $c > 0$, we have

$$\eta(V\tau) = \exp\left\{ -\pi i\left(s(a,c) - \frac{a + d}{12c} \right) \right\} [-i(c\tau + d)]^{\frac{1}{2}} \eta(\tau), \tag{3.7}$$

---

[8] The substitutions of the group determine the form of the uniformizing variable. Thus, for $\Gamma_0(q)$, the uniformizing variable for $\tau = 0$ is $x' = \exp(-2\pi i/q\tau)$. It is only because the functions considered here have an additional property of the type (2.2) that $x = \exp 2\pi i\tau$ can be used in place of x', and it is convenient to do this.

where the sign of the square-root is determined as in (3.6), and where $s(a, c)$ is a "Dedekind sum." The Dedekind sums are fully discussed in [13].

(3.7) is the transformation formula of $\eta(\tau)$ and was discovered by Dedekind and Hermite. (3.6) is a special case of this. (3.5) is obvious from the definition (2.4) of $\eta(\tau)$, and the first two formulae are classical.

4. Modular functions of $\Gamma_0(q)$. Our first object will be to develop a set of modular invariants of $\Gamma_0(11)$. We shall, however, speak more generally of $\Gamma_0(q)$, q being always a prime > 3. We consider, first, modular *forms* of negative dimension, from which modular *functions* (zero dimension) will be obtained by division. Since we need *entire* modular functions, we must have one modular form which is zero-free in the interior of the fundamental region. Such a form is provided by

THEOREM 1. *The functions*

$$\psi_{q,r}(\tau) = \{\eta(q\tau)\eta(\tau)\}^r \tag{4.1}$$

where

$$r(q+1) \equiv 0 \pmod{24}, \tag{4.11}$$

are regular and zero-free in the upper half τ-plane. They have the transformation equations

$$\psi_{q,r}(V\tau) = (a/q)^r(c\tau+d)^r\psi_{q,r}(\tau), \quad V\tau \in \Gamma_0(q). \tag{4.12}$$

$$\psi_{q,r}(-1/q\tau) = (-iq^{1/2}\tau)^r\psi_{q,r}(\tau). \tag{4.13}$$

(a/q) is the Legendre-Jacobi symbol. It will be indicated in this way whenever confusion with the rational number a/q is not to be feared. Otherwise, we shall use the symbol $(a|q)$ or $\dfrac{a}{(q)}$.

Theorem 1 is proved in Part II.

A second set of modular forms, of dimension $-2r$, is given by the Eisenstein series [9, 10]

$$g_r(\tau) = \frac{(2r-1)!}{(2\pi)^{2r}} \sum_{m,n=-\infty}^{+\infty}{}' (m\tau+n)^{-2r}, \qquad r \geqq 2 \tag{4.21}$$

with the well-known property [11]

$$g_r(V\tau) = (c\tau+d)^{2r}g_r(\tau), \quad V\tau \in \Gamma(1). \tag{4.22}$$

---

[9] Σ' means in this case that the term $m = n = 0$ is to be omitted.

[10] These functions can be defined for $r = 1$, also, even though the series on the right is no longer absolutely convergent. See [7], p. 207

[11] [3], pp. 20-22.

They have the Fourier series[12]

(4.23) $$g_r(\tau) = B_r/2r + 2(-1)^r \sum_{n=1}^{\infty} \sigma_{2r-1}(n)x^n,$$

$$\sigma_l(n) = \sum_{d|n} d^l, \quad B_1 = 1/6, \quad B_2 = 1/30, \cdots.$$

The forms (4.21) belong to the full group $\Gamma(1)$ whereas we are interested in functions having the modular property on no group containing $\Gamma_0(q)$ as a proper subgroup. But consider

(4.31) $$G_{q,r}(\tau) = q^r g_r(q\tau) - g_r(\tau), \qquad r \geq 2.$$

Clearly

(4.32) $$G_{q,r}(V\tau) = (c\tau + d)^{2r} G_{q,r}(\tau), \quad V\tau \in \Gamma_0(q),$$

(4.33) $$G_{q,r}(-1/q\tau) = -(q^{1/2}\tau)^{2r} G_{q,r}(\tau).$$

Still another class of modular forms can be obtained from the multiple theta-series. We assume from now on $q \equiv 3 \pmod 4$. The function

(4.41) $$\theta_q(\tau) = \sum_{m,n=-\infty}^{+\infty} x^{Q(m,n)}, \quad Q = m^2 + mn + \frac{q+1}{4} n^2$$

is a modular form of dimension -1 with the properties[12]

(4.42) $$\theta(V\tau) = (c\tau + d)(a/q)\theta(\tau), \qquad (a/q) = \text{Legendre-Jacobi symbol},$$

(4.43) $$\theta(-1/q\tau) = -iq^{1/2}\tau\theta(\tau).$$

For higher dimensions we use either powers of θ_q or theta-series in more variables.[13]

From these modular forms we construct entire modular functions by dividing each form by a $\psi_{q,r}(\tau)$ of appropriate dimension. In particular, we obtain the functions of the following theorem.[14]

---

[12] A convenient method for treating the transformation of θ_q is developed in Hecke [8]. We first reduce Q to a sum of squares: $Q = \frac{1}{4}((2m+n)^2 + qn^2) = \frac{1}{4}(M^2 + qN^2)$, where $M \equiv N(2)$. Hence $Q = \mu\mu'/q$, where μ belongs to the principal ideal $(\sqrt{-q})$ in $K(\sqrt{-q})$. We then have $\theta_q(\tau) = \theta(\tau; 0, (1), \sqrt{-q})$ in Hecke's notation (p. 222). The relations (4.42), (4.43) of the text are equivalent to Sätze 7 and 5, respectively, of Hecke's paper. If $q \equiv 1 \pmod 4$, we must use quadratic forms of determinant $-4q$, rather than $-q$.

[13] The transformation of theta-series in more than two variables can be reduced to that of double theta-series. See [7], pp. 210-11.

[14] The functions which can be formed in this way from modular forms are not necessarily linearly independent. The number of linearly independent modular functions having at most a given valence can be determined from the Riemann-Roch Theorem applied to the Riemann surface which is the map of the fundamental region

THEOREM 2. *The functions*

$$(4.51)\qquad A_q(\tau)=\frac{\theta_q{}^r(\tau)}{\psi_{q,r}(\tau)}\,,\quad B_q(\tau)=\frac{G_{q,r}(\tau)}{\psi_{q,2r}(\tau)}$$

are entire modular functions belonging to the subgroup $\Gamma_0(q)$:

$$(4.52)\quad A_q(V\tau)=A_q(\tau),\quad B_q(V\tau)=B_q(\tau),\quad V\tau\in\Gamma_0(q),\quad q\equiv 3(4).$$

Furthermore,[15]

$$(4.53)\qquad A_q(-1/q\tau)=A_q(\tau),\qquad B_q(-1/q\tau)=-B_q(\tau).$$

The valences of A_q *and* B_q *are* $2s_q$ *and* $4s_q$, *respectively, where* $s_q=r(q+1)/24$.

We need to prove only the statements concerning valence. The valence of a modular function is defined to be the multiplicity of the root of the equation

$$F(\tau)=c,$$

where c is an arbitrary complex number and τ ranges over a point set consisting of the interior of the fundamental region and a complete set of inequivalent boundary points. The multiplicity must be measured in terms of the uniformizing variable. We may take $c=\infty$. Since $A_q(\tau)$ and $B_q(\tau)$ are entire modular functions, they can have poles only at the parabolic points $\tau'=0$, $\tau=i\infty$.

We can find the expansions of A_q and B_q from (4.1), (4.11), (2.4), (3.3), (3.4), (4.31), (4.23), and

$$(4.61)\qquad \theta_q(\tau)=\sum_{n=0}^{\infty}A_Q(n)x^n,\qquad x=\exp 2\pi i\tau,$$

where $A_Q(n)$ is the number of representations[16] of n by the form Q. Thus, for $\tau=i\infty$,

---

of $\Gamma_0(q)$. Since $\Gamma_0(11)$ is of genus one, it follows from the Riemann-Roch Theorem that there are $m-1+1=m$ independent functions on $\Gamma_0(11)$ of valence at most m if $m>1$. In particular, there are 2 independent 2-valent functions (since univalent functions do not exist on Riemann surfaces of positive genus 1) which may be taken as $A_{11}(\tau)$ and $A_{11}(\tau)+1$. Now, we can construct another 2-valent function, namely, $G_{11,1}(\tau)/\psi_{11,2}(\tau)=A'_{11}$, where $G_{11,1}=11g_1(11\tau)-g_1(\tau)$ (see footnote 10). Hence there must be a linear relation between A_{11} and A'_{11}. Indeed, we find by comparing power series that $6A'_{11}=5A_{11}-8$. Similar relations hold among the m-valent functions when $m>2$.

[15] Expressed otherwise, $A_q(\tau)$ is an automorphic function of the group $\Gamma(q)=\Gamma_0(q)+W\cdot\Gamma_0(q)$, $W=-1/q\tau$. Of course, $\Gamma(q)$ is not a subgroup of the modular group.

[16] Counted in the usual way, i. e., two representations are distinct if they differ only in the order or in the sign of the unsquared summands. For special $A_Q(l)$ explicit formulae are known: for $Q=m^2+mn+3n^2$, det. $Q=-11$, we have

$$(4.62)\quad A_q(\tau) = \exp\{-(\pi i\tau/12)r(q+1)\} + \cdots = x^{-s_q} + \cdots,$$

$$(4.63)\quad B_q(\tau) = \exp\{-(\pi i\tau/12)\cdot 2r(q+1)\} + \cdots = x^{-2s_q} + \cdots.$$

At the other parabolic point, we use $\tau' = -1/q\tau$ and (4.53), and have

$$(4.64)\quad A_q(\tau') = A_q(\tau) = x^{-s_q} + \cdots,\quad B_q(\tau') = -B_q(\tau) = -x^{-2s_q} + \cdots.$$

Hence the valences are $2s_q$ for A_q and $4s_q$ for B_q, as stated.

We conclude this section with a proof of the fact, noted in **2**, that $U_q\Phi(\tau; q^2)$ is an entire modular function of $\Gamma_0(q)$. We recall the definition

$$(4.71)\qquad U_q\Phi(\tau; q^2) = q^{-1}\sum_{\lambda=0}^{q-1}\Phi\left(\frac{\tau+\lambda}{q}; q^2\right).$$

Condition (A) in the definition of an entire modular function—**3**—is certainly fulfilled since $U_q\Phi$ is simply a sum of functions, each of which is regular in the upper half-plane.

In the sum of (4.71) we can replace the range of summation of λ by the equivalent condition: λ modulo q. This follows from the fact that Φ is periodic with period 1; replacing λ by $\lambda + q$ leaves each summand unaltered. Hence

$$(4.72)\qquad U_q\Phi(\tau; q^2) = q^{-1}\sum_{\lambda}\Phi\left(\frac{\tau+24\lambda}{q}; q^2\right),\qquad \lambda \bmod q.$$

Using (2.4), (3.3), (3.4), we find

$$(4.73)\qquad \Phi(\tau; q^2) = \frac{\eta(q^2\tau)}{\eta(\tau)} = x^u(1 + x + 2x^2 + \cdots)$$

$$= \sum_{n=u}^{\infty}\phi_n x^n,\quad u = \frac{q^2-1}{24},\qquad \phi_u \neq 0.$$

Introducing (4.73) in (4.72), we have

$$\begin{aligned} U_q\Phi(\tau; q^2) &= q^{-1}\sum_{\lambda, n}\phi_n \exp(2\pi i n(\tau + 24\lambda/q)),\quad \lambda \bmod q,\quad n \geq u \\ &= q^{-1}\sum_{n}\phi_n \exp(2\pi i(n\tau/q))\sum_{\lambda}\exp(2\pi i n(24\lambda/q)) \\ &= \sum_{m}\phi_m \exp 2\pi i(m\tau/q),\quad q\,|\,m. \end{aligned}$$

$$(4.74)\quad U_q\Phi(\tau; q^2) = \sum_{n=v}^{\infty}\phi_{qn}\exp(2\pi i n\tau) = \sum_{n=v}^{\infty}\phi_{qn}x^n,\quad v = [u/q] + 1.$$

$A_Q(l) = 2\{\sum_{d_1|l} d_1 - \sum_{d_2|l} d_2\}$, where $d_1 \equiv$ quad. residue, $d_2 \equiv$ quad. non-residue, (mod 11), and $l \not\equiv 0(11)$. See Dickson, *Introduction to the Theory of Numbers*, 1929, p. 81, ex. 12. For $l = 11k$, we use the substitution $m = M + 6N$, $n = -2M - N$, and deduce easily $A_Q(11k) = A_Q(k)$. Hence for $l = 11^\beta k_1$, $11 \nmid k_1$, $A_Q(l) = A_Q(k_1)$.

Thus for $\tau = i\infty$, $U_q\Phi$ has an expansion of the required type. The effect of U_q is simply to replace a_n by a_{qn}.

For $\tau' = 0$, we write $\tau' = T = -1/\tau$ and apply the transformation equations (3.6) and (3.7). First,

$$\begin{aligned} U_q\Phi(\tau; q^2) &= q^{-1}\sum_{\lambda} \eta(q(\tau + 24\lambda))\eta^{-1}(\tau + 24\lambda/q) \\ &= q^{-1}\eta(q\tau)\sum_{\lambda}\eta^{-1}(T_{24\lambda}\tau), \qquad \lambda \bmod q \end{aligned}$$

where $T_\lambda\tau = (\tau + \lambda)/q$ and where we have made use of (3.5). Hence,

$$qU_q\Phi(\tau'; q^2) = \eta(-q/\tau)\sum_{\lambda=0}^{q-1}\eta^{-1}(T_{24\lambda}T\tau) = \eta(-q/\tau)\{\eta^{-1}(-1/q\tau) + \sum_{\lambda=1}^{q-1}\eta^{-1}(T_{24\lambda}T\tau)\}. \tag{4.81}$$

Now

$$T_{24\lambda}T = V_\mu T_{24\mu}, \tag{4.82}$$

where

$$V_\mu = \begin{pmatrix} 24\lambda & -(24^2\lambda\mu + 1)/q \\ q & -24\mu \end{pmatrix} \tag{4.83}$$

$$24^2\lambda\mu \equiv -1 \pmod{q}, \qquad 0 < \mu < q. \tag{4.84}$$

We see from (4.84) that μ runs with λ over a reduced residue system modulo q. Using (4.82) in (4.81) and applying the transformation equation of $\eta(\tau)$, we get

$$\begin{aligned} qU_q\Phi(\tau'; q^2) &= q^{-1}\eta(\tau/q)\eta^{-1}(q\tau) + q^{-1/2}\eta(\tau/q)\sum_{\mu=1}^{q-1}\eta^{-1}(V_\mu T_{24\mu}\tau) \\ &= q^{-1}\Phi^{-1}(\tau/q; q^2) + q^{-1/2}\eta(\tau/q)\sum_{\mu=1}^{q-1}C_\mu\eta^{-1}(T_{24\mu}\tau), \end{aligned} \tag{4.85}$$

with.

$$C_\mu = \exp\{\pi i(s(24\lambda, q) - 2(\lambda - \mu)/q)\}.$$

A computation similar to [12], (5.4), shows that

$$C_\mu = i^{(1-q)/2}(6\mu/q) = \epsilon_1(\mu/q); \qquad \epsilon_1{}^4 = 1. \tag{4.86}$$

Hence, the sum in the last member of (4.85) is

$$\begin{aligned} \sum_{\mu} &= \epsilon_1 \sum_{n,\mu} \exp\left(-\frac{\pi i}{12}\,\frac{\tau + 24\mu}{q}\right)\cdot(\mu/q)p(n)\exp\left(2\pi i n\,\frac{\tau + 24\mu}{q}\right), \quad n \geq 0, \\ &\hphantom{=} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mu = 1, \cdots, q-1) \\ &= \epsilon_1 \exp(-\pi i\tau/12q)\sum_{n=0}^{\infty}p(n)\exp(2\pi i(n\tau/q))\sum_{\mu=0}^{q-1}(\mu/q) \\ &\qquad \times \exp\{2\pi i\cdot(24n - 1)\mu/q\} \text{—since } (0/q) = 0\text{—} \end{aligned}$$

$$= \epsilon_2 q^{1/2} \exp(-\pi i\tau/12q) \sum_{n=0}^{\infty} \left(\frac{24n-1}{q}\right) p(n) \exp(2\pi i(n\tau/q)), \qquad \epsilon_2^4 = 1$$

where we have made use of the Gaussian sum

$$(4.87) \qquad G(h,q) = \sum_{l} (l/q) \exp(2\pi i(hl/q)) = \epsilon_3 q^{1/2}(h/q), \qquad l \bmod q, \quad \epsilon_3^4 = 1.$$

We replace τ by $q\tau$ and obtain finally

$$\begin{aligned} q^2 U_q \Phi(\tau'; q^2) &= \Phi^{-1}(\tau; q^2) + \epsilon_2 \eta(\tau) \exp(-\pi i\tau/12)(1 + O(x)) \\ &= O(x^{-u}), \qquad \tau' = -1/q\tau, \end{aligned}$$

by (4.73) and (2.4). Hence, $U_q\Phi$ has a pole of order u in $x = \exp 2\pi i\tau$ for $\tau' = 0$, and condition (B) is satisfied.

That $U_q\Phi$ fulfills (C) with $\Gamma = \Gamma_0(q)$ is seen as follows. From Theorem 3, which is proved in Part II, we take the fact $(\alpha = 2)$ that

$$(4.91) \qquad \Phi(V\tau; q^2) = \Phi(\tau; q^2), \qquad V \in \Gamma_0(q^2).$$

We require also the following lemma, proved in Part II.

Lemma 1. *To any* $V \in \Gamma_0(q)$ *and* λ, $0 \leq \lambda < q$, *there exist* μ, $0 \leq \mu < q$, *and* $W_\mu \in \Gamma_0(q^2)$, *such that*

$$(4.92) \qquad T_\lambda V = W_\mu T_\mu, \qquad T_\lambda \tau = (\tau + \lambda)/q.$$

Furthermore, μ *runs with* λ *over a complete residue system modulo* q.

Therefore,

$$\begin{aligned} (4.93) \qquad U_q\Phi(V\tau; q^2) &= q^{-1} \sum_{\lambda} \Phi((V\tau + \lambda)/q; q^2), \qquad \lambda \bmod q \\ &= q^{-1} \sum_{\lambda} \Phi(T_\lambda V\tau; q^2) = q^{-1} \sum_{\mu} \Phi(W_\mu T_\mu \tau; q^2), \qquad \mu \bmod q \\ &= q^{-1} \sum_{\mu} \Phi(T_\mu \tau; q^2) = U_q\Phi(\tau; q^2), \end{aligned}$$

as required. This proves (C) and

Theorem 4. *The function* $U_q\Phi(\tau; q^2)$ *is an entire modular function of the group* $\Gamma_0(q)$.

5. A rational basis for $\Gamma_0(q)$. The next step is to find a basis for the entire modular functions of $\Gamma_0(q)$. For this purpose the following lemma will be useful.[17]

Lemma 2. *Let* $f_1(\tau)$, $f_2(\tau)$ *be modular functions on a group* Γ *having*

---

[17] For proof see [1], Th. 13, pp. 96-97.

the property that an arbitrary pair of values of the functions is assumed at but one point of each fundamental region of Γ, i. e., *the equations*

$$f_1(\tau)=f_1(\tau'), \qquad f_2(\tau)=f_2(\tau')$$

shall imply that τ *and* τ' *are congruent under* Γ. *Then every modular function of* Γ *is a rational function of* f_1 *and* f_2.

We are now ready to prove

THEOREM 5. *If the function* $A_q(\tau)$ *of* (4.51) *is 2-valent on* $\Gamma_0(q)$, *then the functions* $A_q(\tau)$, $B_q(\tau)$ *constitute a rational basis for* $\Gamma_0(q)$.

Proof. By Lemma 2 we have only to show that

$$A_q(\tau)=A_q(\tau'), \qquad B_q(\tau)=B_q(\tau') \tag{5.1}$$

is impossible if $\tau\not\equiv\tau'$ and τ,τ' lie in the same fundamental region of $\Gamma_0(q)$. Since $A_q(\tau)$ is 2-valent it follows from (5.1), (4.53) that $\tau'=-1/q\tau$. But then

$$B_q(\tau)=B_q(-1/q\tau),$$

which, in general, is inconsistent with (4.53), since $B_q(\tau)$ is not identically zero.[18]

As a special case we take $q=11\equiv 3 \pmod 4$, $n=2$, $s_{11}=1$, satisfying the conditions of Theorem 2. Then $A_{11}(\tau)$ is 2-valent on $\Gamma_0(11)$ and we have

COROLLARY.[19] *The functions* $A_{11}(\tau)$, $B_{11}(\tau)$ *form a rational basis for the entire modular functions of* $\Gamma_0(11)$.

Note that

$$\operatorname{val} A_{11}=2, \qquad \operatorname{val}' B_{11}=4. \tag{5.2}$$

6. An identity for modulus eleven. Since $U_{11}\Phi(\tau;11^2)$ is an entire modular function of the group $\Gamma_0(11)$ (Theorem 4), it is a rational function of $A_{11}(\tau)$, $B_{11}(\tau)$:

$$U_{11}\Phi=P(A_{11},B_{11}). \tag{6.1}$$

It will turn out, however, that P is a polynomial. That is, we assume P as a polynomial with undetermined coefficients, and then show how to determine these coefficients so that the difference $U_{11}\Phi-P$, which is an entire modular function, has no poles. Then it must be a constant, which can be evaluated at $\tau = i\infty$, say.

---

[18] This argument shows, incidentally, that $B_q(\tau)$ must vanish at the fixed point of the substitution $\tau'=-1/q\tau$, and indeed at the fixed points of all elliptic substitutions of the group $\Gamma(q)$ (see footnote 15).

[19] See [2], p. 406, (12). Fricke denotes $A(\tau)$ by $\tau(\omega)$ and $B(\tau)$ by $\sigma(\omega)$.

The expansions of $U_{11}\Phi$ at the parabolic points were worked out in the proof of Theorem 4. We repeat them here.

$$U_{11}\Phi(\tau; 11^2) = O(x) \tag{6.2}$$

$$11^2 U_{11}\Phi(\tau'; 11^2) = x^{-5} - x^{-4} - x^{-3} + O(1), \qquad \tau' = -1/11\tau, \tag{6.3}$$

where we have used the expansion

$$\Phi^{-1}(\tau; 11^2) = \frac{\eta(\tau)}{\eta(121\tau)} = x^{-5}(1 - x - x^2 + x^5 + \cdots) \cdot \sum_{n=0}^{\infty} p(n) x^{121n},$$

calculated from (2.4), (3.3), (3.4).

We are now in a position to compare the principal parts of the two members of (6.1). At $\tau = i\infty$, $U_{11}\Phi$ has a zero of the first order, whereas $A_{11} = A$ and $B_{11} = B$ both have poles, by (4.62), (4.63). It will therefore be convenient to construct a new function, $C_{11}(\tau) = C_{11}$, which has a zero of the first order at $\tau = i\infty$. We wish C to have the lowest possible valence, i. e., 2, since the fundamental region of $\Gamma_0(11)$, considered as a closed Riemann surface, is of genus one. In other words, C must have a double pole at $\tau' = 0$. If we add the requirement that the expansion of C at $\tau = i\infty$ shall begin with x, the function is then determined, by the Riemann-Roch Theorem.[20]

From the expansions—see the table on p. 514—

$$A(\tau) = x^{-1} + 6 + 17x + 46x^2 + \cdots \tag{6.41}$$

$$B(\tau) = x^2 + 2x^{-1} - 12 - 116x - 597x^2 + \cdots, \tag{6.42}$$

we find

$$C(\tau) = 2^{-1} \cdot 11^{-2}(A^2(\tau) - 10A(\tau) - 22 - B(\tau)). \tag{6.43}$$

Its expansion at $\tau = i\infty$ is

$$C(\tau) = x + 5x^2 + \cdots = \sum_{n=1}^{\infty} c_n x^n. \tag{6.44}$$

From the last relation and (4.53), we see that

$$\begin{aligned} 2 \cdot 11^2 C(\tau') &= A^2(\tau') - 10A(\tau') - 22 - B(\tau'), \quad \tau' = -1/11\tau \\ &= A^2(\tau) - 10A(\tau) - 22 + B(\tau). \end{aligned} \tag{6.45}$$

---

[20] The number of linearly independent functions having a pole of second order at $\tau = 0$ is $2 - 1 + 1 = 2$. The most general such function contains two arbitrary constants, one of which is used to insure the zero at $\tau = i\infty$, the other to guarantee the coefficient 1 for the linear term.

This gives[21]

$$(6.46) \qquad 11^2C(\tau') = x^{-2} + 2x^{-1} + \cdots = \sum_{n=-2}^{\infty} c'_n x^n.$$

LEMMA 3. *The coefficients c_n, c'_n of* (6.44), (6.46) *are rational integers.*

Proof. Since A, B both belong to $\Gamma_0(11)$, they are connected by an algebraic relation, which is of degree 2 in B and of degree 4 in A, in accordance with the valences (5.2). By writing such a relation with undetermined coefficients and then using the expansions (6.41), (6.42) we find easily

$$(6.51) \qquad B^2 = A^4 - 20A^3 + 56A^2 - 44A.$$

From[21] (6.43)

$$\begin{aligned}(2\cdot 11^2C + B)^2 &= (A^2 - 10A - 22)^2 = A^4 - 20A^3 + 56A^2 - 44A + 484(\\ &\equiv B^2 \pmod{2^2\cdot 11^2},\\ 2\cdot 11^2C + B &\equiv B \pmod{2\cdot 11^2}\\ \text{or} \quad 2\cdot 11^2C &\equiv 0 \pmod{2\cdot 11^2}.\end{aligned}$$

Thus the c_n are integers.

Again, let $C(-1/11\tau) = C'$, etc. Then, from (6.45),

$$\begin{gathered}2\cdot 11^2C' - B = A^2 - 10A - 22,\\ (2\cdot 11^2C' - B)^2 = (A^2 - 10A - 22)^2 \equiv B^2 \pmod 4,\\ 2\cdot 11^2C' - B \equiv B \pmod 2, \quad 11^2C' \equiv B \pmod 1.\end{gathered}$$

This proves the lemma.[22]

For future reference we include another lemma.

LEMMA 4. $AC - 1 \equiv 0 \pmod{11}$.

Proof. First,

$$22A^5 - 55A^4 + 44A^3 \equiv 0 \pmod{11}$$

or

$$\begin{aligned}A^4(A^2 + 2A + 1) &\equiv A^2(A^4 - 20A^3 + 56A^2 - 44A) \pmod{11}\\ &\equiv A^2B^2 \pmod{11}.\end{aligned}$$

---

[21] In what follows we employ the following simple theorem: Let $F = x^{-l} + \Sigma a_\nu x^\nu$, $G = x^{-l} + \Sigma b_\nu x^\nu$, $\nu > -l$; a_ν *and* b_ν *are integers. Then if* $F^2 \equiv G^2 \pmod{k}$, k *odd, we have* $F \equiv G \pmod{k}$; *if* $F^2 \equiv G^2 \pmod{2^\lambda}$, $\lambda > 0$, *then* $F \equiv G \pmod{2^{\lambda-1}}$. In general, we would have $F \equiv \pm G$. The correct sign can be determined from the first terms of the power series for F and G.

[22] The functions C and $C' = C(-1/11\tau)$ do not satisfy so simple a transformation equation as A and B do. However, this equation can be found easily by eliminating A and B from the three relations (6.43), (6.45), and (6.51). The result is

$$11^3C^2C'^2 - 12.11CC' - 11(C + C') - 1 = 0.$$

Hence,

$$A^2(A+1) \equiv AB \pmod{11}$$
$$2^2 \cdot 11^2 A^2(A+1) \equiv 2^2 \cdot 11^2 AB \pmod{2^2 \cdot 11^3}$$
$$A^2(B^2 + 2^2 \cdot 11^2(A+1)) \equiv 2^2 \cdot 11^4 + 2^2 \cdot 11^2 AB + A^2B^2 \pmod{2^2 \cdot 11^3}$$
$$A^2(A^2 - 10A - 22)^2 \equiv (2 \cdot 11^2 + AB)^2 \pmod{2^2 \cdot 11^3}$$

if we use the computation in the lines following (6.51). Hence, taking note of footnote [21], we obtain

$$A(A^2 - 10A - 22) \equiv 2 \cdot 11^2 + AB \pmod{2 \cdot 11^3}$$
$$A(A^2 - 10A - 22 - B) \equiv 2 \cdot 11^2 \pmod{2 \cdot 11^3},$$

and by (6.43)

$$2 \cdot 11^2 AC \equiv 2 \cdot 11^2 \pmod{2 \cdot 11^3},$$

which completes the proof.

Following the procedure outlined at the beginning of this section, we set $U_{11}\Phi$ equal to a polynomial P in A and C with undetermined coefficients. The poles of even order in the expansion (6.3) of $U_{11}\Phi$ at $\tau' = 0$ can be eliminated by powers of C, those of odd order by A times a power of C. Since both $U_{11}\Phi$ and P are bounded at $\tau = i\infty$, it is clear that the term in x^{-1} (at $\tau' = 0$) must drop out automatically. Otherwise, the difference $U_{11}\Phi - P$ would be an entire modular function having a single pole of the first order on $\Gamma_0(11)$, a subgroup of genus one.

We therefore write

$$11^2 U_{11}\Phi(-1/11\tau; 11^2) = x^{-5} - x^{-4} - x^{-3} + O(1)$$
$$= a_{12}AC'^2 + a_{02}C'^2 + a_{11}AC' + a_{01}C' + a_{00}, \qquad C' = C(-1/11\tau)$$
$$11^2 U_{11}\Phi(\tau; 11^2) = O(x) = a_{12}AC^2 + a_{02}C^2 + a_{11}AC + a_{01}C + a_{00}.$$

This gives us the following five equations, which are solved seriatim beginning with the uppermost.

$$(6.52)\qquad \begin{array}{llllll} & a_{12} \cdot 11^{-4} & & & & = 1 \\ & a_{12} \cdot 10 \cdot 11^{-4} & + a_{02} \cdot 11^{-4} & & & = -1 \\ & a_{12} \cdot 21 \cdot 11^{-4} & + a_{02} \cdot 4 \cdot 11^{-4} & + a_{11} \cdot 11^{-2} & & = -1 \\ - & a_{12} \cdot 4 \cdot 11^{-3} & - a_{02} \cdot 20 \cdot 11^{-4} & + a_{11} \cdot 8 \cdot 11^{-2} & + a_{01} \cdot 11^{-2} & = 0 \\ & & & a_{11} & + a_{00} & = 0 \end{array}$$

The values of the a's are: $a_{12} = 11^2$, $a_{02} = -11^3$, $a_{11} = 2 \cdot 11$, $a_{01} = -32 \cdot 11$, $a_{00} = -2 \cdot 11$.

This constitutes the proof of

THEOREM 6. *For* $q=11$, *there is a "Ramanujan identity"*

$$U_{11}\Phi(\tau;121)=11(11AC^2-11^2C+2AC-32C-2). \tag{6.6}$$

Ramanujan's congruence (1.3) *with* $q^\alpha=11$ *follows from* (6.6) *and* (2.6), *which together yield* (2.7).

7. An induction. We now wish to establish identities of the same sort as (6.6) but involving higher powers of 11. But it is not yet clear what functions, analogous to $U_{11}\Phi(\tau;121)$, should appear in the left member of such an identity. Our guide in this respect will be formula (2.6), which connects the partition function with $U_{11}\Phi$. The first step, therefore, will be an appropriate generalization of this formula.

THEOREM 7. *Let*

$$L(\tau;1)=1, \tag{7.11}$$

$$L(\tau;q^{2\beta+1})=U_q\{\Phi(\tau;q^2)\cdot L(\tau;q^{2\beta})\},\qquad \beta\geq 0 \tag{7.12}$$

$$L(\tau;q^{2\beta+2})=U_qL(\tau;q^{2\beta+1}),\qquad \beta\geq 0. \tag{7.13}$$

Then for $\alpha\geq 1$,

$$L(\tau;q^\alpha)=x^t\prod_1^\infty(1-x^{\epsilon m})\sum_{l=0}^\infty p(q^\alpha l+\rho)x^l,\qquad x=\exp 2\pi i\tau, \tag{7.2}$$

where [23]

$$t=t(q^\alpha)=1 \ \textit{for}\ q<24, \tag{7.21}$$

$$\epsilon=\begin{cases}1, & \alpha\ \textit{even}\\ q, & \alpha\ \textit{odd}\end{cases} \tag{7.22}$$

and

$$24\rho\equiv 1\ (\mathrm{mod}\ q^\alpha),\qquad 0<\rho<q^\alpha. \tag{7.23}$$

(2.6) is the case $\alpha=1$ of (7.2).

Proof. We replace (7.2) by two statements,

$$L(\tau;q^\alpha)=q^{-\alpha}\sum_\lambda\eta(\epsilon\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q^\alpha}\right),\qquad \lambda \bmod q^\alpha, \tag{7.31}$$

$$x^t\prod_1^\infty(1-x^{\epsilon m})\sum_{l=0}^\infty p(q^\alpha l+\rho)x^l=q^{-\alpha}\sum_\lambda\eta(\epsilon\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q^\alpha}\right), \tag{7.32}$$

and proceed to prove the first. Suppose $\alpha=1$; then $\beta=0$, and (7.31) reduces to

$$U_q\Phi(\tau;q^2)=q^{-1}\sum_\lambda\eta(q\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q}\right),\qquad \lambda \bmod q. \tag{7.4}$$

---

[23] Actually, the value of t is 1 if α is even and $[q/24]+1$ if α is odd.

Now (see the lines following (4.71))

$$U_q\Phi(\tau;q^2) = q^{-1}\sum_\lambda \Phi\left(\frac{\tau+24\lambda}{q};\,q^2\right), \qquad \lambda \bmod q,$$

$$= q^{-1}\sum_\lambda \eta(q(\tau+24\lambda))\eta^{-1}\left(\frac{\tau+24\lambda}{q}\right) = q^{-1}\sum_\lambda \eta(q\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q}\right),$$

where we have used (3.5). This is (7.4).

Suppose now that (7.31) has been proved for $\alpha = 2\beta$. From (7.12) we have

$$L(\tau;q^{2\beta+1}) = U_q\left\{\frac{\eta(q^2\tau)}{\eta(\tau)}\cdot q^{-2\beta}\sum_\lambda \eta(\tau)\eta^{-1}\left(\frac{\tau+24\lambda}{q^{2\beta}}\right)\right\}, \qquad \lambda \bmod q^\alpha$$

$$= q^{-2\beta}U_q\left\{\eta(q^2\tau)\sum_\lambda \eta^{-1}\left(\frac{\tau+24\lambda}{q^{2\beta}}\right)\right\}$$

$$= q^{-2\beta-1}\sum_\mu \eta\left(q^2\,\frac{\tau+24\mu}{q}\right)\sum_\lambda \eta^{-1}\left(\left(\frac{\tau+24\mu}{q}+24\lambda\right)\Big/q^{2\beta}\right)$$

$$= q^{-2\beta-1}\eta(q\tau)\sum_\nu \eta^{-1}\left(\frac{\tau+24\nu}{q^{2\beta+1}}\right), \qquad \nu \bmod q^{2\beta+1},$$

which establishes the inductive step from 2β to $2\beta+1$. The step from $2\beta+1$ to $2\beta+2$ is accomplished in the same way. Thus (7.31) is proved.

To demonstrate (7.32) we employ the formula

$$\frac{1}{\eta(\tau)} = \exp\left(-\frac{\pi i\tau}{12}\right)\cdot\sum_{n=0}^{\infty} p(n)x^n \tag{7.5}$$

which follows immediately from (2.4) and (3.4). Introducing this in the right member of (7.32) we see that it becomes

$$q^{-\alpha}\eta(\epsilon\tau)\sum_\lambda \exp\left(-\frac{\pi i}{12}\cdot\frac{\tau+24\lambda}{q^\alpha}\right)\sum_{n=0}^{\infty} p(n)\exp\left(2\pi i n\,\frac{\tau+24\lambda}{q^\alpha}\right), \quad \lambda \bmod q^\alpha$$

$$= q^{-\alpha}\eta(\epsilon\tau)\exp\left(-\frac{\pi i\tau}{12q^\alpha}\right)\sum_n p(n)\exp\left(2\pi i\frac{n\tau}{q^\alpha}\right)\sum_\lambda \exp\left(2\pi i\frac{\lambda}{q^\alpha}(24n-1)\right)$$

$$= \exp\left(-\frac{\pi i\tau}{12q^\alpha}+\frac{\pi i\epsilon\tau}{12}\right)\prod_1^\infty (1-x^{\epsilon m})\sum_m p(m)\exp\left(2\pi i\,\frac{m\tau}{q^\alpha}\right),$$

$$24m \equiv 1 \pmod{q^\alpha},$$

from which (7.32) follows readily with

$$t = \frac{24p-1+\epsilon q^\alpha}{24q^\alpha}. \tag{7.6}$$

Clearly t is an integer. For, $\epsilon q^\alpha = q^{2\beta} = (q^\beta)^2 \equiv 1 \pmod{24}$ for $\alpha = 2\beta$, and $\epsilon q^\alpha = q^{\alpha+1} = q^{2\beta+2} = (q^{\beta+1})^2 \equiv 1 \pmod{24}$ for $\alpha = 2\beta+1$; moreover,

q^{α} divides $24\rho - 1$ by (7.23). Since $0 < \rho < q^{\alpha}$ and since we assumed in (7.21) that $q < 24$, we find that

$$0 < t < 1 + q/24 < 2;$$

therefore, as an integer, $t = 1$. This completes the proof of the theorem.

Theorem 7 shows us that we must attempt to construct identities in which one member is $L(\tau; 11^{\alpha})$ while the other is a polynomial in A and C. Obviously we shall proceed by induction, in view of the recursive definition (7.12), (7.13) of the L's. The first step of the induction ($\alpha = 1$, $\beta = 0$) has already been proved; it is Theorem 6.

The nature of the inductive step from α to $\alpha + 1$ will depend upon the parity of α, as the equations (7.12), (7.13) clearly show.[24] However, we can bring these two cases together. Suppose that we have an identity

$$L(\tau; 11^{\alpha}) = P(A(\tau), C(\tau)), \tag{7.7}$$

where P is a polynomial in A and C with rational coefficients, in which A appears at most linearly and always multiplied by a power of C. In order to obtain $L(\tau; 11^{\alpha+1})$, either we apply U_{11} directly to P or we first multiply by $\Phi(\tau; 11^2)$ and then operate with U_{11}. Since U_{11} is linear we may apply it to each term of P or ΦP separately. Thus the most general function which we need to consider is

$$G(\tau; i, j, k) = U_{11}\Phi^{i}(\tau; q^2)A^{j}(\tau)C^{k}(\tau), \qquad i = 0 \text{ or } 1, \quad j = 0 \text{ or } 1, \quad k \geq j. \tag{7.8}$$

The induction will be completely established if we prove that $G(\tau; i, j, k)$ *can be expressed as a polynomial of type* $P(A(\tau), C(\tau))$ *in* (7.7).

The procedure will parallel that used in the case $\alpha = 1$. The whole proof consists in showing that G is an entire modular function of $\Gamma_0(11)$. The expansion of G at $\tau = i\infty$ will not contain any terms with negative exponents. By examining the proof of Theorem 6 we see readily that any such function can be expressed in the form of a polynomial of the desired type.

8. Identities for powers of eleven.

THEOREM 8. *Let* $G(\tau; i, j, k)$ *be defined by* (7.8). *Then* $G(\tau; i, j, k)$ *is an entire modular function of* $\Gamma_0(11)$. *Moreover, we have*

$$G(\tau; i, j, k) = P(A, C)$$

where P *is of the same type as in* (7.7).

---

[24] See [12], pp. 624-625.

Proof. We must show that conditions (A)-(C) of **3** are fulfilled. This is obvious for (A) since all the functions involved in G are regular in the upper τ-half-plane and U_{11} is simply a sum of such functions.

We now consider the expansions of G at the parabolic vertices. At $\tau = i\infty$, we have

$$(8.1) \qquad \Phi^i(\tau; 11^2) = O(x^{5i}), \qquad A^j(\tau) = O(x^{-j}), \qquad C^k(\tau) = O(x^k),$$

by (4.73), (6.41), (6.44). Then

$$\Phi^i A^j C^k(\tau) = O(x^w) = \sum_{n=w}^{\infty} d_n x^n, \qquad w \geq 0.$$

Hence, as we saw in (4.74),

$$(8.2) \qquad G(\tau; i, j, k) = U_{11}\Phi^i A^j C^k = \sum_{n=w'}^{\infty} d_{11n} x^n, \qquad w' \geq 1.$$

For $\tau' = 0$ we write $\tau' = T\tau = -1/\tau$, and have

$$\begin{aligned} G(\tau'; i, j, k) &= 11^{-1} \sum_{\lambda=0}^{10} \Phi^i A^j C^k (T_\lambda T\tau) \\ &= 11^{-1} \sum_{\lambda=0}^{10} \Phi^i A^j C^k (T_{24\lambda} T\tau), \qquad T_\lambda \tau = \frac{\tau + \lambda}{q} \\ &= 11^{-1} \Phi^i A^j C^k (-1/11\tau) + 11^{-1} \sum_{\lambda=1}^{10} \Phi^i A^j C^k (T_{24\lambda} T\tau) \\ &= 11^{-1} \Phi^i A^j C^k (-1/11\tau) + 11^{-1} \sum_{\mu=1}^{10} \Phi^i A^j C^k (V_\mu T_{24\mu}\tau), \end{aligned}$$

if we recall (4.82). The functions A and C are invariant under V_μ, which belongs to $\Gamma_0(11)$, but $\Phi(T_{24\mu}\tau)$ picks up the factor

$11^{-1/2} C_\mu = 11^{-1/2} i (\mu/11)$; where $(\mu/11)$ is the Legendre-Jacobi symbol,

as is seen if we put $q = 11$ in (4.81)-(4.86). This gives

$$(8.31) \qquad 11G(\tau') = \Phi^i A^j C^k (-1/11\tau) + 11^{-1/2} i \sum_{\mu=1}^{10} (\mu/11) \Phi^i A^j C^k (T_{24\mu}\tau).$$

We write

$$(8.32) \qquad \Phi^i(\tau; 11^2) = \sum_{n=5i}^{\infty} \phi_n x^n; \quad A^j(\tau) = \sum_{n=-j}^{\infty} a_n x^n, \quad C^k(\tau) = \sum_{n=k}^{\infty} c_n x^n,$$

where the coefficients are integers. Then

$$\begin{aligned} 11G(\tau') &= \Phi^i A^j C^k (-1/11\tau) \\ &\quad + 11^{-1/2} i \sum_{\mu, m, n, p} (\mu/11) \phi_m a_n c_p \exp \left\{ 2\pi i (m + n + p) \left(\frac{\tau + 24\mu}{11} \right) \right\}. \end{aligned}$$

The sum in the right member is

$$\begin{aligned} &\sum_{m,n,p} \phi_m a_n c_p \exp\left(2\pi i (m+n+p)(\tau/11)\right) \sum_{\mu=0}^{10} (\mu/11) \exp\{2\pi i (m+n+p)(24\mu/11)\} \\ &\qquad = \sum_{m,n,p} \phi_m a_n c_p \cdot \left(\frac{(m+n+p)24}{11} \right) 11^{\frac{1}{2}} \exp\{2\pi i (m+n+p)(\tau/11)\}, \end{aligned}$$

where we have evaluated a Gaussian sum, (4.87). Using (8.32) and replacing τ by 11τ, we now have $\tau' = -1/11\tau$,

$$(8.4) \qquad 11^{i+2k+1}G(\tau') = O(x^{-5i-11j-22k}) + O(x^{5i-j+k}) = O(x^{-5i-11j-22k}).$$

(8.2) and (8.4) show that G has only polar singularities in $x = \exp 2\pi i\tau$ at the parabolic vertices.

It remains to prove that

$$(8.5) \qquad G(V\tau) = G(\tau), \qquad V \epsilon \Gamma_0(11).$$

Here we require (4.92) of Lemma 1.

$$\begin{aligned} G(V\tau) &= 11^{-1}\sum_{\lambda=0}^{10}\Phi^i A^j C^k(T_\lambda V\tau), \qquad && V \epsilon \Gamma_0(11) \\ &= 11^{-1}\sum_{\mu=0}^{10}\Phi^i A^j C^k(W_\mu T_\mu\tau), && W_\mu \epsilon \Gamma_0(11^2) \\ &= 11^{-1}\sum_{\mu=0}^{10}\Phi^i A^j C^k(T_\mu\tau) = G(\tau), \end{aligned}$$

since all three functions are invariant under $\Gamma_0(11^2)$. This completes the proof of the first statement of Theorem 8.

That G can be expressed as a polynomial in A and C now follows in the same way as in the proof of Theorem 6. We set G equal to a polynomial

$$G = \Sigma a_{jk}A^jC^k, \qquad j = 0 \text{ or } 1, \quad k \geq j,$$

with undetermined a_{jk}. Then, equating coefficients of like powers of x in the expansions at the parabolic points, we obtain a set of linear equations analogous to (6.52). The first equation will involve only one coefficient, and each succeeding equation will add one new coefficient. Hence, the coefficients are determined uniquely. They are obviously rational numbers, since the expansions of all functions involved have rational coefficients. This establishes Theorem 8.

Making use of the remark in the lines following (7.8), we see that we have proved

THEOREM 9. *For every integer $\alpha \geq 1$, there is a "Ramanujan identity" of the form*

$$(8.6) \qquad L(\tau; 11^\alpha) = P(A, C),$$

where L is defined in (7.12), (7.13), *$A = A_{11}$ in* (4.51), *and C in* (6.43). *P is a polynomial in A and C with rational coefficients.*

By using Theorem 7 we can put (8.6) in the form

$$(8.7)\qquad \sum_{l=0}^{\infty} p(q^{\alpha}l+\rho)x^{l+1}=P(A,C)\prod_{1}^{\infty}(1-x^{\epsilon m})^{-1},$$

with the ρ, ϵ of (7.23), (7.22). Unfortunately, we have not been able to prove the *integrality* of the coefficients of P except when $\alpha=1,2$. Thus, for higher values of α, (8.7) gives no information concerning Ramanujan's congruence. The discussion for $\alpha=2$ is given in the next section.

The following remark may be of interest. If $F(\tau)$ is an entire modular function of $\Gamma_0(q)$ and we subject U_qF to the transformation $\tau'=-1/q\tau$, we find, following the indications already given,

$$(8.81)\qquad qU_qF(-1/q\tau)-qU_qF(q\tau)=F(-1/q^2\tau)-F(\tau).$$

If, in addition, $F(\tau)$ has expansions in the *same* uniformizing variable (say $x=\exp 2\pi i\tau$) at both parabolic points, then U_qF will be entire modular on $\Gamma_0(q)$. Now suppose that $\Gamma_0(q)$ is a subgroup of genus zero. This is true for $q=5,7,13$. Then there exists a univalent function

$$(8.82)\qquad \Phi_{q,r}(\tau)=\left\{\frac{\eta(q\tau)}{\eta(\tau)}\right\}^{r}$$

where $r(q-1)\equiv 0\ (24)$ and r is minimal. Moreover,

$$(8.83)\qquad \Phi_{q,r}(-1/q\tau)=q^{-r/2}\Phi_{q,r}^{-1}(\tau).$$

$F(\tau)$ and $U_qF(\tau)$ are polynomials in $\Phi_{q,r}(\tau)$, and this is true also of $F(\tau')$ and $U_qF(\tau')$, by (8.83). Substituting these expressions in (8.81) we obtain an algebraic equation connecting $\Phi_{q,r}(\tau)$ and $\Phi_{q,r}(q\tau)$. If this is irreducible, it is the *modular* equation of $\Phi_{q,r}(\tau)$. For the primes mentioned, these equations have been discussed by Rademacher [12].

Another approach to the modular equation is given in a forthcoming paper of the author [10].

9. The Ramanujan congruence for $\alpha=2$. For $\alpha=2$ we can determine the coefficients of P in (8.6) to an extent sufficient to establish the Ramanujan congruence. (6.6) can be written

$$L(\tau;11)=11^2(AC^2-11C^2)+11(2(AC-1)-32C).$$

Also, by (7.13),

$$L(\tau;11^2)=U_{11}L(\tau;11).$$

The application of U_{11} to a power series simply replaces each coefficient a_n by a_{11n}. It is clear, then, that

$$L(\tau;11^2)\equiv 2\cdot 11U_{11}(AC-1)+11U_{11}C\ (\mathrm{mod}\ 11^2).$$

But by Lemma 4, each coefficient of $AC-1$ is a multiple of 11; hence,

$$L(\tau;11^2)\equiv 11U_{11}C \pmod{11^2}.$$

By (7.2) Ramanujan's congruence for $q^a=11^2$ is equivalent to

$$L(\tau;11^2)\equiv 0 \pmod{11^2}.$$

Therefore, in order to prove Ramanujan's congruence for the modulus 11^2, *it is sufficient to show that*

$$U_{11}C\equiv 0 \pmod{11}. \tag{9.1}$$

This was done by a direct computation of the coefficients of the polynomial

$$U_{11}C(\tau)=Q(A(\tau),\ C(\tau))=\sum_{j,k} b_{jk}A^j(\tau)C^k(\tau). \tag{9.2}$$

Since $U_{11}C$, A, and C have expansions (in $x=\exp 2\pi i\tau$) with integral coefficients, the statement (9.1) reduces to

$$b_{jk}\equiv 0 \pmod{11}. \tag{9.21}$$

It is therefore sufficient to determine the b_{jk} modulo 11.

The coefficients b_{jk} will be determined by considering the expansions at the parabolic point $\tau'=0$, except for b_{00}, which is to be determined at $\tau=i\infty$. We therefore replace b_{00} by g, a constant which may vary with each occurrence, and determine g last.

The expansion of $U_{11}C$ at $\tau'=0$ is, by (8.81) and (6.46),

$$11^3U_{11}C(\tau')=x^{-22}+2x^{-11}+O(1),\qquad \tau'=-1/11\tau \tag{9.31}$$

and this we set equal to

$$g+11^3\Sigma b_{jk}A^j(\tau')C^k(\tau')=g+11^3\Sigma b_{jk}A^j(\tau)C^k(\tau'), \tag{9.32}$$
$$j=0 \text{ or } 1,\quad k\geq j,\quad j+k\neq 0.$$

Putting $11^2C(\tau')=E(\tau)$ and $b_{jk}=c_{jk}11^{2k-3}$, we get

$$11^3U_{11}C(\tau')=g+\sum_{j,k}c_{jk}A^j(\tau)E^k(\tau). \tag{9.4}$$

We now see that (9.21) will be completely proved if we can show that

$$c_{j1}\equiv 0 \pmod{11^2},\qquad j=0,1. \tag{9.41}$$

(The constant, g, will be determined separately.) This is true since $E(\tau)$ has an expansion in x with integral coefficients.

Clearly the term of highest degree in E in (9.4) will be E^{11}. We shall therefore need the power series of E and A to 22 terms. To compute E we have from (6.45)

$$2E = A^2 - 10A - 22 + B. \tag{9.5}$$

Since we must compute c_{jk} modulo 11^2, we shall need A and B modulo $2 \cdot 11^2$.

The calculation may be carried out as follows (see the table on p. 514). The coefficients of A are computed from its definition (4.51)

$$A = A_{11} = \frac{\theta_{11}{}^2(\tau)}{\psi_{11,2}(\tau)}, \qquad \psi_{11,2}(\tau) = \eta^2(\tau)\eta^2(11\tau).$$

First, $\eta^{-1}(\tau)\eta^{-1}(11\tau)$ is obtained from (3.4); then this is squared. Next, the coefficients of θ are calculated from (4.61) and footnote [16], and the resulting series squared. From these two series we obtain A and A^2 by multiplication and squaring.

The definition (4.51) of B is

$$B = B_{11} = \frac{G_{11,2}(\tau)}{\psi_{11,4}(\tau)}.$$

$\psi_{11,4}^{-1}$ is simply the square of $\psi_{11,2}^{-1}$. $G_{11,2}$ is defined by (4.31) and (4.23). By multiplying the last two series we get the coefficients of B.

The computation for each series (except that for θ^2 which has very small coefficients) was checked by recomputing the coefficients modulo $2 \cdot 11^3$.

Now, using (9.5), we obtain the coefficients of E, reduced modulo 11^2. We raise E to the eleventh power by the formula $E^{11} = E \cdot E \cdot E^2 \cdot E^4 E^3$, working modulo 11^2. In order to have a check on the coefficients of E^{11}, we observe that, by an elementary theorem on binomial coefficients, if

$$E(\tau) = x^{-2} \sum_{\nu=0}^{\infty} e_\nu x^\nu,$$

then

$$E^{11}(\tau) \equiv x^{-22} \sum_{\nu=0}^{\infty} e_\nu x^{11\nu} \pmod{11}.$$

Therefore,

$$\begin{aligned} E^{11}(\tau) \equiv E(11\tau) &= 11^2 C(-1/121\tau) \\ &= x^{-22} + 2x^{-11} + O(1) \pmod{11}, \end{aligned} \tag{9.6}$$

where we have taken the last expansion from (6.46) with τ replaced by 11τ. (9.6) shows that all the computed coefficients e'_ν of E^{11} must be multiples of 11 except $e'_{-22} \equiv 1 \pmod{11}$ and $e'_{-11} \equiv 2 \pmod{11}$. As a further check we computed E^{12} (modulo 11) in two ways:

$$E^{12} = E \cdot E^{11} = E^3 \cdot E^3 \cdot E^6.$$

Residues of the Coefficients of Certain Power Series

Power	Function								
of x	$x\psi^{-1}_{11,2}$	θ^2	x^2A	x^2A^2	$x^2\psi^{-1}_{11,4}$	$G_{11,2}$	x^2B	x^2E	$x^{22}E^{11}$
	modulo			$2\cdot 11^2$				mod 11^2	
0	1	1		1	1	1	1	1	1
1	2	4	1	12	4	−2	2	2	22
2	5	4	6	70	14	−18	−12	−12	−33
3	10	8	17	54	40	−56	−116	5	−55
4	20	20	46	105	105	96	−113	8	−55
5	36	16	116	72	10	−10	−120	1	0
6	65	32	10	−14	90	−20	−114	7	0
7	110	16	49	116	30	38	110	−11	22
8	−57	36	66	−56	−82	40	10	10	22
9	58	28	25	−4	98	−62	−12	−12	55
10	−3	40	−90	50	−56	−90	−18	−18	11
11	28	4	−39	72	94	−2	−22	−22	2
12	−41	64	6	−35	−107	26	−95	26	−11
13	86	40	−11	0	−82	−40	110	−11	−44
14	23	64	2	−60	90	100	−80	41	11
15	−118	56	75	68	−86	−38	44	44	−22
16	86	68	90	−83	−21	76	−15	−15	0
17	−112	40	30	−38	−114	94	−96	25	−22
18	108	100	76	−32	−6	−74	−66	55	0
19	−48	48	−76	20	−120	74	54	54	33
20	32	104	28	−117	23	−4	87	−34	55
21	−94	80	−36	−4	104	96	36	−46	−11

Setting $c_{jk} = 11f_{jk}$, we can now rewrite (9.4) as

$$(9.7)\qquad 11^2U_{11}C(\tau') \equiv 11^{-1}\cdot E^{11}(\tau) + \Sigma f_{jk}A^j(\tau)E^k(\tau) + g \pmod{11}.$$

We need to compute the f_{jk} only modulo 11, prove

$$(9.71)\qquad f_{j1} \equiv 0 \pmod{11},$$

and determine the constant g.

This is done exactly as in the proof of Theorem 6. We obtain a set of linear equations, analogous to (6.52), which can be solved stepwise for the f_{jk}.

The coefficients of the various powers of A and E are found quite readily since the computations are carried out modulo 11. The result is:

$$(9.8)\qquad 11^2U_{11}C(\tau') \equiv 11^{-1}\cdot E^{11} - (2E^{10}A + 8E^9A + 6E^9 + 4E^8A + 8E^8 + E^7A + 9E^7 + 7E^6A + 9E^6 + 2E^5A + 2E^5) + g \pmod{11}.$$

The computation was checked as follows:

start with:	$2EA + 8A + 6$
multiply by E:	$2E^2A + 8EA + 6E$
add $4A + 8$:	$2E^2A + 8EA + 6E + 4A + 8$
multiply by E:	$2E^3A + 8E^2A + 6E^2 + 4EA + 8E,$

etc. This procedure mixes the numbers quite thoroughly, and it is very unlikely that an error in the check would duplicate one in the original computation.

Recalling that $E(\tau) = 11^2C(\tau')$, we see that (9.8) implies (9.71). Now we have only to compute g. Let $\tau \to \tau'$ in (9.8). Since $A(\tau) = A(\tau')$, we get

$$\begin{aligned} U_{11}C(\tau) &\equiv g + 11Q(A, C) \pmod{11} \\ &\equiv g \pmod{11}, \end{aligned}$$

since $Q(A, C)$ is a polynomial with integer coefficients. Since

$$U_{11}C(\tau) = O(x)$$

we have

$$g \equiv 0(11),$$

which proves (9.1). By the remark in the lines preceding (9.1), this establishes

THEOREM 10. $p(121l + 16) \equiv 0 \pmod{121}$, $\qquad (l = 0, 1, 2, \cdots)$.

II. *Proofs of certain theorems.*

10. Proof of Theorem 1.

That the functions

$$\psi_{q,r}(\tau) = \{\eta(q\tau)\eta(\tau)\}^r$$

are regular and zero-free in the interior of the upper half-τ-plane is clear from the definition (2.4) of $\eta(\tau)$. We consider their behavior under $V = (a\tau + b)/(c\tau + d) \in \Gamma_0(q)$. First, if $c = 0$, $V\tau = \tau + b$, and using (3.5) we obtain

$$\psi_{q,r}(V\tau) = \exp\{(\pi i/12)br(q+1)\}\psi_{q,r}(\tau) = \psi_{q,r}(\tau),$$

because of condition (4.11). Thus (4.12) is proved for this case.

If $c > 0$, we apply (3.7) and find

$$\psi_{q,r}(V\tau) = N(a,b,c,d)(-i(c\tau+d))^r\psi_{q,r}(\tau)$$

with

(10.11) $$(-i)^r N(a,b,c,d) = \exp(-\pi i r A)$$

(10.12) $$A = (s(a,c) - (a+d)/12c) + (s(a,c/q) - (a+d)/(12c/q)) + \tfrac{1}{2}.$$

Therefore, in order to establish (4.12), we must prove

(10.13) $$rA \equiv r\cdot\tfrac{1}{2}(1-(a/q)) \pmod 2.$$

a/q is the Legendre-Jacobi symbol.

We shall need the following properties of $s(h,k)$:
For $(h,k)=1$, we have

(10.21) $$12hks(h,k) \equiv h^2+1 \pmod{\theta\cdot k}$$

(10.22) $$12ks(h,k) \equiv 0 \pmod 3 \quad \text{if } 3\nmid k.$$

Here $\theta = 1$ for $3\nmid k$ and $\theta = 3$ for $3\mid k$.
For odd k we have

(10.23) $$12ks(h,k) \equiv k+1-2(h/k) \pmod 8.$$

Let $k = 2^\lambda l$, $\lambda \geqq 0$, and let l and h be odd. Then

(10.24) $$12hks(h,k) \equiv h^2+k^2+3k+1+2k(k/h) \pmod{2^{\lambda+3}}.$$

These properties appear as Theorems 17, 18, 19 in [13].

From (10.21) with $k = c, c/q$, we get

$$12ac\{s(a,c)-(a+d)/12c\} \equiv a^2+1-a(a+d) \equiv -bc \pmod{\theta c}$$

$$12ac\{s(a,c/q)-(a+d)/(12c/q)\} \equiv q(a^2+1)-qa(a+d) \equiv -qbc \pmod{\theta c}.$$

Hence, by addition,

$$12acrA \equiv -r(q+1)bc \equiv 0 \pmod{\theta c}.$$

Also, by (10.22)

$$12crA \equiv 0 \pmod 3 \quad \text{if } 3\nmid c.$$

Therefore, we have

(10.31) $$12crA \equiv 0 \pmod{3c}.$$

Now suppose that c is odd. Then from (10.23) we obtain

$$\begin{aligned}&12c\{s(a,c)-(a+d)/12c\}\equiv c+1-2(a/c)-(a+d)\pmod 8,\\&12c\{s(a,c)-(a+d)/(12c/q)\}\\&\qquad\equiv c+q-2q(a|c/q)-q(a+d)\pmod 8.\end{aligned}$$

If we note that the Legendre-Jacobi symbols satisfy

$$(a/(c/q))=(a/(c/q))(a/q^2)=(a/cq)=(a/c)(a/q),$$

we obtain.

$$\begin{aligned}12crA&\equiv 2cr+r(q+1)-2r(a/c)(q(a/q)+1)\\&\qquad -r(q+1)(a+d)+6rc\pmod 8\\&\equiv -2r(a/c)(q(a/q)+1)\pmod 8\\&\equiv -2r(a/c)(1-(a/q))\pmod 8\end{aligned}$$

by virtue of (4.11)

$$\equiv -2rc(a/c)(1-(a/q))\pmod 8$$

since c is odd and $4|2(1-(a/q))$

(10.32) $$12crA\equiv -2rc(1-(a/q))\equiv 6rc(1-(a/q))\pmod 8.$$

By (10.31)

(10.33) $$12crA\equiv 6rc(1-(a/q))\pmod{3c}.$$

From (10.32) and (10.33) we infer

$$12crA\equiv 6rc(1-(a/q))\pmod{24c},$$

which is equivalent to (10.13).

Now let $c=2^\lambda\gamma$, $\lambda>0$, γ odd. Then, (10.24),

$$\begin{aligned}&12ac\{s(a,c)-(a+d)/12c\}\\&\qquad\equiv a^2+c^2+3c+1+2c(c/a)-a(a+d)\pmod{2^{\lambda+3}}\\&12ac\{s(a,c/q)-(a+d)\,|\,(12c/q)\}\\&\qquad\equiv qa^2+c^2/q+3c+q+2c(c/q|a)-qa(a+d)\\&\qquad\qquad\pmod{2^{\lambda+3}},\end{aligned}$$

from which it follows that

$$\begin{aligned}12acrA&\equiv rc^2/q(q+1)+6cr-rbc(q+1)\\&\qquad+2cr(c/a)((q/a)+1)+6acr\pmod{2^{\lambda+3}}\\&\equiv\pm 2cr((q/a)+1)+6cr(a+1)\pmod{2^{\lambda+3}}\\&\equiv\pm 2^{\lambda+1}r\gamma\{\pm(1+(q/a))-(a+1)\}(2^{\lambda+3}),\end{aligned}$$

(10.41) $$12acrA\equiv 2^{\lambda+1}r\gamma\{(q/a)-a\}\pmod{2^{\lambda+3}}.$$

This shows that

(10.42) $$12crA\equiv 0\,(2^{\lambda+3})\ \text{ if } r \text{ is even},$$

since a must be odd if c is even, because of $(a, c) = 1$.

If r is odd, we must have $q \equiv -1 (4)$ and therefore

$$(q/a) = (a/q)(-1)^{(a-1)/2} \equiv a(a/q) \pmod{4}.$$

Hence, from (10.41),

$$12acrA \equiv 2^{\lambda+1} r\gamma a((a/q) - 1) \pmod{2^{\lambda+3}},$$

so that

$$(10.43) \quad 12crA \equiv \begin{cases} 0 \pmod{2^{\lambda+3}}, \text{ if } r \text{ is even or if } (a/q) = +1 \\ 2^{\lambda+2} \pmod{2^{\lambda+3}}, \text{ if } (a/q) = -1 \text{ and } r \text{ is odd.} \end{cases}$$

The congruence (10.43) proves (10.13) in case c is even. Thus the transformation equation (4.12) is established in all cases.

To prove (4.13) we use the formula (3.6).

$$\psi_{q,r}(-1/\tau) = \{\eta(-q/\tau)\eta(-1/\tau)\}^r = (-i\tau/q)^{r/2}(-i)^{r/2}\{\eta(\tau/q)\eta(\tau)\}^r$$
$$= (-iq^{-i/2}\tau)^r \psi_{q,r}(\tau/q).$$

(4.13) now follows upon replacing τ by $q\tau$. This completes the proof of Theorem 1.

THEOREM 3. *Let q be a prime > 3, α an even integer, and*

$$\Phi(\tau; q^\alpha) = \frac{\eta(q^\alpha \tau)}{\eta(\tau)}, \qquad \Im\tau > 0.$$

Then

$$\Phi(V\tau) = \Phi(\tau)$$

for every $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *belonging to* $\Gamma_0(q^\alpha)$, i. e., *with* $c \equiv 0 \pmod{q^\alpha}$.

Proof. We shall make use of the transformation equations (3.5) and (3.7). If $c = 0$, $V\tau = \tau + b$, and

$$(10.5) \qquad \Phi(V\tau) = \exp(\pi i/12)(q^\alpha - 1) \cdot \Phi(\tau) = \Phi(\tau),$$

since $24 \mid (q^\alpha - 1)$ by virtue of the conditions placed on q and α.

For $c > 0$, we apply (3.7) and find

$$(10.61) \qquad \Phi(V\tau) = M(a, b, c, d)\Phi(\tau)$$

with

$$(10.62) \qquad M = \exp(\pi i H)$$

and

$$(10.63) \quad H = (s(a, c) - (a + d)/12c) - (s(a, c/q^\alpha) - (a + d)/(12c/q^\alpha)).$$

We have therefore to show that

$$H \equiv 0 \pmod{2}. \tag{10.64}$$

The proof of (10.64) is word for word the same as that of Lemma 5 in Rademacher [12] if we merely replace 2 by α.

Proof of Lemma 1. With

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad W_{\mu} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},$$

we must have

$$\begin{pmatrix} a + \lambda c & b + \lambda d \\ qc & qd \end{pmatrix} = \begin{pmatrix} a_1 & a_1 + qb_1 \\ c_1 & c_1 + qd_1 \end{pmatrix},$$

or

$$\begin{aligned} a_1 &= a + \lambda c & \qquad a_1\mu + qb_1 &= b + \lambda d \\ c_1 &= qc & c_1\mu + qd_1 &= qd. \end{aligned} \tag{10.7}$$

Equations (10.7) determine c_1 and a_1 uniquely with $c \equiv 0 \pmod{q^2}$ and $(a_1, c_1) = 1$. From the equations on the right we find

$$\mu a \equiv b + \lambda d \pmod{q}. \tag{10.8}$$

Since $q \mid c$, $q \nmid a$, and (10.8) together with the condition $0 \leq \mu < q$ determines μ uniquely. Furthermore, $\mu_1 = \mu_2$ implies $(\lambda_1 - \lambda_2)d \equiv 0 \pmod{q}$, which, since $q \nmid d$, is equivalent to $\lambda_1 \equiv \lambda_2 \pmod{q}$, i.e., $\lambda_1 = \lambda_2$.

Now we find d_1 from (10.7):

$$d_1 = d - \mu c_1/q = d - \mu c.$$

If $c > 0$, we have

$$b_1 = \frac{a_1 d_1 - 1}{c_1} = \frac{(a + \lambda c)(d - \mu c) - 1}{qc} = \frac{c(b + \lambda d - \mu a) - \mu\lambda c^2}{qc}, \tag{10.9}$$

since $ad - bc = 1$. b_1 is an integer, as we verify from (10.8) and (10.9). If $c = 0$, we necessarily have $a = d = 1$, $a_1 = 1$, $d_1 = 1$, $c_1 = 0$, $\mu \equiv b + \lambda (q)$. From (10.7), $b_1 = (b + \lambda - \mu)/q$, an integer, and $a_1 d_1 - b_1 c_1 = 1$. Thus the lemma is proved.

CORNELL UNIVERSITY,
ITHACA, N. Y.

BIBLIOGRAPHY.

1. L. R. Ford, *Automorphic Functions*, New York, 1929.
2. R. Fricke, *Die elliptischen Funktionen und ihre Anwendungen*, vol. 2, Leipzig and Berlin, 1922.

3. R. Fueter, *Vorlesungen über die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen*, vol. 1, Leipzig, 1924.
4. G. H. Hardy, *Ramanujan*, Cambridge, 1940.
5. E. Hecke, "Die Primzahlen in der Theorie der elliptischen Modulfunktionen," *Det Kgl. Danske Videnskabernes Selskab*, XIII (1935), no. 10.
6. E. Hecke, "Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung," *Mathematische Annalen*, vol. 112 (1936), pp. 664-699.
7. E. Hecke, "Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktiontheorie und Arithmetik," *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, vol. 5 (1927), pp. 199-224.
8. E. Hecke, "Zur Theorie der elliptischen Modulfunktionen," *Mathematischen Annalen*, vol. 97 (1927), pp. 210-242.
9. D. H. Lehmer, "An application of Schläfli's modular equation to a conjecture of Ramanujan," *Bulletin of the American Mathematical Society*, vol. 44 (1938), pp. 84-90.
10. J. Lehner, "On certain irrational modular equations," to be published.
11. H. Rademacher and H. S. Zuckerman, "A new proof of two of Ramanujan's identities," *Annals of Mathematics*, vol. 40 (1939), pp. 473-489.
12. H. Rademacher, "The Ramanujan identities under modular substitutions," *Transactions of the American Mathematical Society*, vol. 51 (1942), pp. 609-636.
13. H. Rademacher and A. L. Whiteman, "Theorems on Dedekind sums," *American Journal of Mathematics*, vol. 63 (1941), pp. 377-407.
14. *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927.
15. G. N. Watson, "Ramanujans Vermutung über zerfallungsanzahlen," *Journal für die reine und angewandte Mathematik*, vol. 179 (1938), pp. 97-128.
16. H. Zuckerman, "Identities analogous to Ramanujan's identities involving the partition function," *Duke Mathematical Journal*, vol. 5 (1939), pp. 88-110.

CALCUTTA UNIVERSITY CENTRAL LIBRARY

A NEW KIND OF RELATIONSHIP BETWEEN MATRICES.*

By Claude Chevalley.

The consideration of algebraic Lie groups and of their Lie algebras has led me to the study of a new kind of relationship between matrices, which seems to be of some interest by itself. The present paper is concerned with the study of this relationship. The connection with the theory of Lie algebras will be explained in a later paper.

1. The matrices $X_{r,s}$. The notion of replica. Let X, Y be two square matrices of degrees m, n respectively. If k is any integer, we shall denote by E_k the unit matrix of degree k. We set

$$X \boldsymbol{+} Y = X \boldsymbol{\times} E_n + E_m \boldsymbol{\times} Y$$

where the sign $\boldsymbol{\times}$ is used to represent the Kronecker multiplication of matrices. The matrix $X \boldsymbol{+} Y$ is called the Kronecker sum of X and Y.

If X is any square matrix, we denote by ${}^tX$ the transpose of X and by X^* the matrix $-{}^tX$.

The Kronecker addition of matrices is obviously a distributive operation with respect to both of its arguments. It is easy to recognize that it is associative. But it is in general not commutative.

If X is any matrix, we set

$$X_{r,s} = \underbrace{X^* \boldsymbol{+} X^* \boldsymbol{+} \cdots \boldsymbol{+} X^*}_{r \text{ times}} \boldsymbol{+} \underbrace{X \boldsymbol{+} X \boldsymbol{+} \cdots \boldsymbol{+} X}_{s \text{ times}}.$$

If X is considered as an infinitesimal operation on the vectors of a certain vector space, $X_{r,s}$ indicates how X operates on the r times contravariant and s times covariant tensors.

By a vector of type (r,s) we understand a column of m^{r+s} numbers (a "number" is an element of the basic field out of which the coefficients of our matrices are selected). A vector $\vec{e}$ of type (r,s) is said to be an *invariant* of X if we have

$$X_{r,s}\vec{e} = 0.$$

Definition 1. *A matrix Y is said to be a replica of X if every invariant of X is also an invariant of Y.*

---

* Received August 14, 1942.

Let K be our basic field. It will be convenient to make use of a vector space $\mathfrak{M}$ of dimension n over K. If a base $(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n)$ is selected in $\mathfrak{M}$, the matrices of degree n may be considered as linear endomorphisms of $\mathfrak{M}$. We denote by $\mathfrak{M}^*$ the dual space of $\mathfrak{M}$ (i. e. the space of linear functions on $\mathfrak{M}$ with values in K). If λ_i is the linear function defined by $\lambda_i(\vec{e}_j) = \delta_{ij}$, the elements $\lambda_1, \lambda_2, \cdots, \lambda_n$ form a base of $\mathfrak{M}^*$, the dual of the base $(\vec{e}_1, \cdots, \vec{e}_n)$ of $\mathfrak{M}$. On the other hand, if $\mathfrak{N}, \mathfrak{P}$ are two vector spaces of dimensions n, p over K, we denote by $\mathfrak{N} \mathbf{X} \mathfrak{P}$ the Kronecker product of $\mathfrak{N}$ and $\mathfrak{P}$. To every pair $(\vec{f}, \vec{g})$ composed of a vector $\vec{f} \in \mathfrak{N}$ and of a vector $\vec{g} \in \mathfrak{P}$ there corresponds a vector $\vec{f} \mathbf{X} \vec{g} \in \mathfrak{N} \mathbf{X} \mathfrak{P}$; $\vec{f} \mathbf{X} \vec{g}$ is a linear function of each of its arguments when the other is kept fixed. If $(\vec{f}_1, \vec{f}_2, \cdots, \vec{f}_n)$, $(\vec{g}_1, \cdots, \vec{g}_p)$ are bases for $\mathfrak{N}, \mathfrak{P}$ respectively, the vectors $\vec{f}_i \mathbf{X} \vec{g}_j$ form a base for $\mathfrak{N} \mathbf{X} \mathfrak{P}$.

We set

$$\mathfrak{M}_{r,s} = \underbrace{\mathfrak{M}^* \mathbf{X} \mathfrak{M}^* \mathbf{X} \cdots \mathbf{X} \mathfrak{M}^*}_{r \text{ times}} \mathbf{X} \underbrace{\mathfrak{M} \mathbf{X} \mathfrak{M} \mathbf{X} \cdots \mathbf{X} \mathfrak{M}}_{s \text{ times}}.$$

If $\mu_1, \mu_2, \cdots, \mu_r$ are r elements of $\mathfrak{M}^*$ and $\vec{f}_1, \cdots, \vec{f}_s$ s elements of $\mathfrak{M}$, the element $\mu_1 \mathbf{X} \mu_2 \mathbf{X} \cdots \mathbf{X} \mu_r \mathbf{X} \vec{f}_1 \mathbf{X} \cdots \mathbf{X} \vec{f}_s$ is an element of $\mathfrak{M}_{r,s}$ (a tensor of type (r, s) !). The m^{r+s} vectors $\lambda_{i_1} \mathbf{X} \lambda_{i_2} \mathbf{X} \cdots \mathbf{X} \lambda_{i_r} \mathbf{X} \vec{e}_{j_1} \mathbf{X} \vec{e}_{j_2} \mathbf{X} \cdots \mathbf{X} \vec{e}_{j_s}$ $(1 \leq i_\alpha \leq m,\ 1 \leq j_\beta \leq m)$ form a base of $\mathfrak{M}_{r,s}$. The matrix $X_{r,s}$ defines a linear endomorphism of $\mathfrak{M}_{r,s}$; more specifically, we have

$$\begin{aligned} X_{r,s}\mu_1 \mathbf{X} \mu_2 \mathbf{X} \cdots \mathbf{X} \mu_n \mathbf{X} \vec{f}_1 \mathbf{X} \vec{f}_2 \mathbf{X} \cdots \mathbf{X} \vec{f}_s & \\ = \sum_{i=1}^{r} \mu_1 \mathbf{X} \cdots \mathbf{X} X^*\mu_i \mathbf{X} \cdots \mathbf{X} \mu_r \mathbf{X} \vec{f}_1 \mathbf{X} \vec{f}_2 \mathbf{X} \cdots \mathbf{X} \vec{f}_s & \\ + \sum_{j=1}^{s} \mu_1 \mathbf{X} \mu_2 \mathbf{X} \cdots \mathbf{X} \vec{\mu}_r \mathbf{X} \vec{f}_1 \mathbf{X} \cdots \mathbf{X} X\vec{f}_j \mathbf{X} \cdots \mathbf{X} \vec{f}_s & \end{aligned}$$

where $X^*\mu_i$ is the linear function defined by $X^*\mu_i(\vec{e}) = -\mu_i(X\vec{e})$ for any vector $\vec{e}$.

Let us consider, in particular, the space $\mathfrak{M}^* \mathbf{X} \mathfrak{M} = \mathfrak{M}_{1,1}$. There exists a natural isomorphism I between $\mathfrak{M}_{1,1}$ and the space of linear endomorphisms of $\mathfrak{M}$, which is defined in the following way: if $\mu \in \mathfrak{M}^*$, $\vec{f} \in \mathfrak{M}$, $I(\mu \mathbf{X} f)$ is the endomorphism ϕ which is defined by $\phi(\vec{e}) = \mu(\vec{e})f$ for any vector $\vec{e}$. From now on we shall, whenever it is convenient, identify the linear endomorphisms of $\mathfrak{M}$ (i. e. also the matrices of degree m) with the corresponding elements of $\mathfrak{M}_{1,1}$.

Let ϕ be an element of $\mathfrak{M}_{1,1}$ which is of the form $\mu \mathbf{X} \vec{f}$. We have

$$X_{1,1}\phi = (X^*\mu) \mathbf{X} \vec{f} + \mu \mathbf{X} \overrightarrow{Xf}$$

whence easily

$$X_{11}\phi = X\phi - \phi X.$$

By linearity, the same formula holds for any ϕ.

It follows that a matrix ϕ is an invariant (of type $(1,1)$) of X if, and only if, ϕ commutes with X. Therefore, if Y is a replica of X, Y commutes with every matrix which commutes with X. Making use of a well known theorem, this gives the following result:

THEOREM 1. *Any replica of a matrix X may be written in the form of a polynomial in X.*

2. Semi-simple and nilpotent components of a matrix. Using the same notations as in **1**, we shall say that a sub-space $\mathfrak{N}$ of $\mathfrak{M}$ is an $\{X\}$-module if $X\mathfrak{N} \subset \mathfrak{N}$. The $\{X\}$-module $\mathfrak{N}$ is said to be simple if it is $\neq \{0\}$ and if it does not contain any other $\{X\}$-modules than $\{0\}$ and itself. The space $\mathfrak{M}$ is said to be semi-simple (with respect to X) if $\mathfrak{M}$ is a sum of simple sub-modules. It is well known that, under this condition, $\mathfrak{M}$ can be represented as a direct sum of simple sub-modules.

If $\mathfrak{M}$ is semi-simple with respect to X, we shall also say that X is semi-simple over K. Well known examples show that X may be semi-simple over a certain field K and not be semi-simple over a larger field K' (this however only in the case where K'/K is not separable).

THEOREM 2. *A matrix X over a perfect field K may be represented in one and only one way in the form $X = Y + Z$, where Y is semi-simple over K, Z is nilpotent and Y, Z commute with X. Furthermore, Y and Z may be expressed as polynomials in X.*

This theorem is only a different form of the well known theorem of the reduction of a matrix to semi-diagonal form. We present here a new proof of it which is better adapted to our purposes.

Let us say that a polynomial $f(x)$ with coefficients in K is semi-simple (over K) if it is a product of *distinct* irreducible factors. We shall say that X satisfies a semi-simple equation in K if $f(X) = 0$ with some semi-simple f.

LEMMA 1. *If X satisfies a semi-simple equation, X is semi-simple.*

Assume that $f_1(X)f_2(X)\cdots f_r(X) = 0$, with f_i irreducible $(1 \leq i \leq r)$, f_i and f_j being relatively prime for $i \neq j$. Let $g_i = \prod_{j \neq i} f_i$; the h. c. d. of $g_1, g_2, \cdots, g_r$ is 1, and therefore we can find polynomials u_i such that

$$\Sigma_i u_i(x)g_i(x) = 1. \tag{1}$$

We set $u_i(X)\mathfrak{M} = \mathfrak{M}_i$; $\mathfrak{M}_i$ is an $\{X\}$ module, and it follows from (1) that $\mathfrak{M} = \Sigma_i \mathfrak{M}_i$. We have $f_i(X)\mathfrak{M}_i = \{0\}$.

Let $\vec{e}$ be a vector $\neq 0$ in $\mathfrak{M}_i$ (if any), and assume that $f_i(x) = x^{m_i} + a_1x^{m_i-1} + \cdots + a_{m_i}$. The vectors $\vec{e}$, $X^h\vec{e}$ $(1 \leq h \leq m_i)$ span a subspace $\mathfrak{N}$ of $\mathfrak{M}_i$ which is clearly an $\{X\}$-module. If $p(x)$ is a polynomial of degree $< m_i$, $p(x)$ is relatively prime to f_i: if $u(x)p(x) + v(x)f_i(x) = 1$, we have $\vec{e} = u(X)p(X)\vec{e}$, whence $p(X)\vec{e} \neq 0$, which proves that $\vec{e}, Xe, \cdots, X^{m_i-1}\vec{e}$ are linearly independent and that $\dim \mathfrak{N} = m_i$. Furthermore $\mathfrak{N}$ is clearly the smallest $\{X\}$-module to contain $\vec{e}$. In the same way, we see that the smallest $\{X\}$-module to contain any vector $\vec{e}' \neq 0$ in $\mathfrak{N}$ is of dimension m_i, and therefore coincides with $\mathfrak{N}$. This means that $\mathfrak{N}$ is a simple $\{X\}$-module; therefore every element $\neq 0$ of $\mathfrak{M}_i$ belongs to a simple $\{X\}$-module, which proves Lemma 1.

Let now X be any matrix with coefficients in a given perfect field K. It satisfies at least one equation $f(X) = 0$, where f is a polynomial with coefficients in K. We can find a semi-simple polynomial g which divides f and is such that f divides some power g^r of g (take for g the product of the distinct irreducible factors $f_1, f_2, \cdots, f_s$ of f).

We prove by induction on k the existence of a polynomial h_k such that $g(x - h_k(x)g(x)) \equiv 0 \pmod{g^k}$. We may take $h_1 = 0$; since K is perfect, we have $dg/dx \neq 0$ and dg/dx is relatively prime to g (this follows at once from $g = f_1f_2\cdots f_s$, $df_j/dx \neq 0$). For any h we have

$$g(x - h(x)g(x)) \equiv g(x) - (dg/dx)h(x)g(x) \quad (\text{mod. } g^2).$$

Since dg/dx is relatively prime to g, the identity of Bezout furnishes us with a polynomial h_2 such that $1 - (dg/dx)h_2 \equiv 0 \pmod{g}$ whence $g(x - h_2g) \equiv 0 \pmod{g^2}$.

Suppose now that the existence of h_k has been proved for some $k \geq 2$. We set $g(x - h_k(x)g(x)) = g^k(x)\phi(x)$, $y = x - h_2(x)g(x)$, whence $g(y - h_k(y)g(y)) = g^k(y)\phi(y) \equiv 0$ (mod. $g^{2k}(x)$); on the other hand

$y - h_k(y)g(y) \equiv x \pmod{g(x)}$, and we can therefore write $y - h_k(y)g(y) = x - h_{k+1}(x)g(x)$; then $g(x - h_{k+1}(x)g(x))$ is divisible by $g^{2k}(x)$ and, *a fortiori*, by $g^{k+1}(x)$, which proves the existence of h_{k+1}.

We set now $h(x) = h_r(x)$, $Y = X - h(X)g(X)$, $Z = h(X)g(X)$. We have $X = Y + Z$, and Y, Z commute with X; since $g^r(X) = 0$, we have $g(Y) = 0$, and Y is semi-simple; for the same reason, we have $Z^r = 0$, and Z is nilpotent. The existence of a decomposition of the type prescribed in Theorem 2 is thereby proved. There remains to prove the statement of uniqueness.

Let us consider any decomposition $X = Y' + Z'$ with Y' semi-simple and Z' nilpotent, both commuting with X. We have $Y' - Y = Z - Z'$. Since Y', Z' commute with X, they also commute with Y and Z. Therefore, in particular, $Z - Z' = U$ is nilpotent as the difference of two commuting nilpotent matrices. By the Taylor expansion, we have

$$g(Y') = U((dg/dx)(Y) + Ug_2(Y) + \cdots + U^{N-1}g_N(Y)) \tag{2}$$

where $g_2, \cdots, g_N$ are polynomials. It follows that $g(Y')$ is nilpotent; $g^M(Y') = 0$ if $U^M = 0$. Since Y' is semi-simple, we have $\mathfrak{M} = \Sigma_i \mathfrak{M}'_i$ where each $\mathfrak{M}'_i$ is a simple $\{Y'\}$-module. Let f'_i be a polynomial of smallest possible degree such that $f'_i(Y')\mathfrak{M}' = \{0\}$. If $f'_i = pq$, p and q being polynomials, p of degree less than f'_i, the set of vectors $\vec{e} \in \mathfrak{M}'_i$ such that $p(Y')\vec{e} = 0$ is a $\{Y'\}$-module which is $\neq \mathfrak{M}'_i$; therefore this set reduces to $\{0\}$, and $p(Y')$ induces a regular linear endomorphism in $\mathfrak{M}'_i$ (i. e. an endomorphism which has an inverse). It follows immediately that $\phi(Y')\mathfrak{M}'_i = \{0\}$ whence the degree of $p = 0$: f'_i is irreducible. If now g' is the l. c. m. of the polynomials f'_i, we have $g'(Y') = 0$ and g' is semi-simple. Comparing the equations $g^M(Y') = 0$, $g'(Y') = 0$, we obtain easily $g(Y') = 0$.

Suppose now that $U^M = 0$, $M > 1$. Then (2) gives $U^{M-1}(dg/dx)(Y) = 0$. On the other hand, we have $h_2(x)(dg/dx) \equiv 1 \pmod{g(x)}$, whence $h_2(Y)(dg/dx)(Y) = E$ (the unit matrix) and $U^{M-1} = 0$. Therefore we have $U = 0$, $X = X'$, $Y = Y'$, which completes the proof of Theorem 2.

Remark. We have proved incidentally the converse of Lemma 1, i. e. that a semi-simple matrix satisfies a semi-simple equation.

If the field K is perfect, a polynomial which is semi-simple over K remains semi-simple over any larger field. Therefore we have the following result:

LEMMA 2. *A matrix which is semi-simple over a perfect field remains such over any larger field.*

Remark. The assertion of existence in Theorem 2 could be deduced easily from the theorem that any algebra over a perfect field is the direct sum of the radical and of a semi-simple algebra.

We shall say that the matrices Y, Z whose existence and uniqueness are asserted in Theorem 2 are the *semi-simple and nilpotent components* of X.

Using the notations of **1**, we have

$$X_{r,s} = Y_{r,s} + Z_{r,s}. \tag{3}$$

Observe now that, if Z, Z' are nilpotent matrices, the matrix $Z \boldsymbol{+} Z' = Z \boldsymbol{\times} E' + E \boldsymbol{\times} Z'$ is nilpotent as the sum of two nilpotent matrices which commute with each other (E, E' are unit matrices of suitable degrees). It follows immediately that $Z_{r,s}$ is nilpotent. We shall prove that $Y_{r,s}$ is semi-simple.

Since Y is semi-simple over a perfect field K, it remains semi-simple over the algebraic closure $\bar{K}$ of K. Hence we may write $Y = \Gamma D \Gamma^{-1}$, where D is a diagonal matrix and Γ a regular matrix, the coefficients of D and Γ belonging to $\bar{K}$. (This follows immediately from the fact that a simple $\{Y\}$-module over $\bar{K}$ is necessarily of dimension 1). Therefore it is sufficient to prove that $D_{r,s}$ is semi-simple. But $D_{r,s}$ may be computed directly without any difficulty; it turns out that $D_{r,s}$ is again diagonal (i. e. semi-simple). Moreover, if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the elements in the main diagonal of D, the elements of the main diagonal of $D_{r,s}$ are all the quantities

$$-\Sigma_1^r \alpha_{i_\lambda} + \Sigma_1^s \alpha_{j_\mu} \qquad \begin{array}{l}(1 \leq i_\lambda \leq n,\ 1 \leq j_\mu \leq n,\\ 1 \leq \lambda \leq r,\ 1 \leq \mu \leq s).\end{array} \tag{4}$$

Since the matrices $Y_{r,s}$, $Z_{r,s}$ in (3) obviously commute with $X_{r,s}$, they are the semi-simple and nilpotent components of $X_{r,s}$.

3. Some further results on replicas of matrices.

LEMMA 1. *If Y, Z are the semi-simple and nilpotent components of a matrix X over a perfect field, and if $\vec{e}$ is a vector such that $X\vec{e} = 0$, then $Y\vec{e} = 0$, $Z\vec{e} = 0$.*

We have $Y\vec{e} = -Z\vec{e}$; since Y, Z commute, it follows easily that $Y^k\vec{e} = (-1)^k Z^k \vec{e}$ for all k, whence $Y^M \vec{e} = 0$ for some integer M. But we have also $g(Y)\vec{e} = 0$ with a semi-simple polynomial g. If d is the h. c. d. of $g(y)$ and y^M, we have, by the Bezout identity, $d(Y)\vec{e} = 0$. But $d(y) = y$ since g is semi-simple, which proves Lemma 1.

Since $Y_{r,s}$ and $Z_{r,s}$ are the semi-simple and the nilpotent components of $X_{r,s}$ and $Z_{r,s}$ it follows immediately that we have

LEMMA 2. *If X is a matrix in a perfect field K, the semi-simple and the nilpotent components of X are replicas of X.*

We shall now determine the replicas of a semi-simple matrix Y over a perfect field K. A replica of Y is of the form $p(Y)$, p being some polynomial. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the characteristic roots of Y, so that Y may be transformed (in the algebraic closure $\bar{K}$ of K) into the diagonal matrix

$$D = \begin{bmatrix} \alpha_1 & & & & 0 \\ & \alpha_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \alpha_n \end{bmatrix}.$$

The same transformation carries $p(Y)$ into the matrix

$$D' = \begin{bmatrix} p(\alpha_1) & & & & 0 \\ & p(\alpha_2) & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & p(\alpha_n) \end{bmatrix}.$$

The matrix $D_{r,s}$ is a diagonal matrix whose diagonal elements are given by (4). Since every vector which is mapped on 0 under $D_{r,s}$ is also mapped on 0 under $D'_{r,s}$, we see that the condition $-\Sigma_1{}^r\alpha_{i_\lambda} + \Sigma_1{}^s\alpha_{j_\mu} = 0$ implies $-\Sigma_1{}^r p(\alpha_{i_\lambda}) + \Sigma_1{}^s p(\alpha_{j_\mu}) = 0$, and conversely this condition is sufficient to make $D'_{r,s}$ a replica of $D_{r,s}$.

Therefore we have the following result:

THEOREM 3. *Let Y be a semi-simple matrix over a perfect field K, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the characteristic roots of Y. If p is a polynomial with coefficients in K, the matrix $p(Y)$ is a replica of Y if and only if the following condition is realized: if $\Sigma_i k_i \alpha_i = 0$ is any linear relation between the α_i's with rational integral coefficients, then the corresponding relation $\Sigma_i k_i p(\alpha_i) = 0$ holds between the elements $p(\alpha_i)$.*

An easy consequence of Theorem 3 is

THEOREM 4. *Let K be a field of characteristic 0. A necessary and sufficient condition for a matrix X to be nilpotent is that $Sp.\ XX' = 0$ for every replica X' of X.*

If X is nilpotent, and if X' is a replica of X, X' commutes with X and therefore XX' is nilpotent, whence Sp. $XX' = 0$.

Assume that X is not nilpotent. It has a semi-simple component $Y \neq 0$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the characteristic roots of Y; they generate a vector space P of finite dimension over the field R of rational numbers. Let $\{\rho_1, \rho_2, \cdots, \rho_r\}$ be a base of P: we have $\alpha_i = \sum_{j=1}^{r} a_{ij}\rho_i$ and one at least of the numbers a_{ij}, say $a_{i_0 j_0}$, is $\neq 0$. We set $\beta_i = a_{ij_0}$; if $\Sigma_i k_i \alpha_i = 0$ is a linear relation with integral coefficients between $\alpha_1, \alpha_2, \cdots, \alpha_n$ we have $\Sigma_i k_i a_{ij} = 0$ $(1 \leq j \leq h)$, whence $\Sigma_i k_i \beta_i = 0$. We can find a polynomial $p(x)$ (whose coefficients are algebraic over K) such that $p(\alpha_i) = \beta_i$ $(1 \leq i \leq n)$. The matrix $X'' = p(X)$ is a replica of Y. Let K' be the field generated by the coefficients of X''; if $\{\omega_1, \cdots, \omega_s\}$ is a linear base of K' over K, we may write $X'' = \Sigma_k \omega_k X'_k$, where each X'_k is a matrix with coefficients in K. Each X'_k is clearly a replica of X''. On the other hand, we have Sp. $XX'' =$ Sp. YX'' $+$ Sp. ZX''; since Z commutes with Y and X'' is a replica of Y, Z commutes with X'', and Sp. $ZX'' = 0$, whence Sp. $XX'' =$ Sp. $YX'' = \Sigma_i \alpha_i \beta_i = \Sigma_i a^2_{ij_0}\rho_{j_0} + \sum_{i, j \neq j_0} a_{ij} a_{ij_0} \rho_j \neq 0$ since $\Sigma_i a^2_{ij_0} \neq 0$. It follows that Sp. $XX'_k \neq 0$ for at least one k, which proves Theorem 4.

LEMMA 3. *If the matrix X' is a replica of the matrix X, $X'_{r,s}$ is a replica of $X_{r,s}$.*

Consider the space $(\mathfrak{M}_{r,s})_{r',s'}$. If $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{Q}$ are vector-spaces, there exist natural isomorphisms between the spaces $\mathfrak{N} \times \mathfrak{P} \times \mathfrak{Q}$, $(\mathfrak{N} \times \mathfrak{P}) \times \mathfrak{Q}$, $\mathfrak{N} \times (\mathfrak{P} \times \mathfrak{Q})$ and also between the spaces $\mathfrak{N} \times \mathfrak{P}$, $\mathfrak{P} \times \mathfrak{N}$; there exists also a natural isomorphism between $(\mathfrak{N}^*)^*$ and $\mathfrak{N}$. It follows easily from these remarks that there exists a natural isomorphism between $(\mathfrak{M}_{r,s})_{r',s'}$ and $\mathfrak{M}_{rs'+sr', rr'+ss'}$; and that the endomorphisms $(X_{r,s})_{r',s'}$ and $X_{rs'+sr', rr'+ss'}$ correspond to each other under this isomorphism. Since every invariant of type $(rs' + sr', rr' + ss')$ of X is also an invariant of the same type of X', Lemma 3 is proved.

It follows by Theorem 1, that $X'_{r,s}$ may be written as a polynomial $P_{r,s}(X_{r,s})$ in $X_{r,s}$. We may furthermore assume that $P_{r,s}(0) = 0$. In fact, if $X_{r,s}$ is a regular matrix, it satisfies at least one equation $g(X_{r,s}) = 0$ with a g such that $g(0) \neq 0$, and therefore the unit matrix $E_{r,s}$ may be written as a polynomial $e_{r,s}(X_{r,s})$ with $e_{r,s}(0) = 0$, which proves our assertion in this case. If $X_{r,s}$ is not regular, there exists at least one vector $\vec{e} \neq 0$ in $\mathfrak{M}_{r,s}$ such that $X_{r,s}\vec{e} = 0$, whence $P_{r,s}(X_{r,s})\vec{e} = P_{r,s}(0)\vec{e} = 0$, $P_{r,s}(0) = 0$. Hence we have

LEMMA 4. *If X' is a replica of X, $X'_{r,s}$ may be written in the form $p_{r,s}(X_{r,s})$ where $p_{r,s}$ is a polynomial such that $p_{r,s}(0) = 0$.*

Remark. We see at once that the condition expressed in Lemma 4 is not only necessary but also sufficient for X' to be a replica of X.

THEOREM 5. *Let X, X' be matrices over a perfect field K; let Y, Y' be the semi-simple components of X, X' and Z, Z' be their nilpotent components. In order for X' to be a replica of X, it is necessary and sufficient that Y' be a replica of Y and Z' a replica of Z.*

1) The condition is sufficient. Assume that the condition is satisfied, and let $\vec{e}$ be an invariant of type (r, s) of X. We have seen above that it is also an invariant of Y and of Z (cf. Lemma 1, **3**); therefore $\vec{e}$ is an invariant of Y' and of Z', and also of X': X' is a replica of X.

2) The condition is necessary. Assume that X' is a replica of X. Let $\vec{e}$ be an invariant of Y, of type (r, s): we have $X_{r,s}\vec{e} = Z_{r,s}\vec{e}$. Since $X_{r,s}$ commutes with $Z_{r,s}$, we have also $X^k{}_{r,s}\vec{e} = Z^k{}_{r,s}\vec{e}$ $(k = 1, 2, \cdots)$, whence $X^M{}_{rs}\vec{e} = 0$ for some M. By induction on k, we prove that $X^{M-k}_{r,s} Y'^{k}_{r,s}\vec{e} = 0$ $(k = 0, 1, \cdots, M)$. It is true for $k = 0$; assume that it holds for some $k < M$; we have $X_{r,s}(X^{M-k-1}_{r,s} Y'^{k}_{r,s} e) = 0$, whence $Y'_{r,s}(X^{M-k-1}_{r,s} Y'^{k}_{r,s} e) = 0$ because Y' is a replica of X' which is a replica of X, so that Y' is a replica of X. It follows that $X^{M-k-1}_{r,s} Y'^{k+1}_{r,s} e = 0$ because $Y'_{r,s}$ commutes with $X_{r,s}$, which proves our assertion for $k + 1$. Hence we have $Y'^{M}_{r,s}\vec{e} = 0$. Using (as we have done already) the fact that Y' is semi-simple, it follows that $Y'_{r,s}\vec{e} = 0$, which shows that Y' is a replica of Y.

Consider now the sub-space $\mathfrak{N}$ of $\mathfrak{M}_{r,s}$ formed by the invariants of Z. Since $Z_{r,s}$ may be written as a polynomial in $X_{r,s}$ and $X'_{r,s}$ commutes with $X_{r,s}$, $X'_{r,s}$ commutes with $Z_{r,s}$ whence $X'_{r,s}\mathfrak{N} \subset \mathfrak{N}$. Let $\bar{X}_{r,s}$, $\bar{X}'_{r,s}$, $\bar{Y}_{r,s}$ denote the endomorphisms induced on $\mathfrak{N}$ by $X_{r,s}$, $X'_{r,s}$, $Y_{r,s}$: we have $\bar{X}_{r,s} = \bar{Y}_{r,s}$. But we know that $Y_{r,s}$ is semi-simple; hence the same holds for $\bar{Y}_{r,s}$ and $\bar{X}_{r,s}$. We have $\bar{X}'_{r,s} = p_{r,s}(\bar{X}_{r,s})$, where $p_{r,s}$ is a polynomial, and it is easy to see that every polynomial in a semi-simple matrix is itself a semi-simple matrix. Therefore $\bar{X}'_{r,s}$ is semi-simple.

Since $Y'_{r,s}$, $Z'_{r,s}$ are polynomials in $X'_{r,s}$, they map $\mathfrak{N}$ into itself and induce endomorphisms $\bar{Y}'_{r,s}$, $\bar{Z}'_{r,s}$ on $\mathfrak{N}$; $\bar{Y}'_{r,s}$ is semi-simple, $\bar{Z}'_{r,s}$ is nilpotent; we have $\bar{X}'_{r,s} = \bar{Y}'_{r,s} + \bar{Z}'_{r,s}$ and $\bar{Y}'_{r,s}$, $\bar{Z}'_{r,s}$ commute with $\bar{X}'_{r,s}$; by the uniqueness of the decomposition in semi-simple and nilpotent components, we have $\bar{Z}'_{r,s} = 0$, $Z'_{r,s}\mathfrak{N} = \{0\}$, which proves that every invariant of Z is also an invariant of Z'.

Theorem 5 reduces the problem of finding all replicas of a matrix X to the same problem with the supplementary assumption that X is either semi-simple or nilpotent. The problem is solved in the semi-simple case by Theorem 3; there remains to consider the nilpotent case.

THEOREM 6. *If Z is a nilpotent matrix over a field K of characteristic 0, the only replicas of Z are the matrices tZ, $t \in K$.*

In order to prove Theorem 6, we shall make use of a particular case of a theorem due to Ado. The original proof being written in Russian, we give here a proof of the results which we need.

The elements of $\mathfrak{M}_{r,0} = \mathfrak{M}^* \times \mathfrak{M}^* \times \cdots \times \mathfrak{M}^*$ are the r-linear forms on $\mathfrak{M}$. Among these, we consider only those which are symmetric: they form a vector sub-space $\mathfrak{S}_r$ of $\mathfrak{M}_{r,0}$. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be a base of $\mathfrak{M}^*$ over K. It is well known that $\mathfrak{S}_r$ has a natural isomorphism with the set Π_r of the homogeneous polynomials of degree r in $\lambda_1, \lambda_2, \cdots, \lambda_n$. The operation $Z_{r,0}$ may therefore be considered as an operation on Π_r. This being true for every r, Z defines an operation D_Z in the ring $K[\lambda_1, \lambda_2, \cdots, \lambda_n]$ of all polynomials in $\lambda_1, \lambda_2, \cdots, \lambda_n$. It is clear that D_Z is a linear operation. An easy computation shows that it is a *derivation*, i. e.

$$D_Z(PQ) = D_Z(P) \cdot Q + P \cdot D_Z Q$$

if $P, Q \in K[\lambda_1, \cdots, \lambda_n]$. As such, D_Z may be extended to a derivation of the field $\Sigma = K(\lambda_1, \cdots, \lambda_n)$ of the rational functions in $\lambda_1, \lambda_2, \cdots, \lambda_n$. An element $R \in \Sigma$ is called an *invariant* of Z if $D_Z R = 0$. The invariants of Z clearly form a subfield J of Σ.

Since Z is nilpotent, it may be assumed that the base $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ has been selected in such a way that $D_Z\lambda_1 = 0$, and that $D_Z\lambda_{k+1}$ is a linear combination of $\lambda_1, \lambda_2, \cdots, \lambda_k$ $(1 \leq k < n)$. We shall prove that we can find elements $\lambda'_1, \lambda'_2, \cdots, \lambda'_n \in \Sigma$ with the following properties:

a) The field $K(\lambda'_1, \lambda'_2, \cdots, \lambda'_k)$ coincides with $\Sigma_k = K(\lambda_1, \lambda_2, \cdots, \lambda_k)$ $(1 \leq k \leq n)$;

b) $D_Z\lambda'_k$ is either equal to 0 or to 1.

c) If $\mathfrak{o}_k = K[\lambda'_1, \lambda'_2, \cdots, \lambda'_k]$, we have $\lambda_i \in \mathfrak{o}_k$ $(1 \leq i \leq k)$.

d) If $D_Z\mathfrak{o}_k$ is $\neq \{0\}$, it coincides with the whole ring $\mathfrak{o}_k$.

We set $\lambda'_1 = \lambda_1$: the conditions a), b), c), d) are satisfied for $k = 1$, as $D_Z\mathfrak{o}_1 = \{0\}$.

Assume that $\lambda'_1, \cdots, \lambda'_k$ have been constructed. We set $\mu_{k+1} = D_Z\lambda_{k+1}$; by c) we have $\mu_{k+1} \epsilon \mathfrak{o}_k$. If $D_Z\mathfrak{o}_k = \{0\}$, $\mu_{k+1} = 0$, we may take $\lambda'_{k+1} = \lambda_{k+1}$. If $D_Z\mathfrak{o}_k = \{0\}$, $\mu_{k+1} \neq 0$, we take $\lambda'_{k+1} = \mu_{k+1}^{-1}\lambda_{k+1}$; we have $K(\lambda'_1, \cdots, \lambda'_{k+1}) = K(\lambda_1, \cdots, \lambda_k, \lambda'_{k+1}) = \Sigma_{k+1}$. Since $D_Z\mu_{k+1} = 0$, we have $D_Z\lambda'_{k+1} = 1$. We have $\lambda_{k+1} = \mu_{k+1}\lambda'_{k+1} \epsilon \mathfrak{o}_k[\lambda'_{k+1}] = \mathfrak{o}_{k+1}$; finally let $P = \Sigma p_i\lambda'^{i}_{k+1}$ be an element of $\mathfrak{o}_{k+1}$, with $p_i \epsilon \mathfrak{o}_k$: since $D_Zp_i = 0$, we have $P = D_Z(\Sigma(i+1)^{-1}p_i\lambda'^{i+1}_{k+1})$.

Assume now that $D_Z\mathfrak{o}_k \neq \{0\}$, whence $D_Z\mathfrak{o}_k = \mathfrak{o}_k$. There exists an element $\nu_{k+1} \epsilon \mathfrak{o}_k$ such that $D_Z\nu_{k+1} = \mu_{k+1}$; we set in this case $\lambda'_{k+1} = \lambda_{k+1} - \nu_{k+1}$, whence $D_Z\lambda'_{k+1} = 0$, $K(\lambda'_1, \cdots, \lambda'_{k+1}) = K(\lambda_1, \cdots, \lambda_k, \lambda'_{k+1}) = \Sigma_{k+1}$, $\lambda_{k+1} \epsilon \mathfrak{o}_k[\lambda'_{k+1}] = \mathfrak{o}_{k+1}$. Let $P = \Sigma_i p_i\lambda'^{i}_{k+1}$ be any element of $\mathfrak{o}_{k+1}$, with $p_i \epsilon \mathfrak{o}_k$; since $D_Z\mathfrak{o}_k = \mathfrak{o}_k$ we have $p_i = D_Zq_i$, $q_i \epsilon \mathfrak{o}_k$ and $P = D_ZQ$ with $Q = \Sigma_i q_i\lambda'^{i}_{k+1} \epsilon \mathfrak{o}_{k+1}$.

Moreover, our induction gives also the following results: we have $D_Z\lambda'_k = 1$ for at most one k; each λ'_k may be written as a rational function of $\lambda_1, \lambda_2, \cdots, \lambda_n$ whose denominator is an invariant (in fact, a power of μ_{k+1}, if $D_Z\lambda'_{k+1} = 1$). Assuming that $Z \neq 0$, we have $D_Z\lambda'_k = 1$ for exactly one value k_0 of k; we set $u = \lambda'_{k_0}$, and we have $\Sigma = J(u)$. The field J is generated over K by the λ'_k, $k \neq k_0$, which shows that every element of J may be written as quotient of two polynomials in $\lambda_1, \lambda_2, \cdots, \lambda_n$ which are both invariants.

Let now Z' be a replica of Z. A polynomial in $\lambda_1, \lambda_2, \cdots, \lambda_n$ which is an invariant of Z is also an invariant of Z'; therefore we have $D_{Z'}J = \{0\}$. If we set $v = D_{Z'}u$, we have

$$D_{Z'}\phi = vD_Z\phi$$

for any $\phi \epsilon \Sigma$. In particular, we have $D_{Z'}\lambda_i = vD_Z\lambda_i$ $(1 \leq i \leq n)$. Since $D_Z\lambda_i$ and $D_{Z'}\lambda_i$ are both linear forms in $\lambda_1, \lambda_2, \cdots, \lambda_n$, we see that if $D_Z\lambda_i$, $D_Z\lambda_j$ are linearly independent for at least one pair (i, j) of indices, we may conclude that $v \epsilon K$, whence $-{}^tZ' = v(-{}^tZ)$, $Z' = vZ$. There remains to consider the case where any two of the forms $D_Z\lambda_i$ are linearly dependent. It may then be assumed without loss of generality that $D_Z\lambda_i = 0$ for $1 \leq i \leq n-1$, $D_Z\lambda_n = \lambda_{n-1}$, and Z^* maps $\mathfrak{M}^*$ onto the sub-space spanned by λ_{n-1}. Since $(Z')^*$ is a polynomial in Z^*, we have $D_{Z'}\lambda_i = 0$, $D_{Z'}\lambda_n = v\lambda_{n-1}$, $v \epsilon K$ whence again $Z' = tZ$ which completes the proof of Theorem 6.

PRINCETON UNIVERSITY.

ON CONJUGATE TRIGONOMETRIC POLYNOMIALS.*

By G. Szegö.

1. In a joint paper with A. C. Schaeffer[1] we discussed the following question: *Let D be a closed domain in the complex z-plane and z_0 a fixed point of D. Let us consider all polynomials $f(z)$ of given degree n for which $|\Re f(z)| \leqq 1$ in D and $f(z_0)$ is real. What is the largest possible value of $|\Im f(z)|$ if $f(z)$ is an arbitrary polynomial of the kind mentioned and z is arbitrary in D?* Under certain conditions on D and z_0 the inequality $|\Im f(z)| < A \log n$ was established where $A > 0$ depends only on D and z_0.

This theorem was well-known in the special case when D is a circle and z_0 is an interior point of D. However Mr. P. Erdös has called my attention to the fact that the best possible bound of $|\Im f(z)|$ is not known in this case.

In the present paper I wish to determine the best possible bound in case D is a circle and z_0 is the center of D. I prove the following

THEOREM. *Let $f(z)$ be a polynomial of degree n, $f(0)$ real, and*

$$(1.1) \qquad |\Re f(z)| \leqq 1, \qquad |z| \leqq 1.$$

Then

$$(1.2) \qquad |\Im f(z)| \leqq 2(n+1)^{-1} \sum_{1 \leqq m \leqq n,\ m \text{ odd}} \operatorname{cotg} \frac{m\pi}{2(n+1)} = M_n, \qquad |z| \leqq 1.$$

This constant M_n can not be replaced by a smaller one.

The asymptotic value of M_n as $n \to \infty$ is $(2/\pi) \log n$.

2. Another formulation of this theorem is as follows:

Let

$$(2.1) \qquad u(\phi) = a_0 + a_1 \cos \phi + b_1 \sin \phi + \cdots + a_n \cos n\phi + b_n \sin n\phi$$

be a trigonometric polynomial of degree n with real coefficients satisfying the condition

$$(2.2) \qquad |u(\phi)| \leqq 1, \qquad \phi \text{ real.}$$

Then the conjugate polynomial

---

* Received March 9, 1942.

[1] A. C. Schaeffer and G. Szegö, "Polynomials whose real part is bounded on a given curve in the complex plane," *American Journal of Mathematics*, vol. 62 (1940), pp. 868-876.

(2.3) $v(\phi) = -b_1 \cos\phi + a_1 \sin\phi - \cdots - b_n \cos n\phi + a_n \sin n\phi$

satisfies the inequality

$$|v(\phi)| \leqq M_n, \qquad \phi \text{ real}, \tag{2.4}$$

where M_n has the same meaning as in (1.2). *This bound can not be replaced by a smaller one.*

Further discussion is devoted to the problem of determining all the cases in which $|v(\phi_0)| = M_n$ holds, ϕ_0 being a fixed real value. As we shall prove, the solution of this problem is different according as n is even or odd.

For n even $|v(\phi_0)| = M_n$ holds if, and only if, $u(\phi) = \pm u_0(\phi - \phi_0)$ where

$$u_0(\phi) = \lambda_0(\phi) + \gamma \frac{1 + \cos(n+1)\phi}{1 + \cos\phi}; \tag{2.5}$$

here $\lambda_0(\phi)$ is the special sine polynomial (6.1) and γ is real, $|\gamma| \leqq (n+1)^{-2}$.

For n odd $|v(\phi_0)| = M_n$ holds if and only if $u(\phi) = \pm \lambda_0(\phi - \phi_0)$.

3. It is sufficient to prove (2.4) for $\phi = 0$. Furthermore the sine polynomial

(3.1) $\lambda(\phi) = \frac{1}{2}(u(\phi) - u(-\phi)) = b_1 \sin\phi + b_2 \sin 2\phi + \cdots + b_n \sin n\phi$

satisfies the inequality $|\lambda(\phi)| \leqq 1$ and its conjugate is

(3.2) $\mu(\phi) = \frac{1}{2}(v(\phi) + v(-\phi)) = -b_1 \cos\phi - b_2 \cos 2\phi - \cdots - b_n \cos n\phi$

so that $\mu(0) = v(0)$. Thus our theorem is equivalent to the following:

Let the sine polynomial

$$\lambda(\phi) = b_1 \sin\phi + b_2 \sin 2\phi + \cdots + b_n \sin n\phi \tag{3.3}$$

satisfy the condition $|\lambda(\phi)| \leqq 1$, ϕ real. Then (with the same M_n as before)

$$\left| \sum_{\nu=1}^{n} b_\nu \right| \leqq M_n. \tag{3.4}$$

This bound can not be replaced by a smaller one.

4. Next we prove the following

LEMMA. *By using the notation* (3.3) *we have the identity*

$$\sum_{\nu=1}^{n} b_\nu = 2(n+1)^{-1} \sum_{1 \leqq m \leqq n,\ m \text{ odd}} \lambda\left(\frac{m\pi}{n+1}\right) \operatorname{cotg} \frac{m\pi}{2(n+1)}. \tag{4.1}$$

Indeed, for any trigonometric polynomial $U(\phi)$ of degree $2n+1$ we have as well-known

$$\pi^{-1}\int_{-\pi}^{+\pi} U(\phi)\,d\phi = (n+1)^{-1}\sum_{m=-n}^{n+1} U\left(\frac{m\pi}{n+1}\right). \tag{4.2}$$

Apply this to the cosine polynomial [2]

$$U(\phi) = \lambda(\phi)\sum_{\nu=1}^{n}\sin\nu\phi = \lambda(\phi)\,\frac{\sin(n/2)\phi\,\sin[(n+1)/2]\phi}{\sin(\phi/2)}. \tag{4.3}$$

Then $[U(0)=U(\pi)=0,\ U(-\phi)=U(\phi)]$

$$\begin{aligned}\sum_{\nu=1}^{n} b_\nu &= \pi^{-1}\int_{-\pi}^{+\pi}\lambda(\phi)\left(\sum_{\nu=1}^{n}\sin\nu\phi\right)d\phi \\ &= 2(n+1)^{-1}\sum_{m=1}^{n}\lambda\left(\frac{m\pi}{n+1}\right)\frac{\sin\left(\dfrac{n}{2}\,\dfrac{m\pi}{n+1}\right)\sin\dfrac{m\pi}{2}}{\sin\left(\dfrac{1}{2}\,\dfrac{m\pi}{n+1}\right)}.\end{aligned} \tag{4.4}$$

This can easily be identified with (4.1).

5. The inequality $\left|\sum_{\nu=1}^{n} b_\nu\right| \leqq M_n$ follows immediately from the Lemma. The only remaining question is whether the equality sign holds for a suitable sine polynomial. This is the case if and only if

$$\lambda\left(\frac{m\pi}{n+1}\right) = +1 \ (\text{or } -1), \quad 1 \leqq m \leqq n,\ m \text{ odd}. \tag{5.1}$$

These equations in combination with $|\lambda(\phi)| \leqq 1$ furnish

$$\lambda'\left(\frac{m\pi}{n+1}\right) = 0, \quad 1 \leqq m \leqq n,\ m \text{ odd}. \tag{5.2}$$

The total number of these conditions is n or $n+1$ according as n is even or odd. They determine the sine polynomial $\lambda(\phi)$ uniquely in the first case whereas in the second case the existence of such a polynomial has to be shown. Finally in both cases the inequality $|\lambda(\phi)| \leqq 1$ must be verified.

6. The special sine polynomial $\lambda_0(\phi)$ satisfying conditions (5.1) and (5.2) has the character of the so called step-parabolas. It can be obtained by the Hermite interpolation: [3]

$$\lambda_0(\phi) = 2(n+1)^{-2}\sin\phi\sum_{1\leqq m\leqq n,\ m\text{ odd}}\frac{1+\cos(n+1)\phi}{\left(\cos\phi-\cos\dfrac{m\pi}{n+1}\right)^2}\sin\frac{m\pi}{n+1}. \tag{6.1}$$

---

[2] See, for instance, G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1 (Berlin, 1925), p. 77, p. 269, problem 16.

[3] Cf. for instance, G. Szegö, *Orthogonal Polynomials*, Colloquium Publications, vol. 23 (New York, 1939), p. 324, (14.1.9).

This is indeed a sine polynomial of degree n. The verification of the conditions mentioned does not cause any trouble. We only have to show that the polynomial

$$2(n+1)^{-2}\sin\phi\,\frac{1+\cos(n+1)\phi}{(\cos\phi-\cos m\pi/(n+1))^2}\sin m\pi/(n+1)$$

$$=2(n+1)^{-2}(1-x_m^2)^{1/2}(1-x^2)^{1/2}\frac{1+T_{n+1}(x)}{(x-x_m)^2}=t(x), \tag{6.2}$$

$$x=\cos\phi,\ x_m=\cos m\pi/(n+1),\qquad T_{n+1}(x)=\cos(n+1)\phi,$$

satisfies the conditions $t(x_m)=1$, $t'(x_m)=0$. But from the well-known differential equation

$$(1-x^2)T''_{n+1}(x)-xT'_{n+1}(x)+(n+1)^2T_{n+1}(x)=0 \tag{6.3}$$

we conclude that (m odd)

$$T''_{n+1}(x_m)=(n+1)^2(1-x_m^2)^{-1},\ T'''_{n+1}(x_m)=3(n+1)^2x_m(1-x_m^2)^{-2}. \tag{6.4}$$

Taking into account the relation

$$1+T_{n+1}(x)=\tfrac{1}{2}(x-x_m)^2T''_{n+1}(x_m)+\tfrac{1}{6}(x-x_m)^3T'''_{n+1}(x_m)+\cdots \tag{6.5}$$

we obtain

$$\begin{aligned} t(x)&=2(n+1)^{-2}(1-x_m^2)^{1/2}(1-x^2)^{1/2}\{\tfrac{1}{2}(n+1)^2(1-x_m^2)^{-1}\\ &\quad+\tfrac{1}{2}(n+1)^2x_m(1-x_m^2)^{-2}(x-x_m)+\cdots\}\\ &=\{1-x_m(1-x_m^2)^{-1}(x-x_m)+\cdots\}\\ &\quad\cdot\{1+x_m(1-x_m^2)^{-1}(x-x_m)+\cdots\} \end{aligned} \tag{6.6}$$

from which the assertion follows.

7. Further we need to show that $|\lambda_0(\phi)|\leqq 1$ holds, ϕ real. Condition (5.2) furnishes $[\frac{1}{2}(n+1)]$ roots of $\lambda'_0(\phi)$ in $0<\phi<\pi$, condition (5.1) in combination with Rolle's theorem $[\frac{1}{2}(n+1)]-1$ further roots of $\lambda'_0(\phi)$ in the same interval. The total number is $2[\frac{1}{2}(n+1)]-1=n-1$ or n according as n is even or odd. But in the first case π is also a (double) root of $\lambda'_0(\phi)$ so that in both cases we find the maximum number of roots for the cosine polynomial $\lambda'_0(\phi)$ in $0<\phi<\pi$. Thus all these roots of $\lambda'_0(\phi)$ are simple and these are the only roots of $\lambda'_0(\phi)$ in $0<\phi<\pi$. Hence $\lambda_0(\phi)$ is monotonic in the intervals determined by those roots. Taking into account $\lambda_0(0)=0$ and the conditions (5.1) we find that $\lambda_0(\phi)\leqq 1$ in $0\leqq\phi\leqq\pi$. From (6.1) we see that in the same interval $\lambda_0(\phi)\geqq 0$.

This establishes the proof of our Theorem.

8. Finally we wish to determine all the cases in which $|v(0)| = M_n$ holds. The previous argument shows that in this case $u(\phi) = \pm u_0(\phi)$ where

$$u_0(\phi) = \lambda_0(\phi) + c_0(\phi); \tag{8.1}$$

here $c_0(\phi)$ is a certain cosine polynomial with real coefficients. Because of (2.2), (5.1) and (5.2) we must have

$$c_0\left(\frac{m\pi}{n+1}\right) = c'_0\left(\frac{m\pi}{n+1}\right) = 0, \qquad 1 \leqq m \leqq n,\ m \text{ odd}. \tag{8.2}$$

The number of these conditions is n or $n+1$ according as n is even or odd. Hence in the second case $c_0(\phi) \equiv 0$.

Let n be even. Then

$$c_0(\phi) = \gamma \frac{1+\cos(n+1)\phi}{1+\cos\phi}, \qquad \gamma \text{ a real constant}, \tag{8.3}$$

are the only polynomials satisfying (8.2). We have $|u_0(\pi)| = |c_0(\pi)| = |\gamma|\,(n+1)^2 \leqq 1$. Now we show that

$$0 \leqq h(\phi) = \lambda_0(\phi) + (n+1)^{-2}\frac{1+\cos(n+1)\phi}{1+\cos\phi} \leqq 1, \quad 0 \leqq \phi \leqq \pi. \tag{8.4}$$

Indeed,

$$\begin{aligned} &h[m\pi/(n+1)] = -h\{-[m\pi/(n+1)]\} = 1, \\ &h'[m\pi/(n+1)] = h'\{-[m\pi/(n+1)]\} = 0, \quad 1 \leqq m \leqq n,\ m \text{ odd}. \end{aligned} \tag{8.5}$$

Also $h(\pi) = 1$, $h'(\pi) = 0$. Furthermore Rolle's Theorem furnishes $n/2$ roots of $h'(\phi)$ in $0 < \phi < \pi$ and $n/2 - 1$ roots in $-\pi < \phi < 0$. Thus we find altogether

$$n/2 + 1 + n/2 + n/2 + n/2 - 1 = 2n$$

(mod 2π) distinct roots for $h'(\phi)$. Hence all these roots of $h'(\phi)$ are simple and these are the only roots of $h'(\phi)$. Consequently $h(\phi)$ is monotonic in the intervals defined by those roots. Since $h(0) = (n+1)^{-2}$ an easy discussion furnishes (8.4).

Stanford University,
California.

RADICAL IDEALS.*

By Reinhold Baer.

The radical has been introduced into the theory of rings with the intent of constructing a two-sided nilideal modulo which there do not exist nilpotent right-ideals different from 0. Thus it seems justified to define as a radical ideal every ideal meeting these requirements. Every ring possesses at least one radical ideal; and both the cross-cut and the sum of all the radical ideals are themselves radical ideals which may be called the lower and the upper radical respectively. It is possible that the upper and lower radicals are different and that neither of them is nilpotent. If all the radical ideals are equal, then we say that the radical exists; and this happens e. g., if every right-ideal, not 0, in the quotient ring modulo the lower radical contains a smallest right-ideal different from 0. If this latter condition is satisfied by every quotient ring of the ring under consideration, then a finite or transfinite power of the radical is 0.

Several applications of the theory of radical ideals are given: we prove a criterion for the existence of the identity element in rings that need not satisfy the minimum condition for right- (or left-) ideals; and we deduce the double chain condition for right-ideals from properties considerably weaker than the minimum condition.

It should be noted that we have restricted our attention throughout to the consideration of right-ideals.

1. Existence of the upper and lower radicals. The element x in the ring R is a *nilelement*, if $x^i = 0$ for some exponent i; and the right-ideal J in R may be termed a *nilideal*, if every element in J is a nilelement. The right-ideal J is *nilpotent*, if $J^i = 0$ for some exponent i. It is clear that nilpotent right-ideals are nilideals, though the converse need not be true.[1] It is well known [2] that the sum of a finite number of nilpotent right-ideals is a nilpotent right-ideal; and it is obvious that xJ is a nilpotent right-ideal, whenever J is a nilpotent right-ideal and x an element in R. From these two facts one readily deduces the following well known statement.

---

* Received May 15, 1942. Presented to the American Mathematical Society, April 16, 1942.

[1] For an example cf. Köthe (1), p. 165 (the numbers in parentheses refer to the bibliography at the end of the paper).

[2] Cf. e. g. v. d. Waerden (1), p. 154.

LEMMA 1.1. *The sum $N = N(R)$ of all the nilpotent right-ideals in R is a two-sided nilideal in R.*

Note that the ideal $N(R)$ need not be nilpotent, as follows from an example due to Köthe.[3]

The ideal P in the ring R shall be termed *a radical ideal,*[4] if

(1.a) P is a two-sided ideal;

(1.b) P is a nilideal;

(1.c) the quotient-ring R/P does not contain nilpotent right-ideals different from 0.

There exist ideals in R which meet the requirements (1.a) and (1.b), e.g. the null-ideal. Thus we may form the sum[5] $U = U(R)$ of all the ideals P in R satisfying conditions (1.a) and (1.b). This ideal U is clearly a two-sided ideal in R and shall be called *the upper radical of R.*

There exist ideals in R which meet the requirements (1.a) and (1.c), e.g. the ideal $P = R$. Thus we may form the cross-cut $L = L(R)$ of all the ideals P in R which satisfy conditions (1.a) and (1.c). This ideal L is obviously a two-sided ideal in R and shall be termed *the lower radical of R.*

It is an immediate consequence of our definitions that every radical ideal is situated between the upper and the lower radical. But the main justification for our terminology may be seen in the following fact.

THEOREM 1.2. *The upper and the lower radical of the ring R are radical ideals in R.*

Proof. It has been shown by Köthe[6] that the sum of all the two-sided nilideals in R is a nilideal. Denote by W the uniquely determined two-sided ideal in R which satisfies: $U \leq W$ and $W/U = N(R/U)$. There exists, by Lemma 1.1, to every element x in W a positive integer i such that x^i is an element in the nilideal U; and hence every element in W is a nilelement. Now it follows from the definition of the upper radical that $W = U$, $N(R/U) = 0$; and thus we have shown that the upper radical of R is a radical ideal in R.

$L \leq U$, since we have just shown that the upper radical U is a radical ideal, and since the lower radical L is certainly part of every radical ideal in R. Consequently L is a nilideal, since U is a nilideal. Every nilpotent right-ideal in R/L has the form S/L for S a suitable right-ideal between L and R; and from the nilpotence of S/L we infer the existence of a positive integer i such that $S^i \leq L$. Suppose now that condition (1. c) is satisfied by the two-sided ideal T in R. Then $L \leq T$ and $((T+S)/T)^i = 0$, since $(T+S)^i \leq T + S^i \leq T + L = T$; and it follows from (1. c) that $(T+S)/T = 0$ or $T + S = T$ or $S \leq T$. Thus we have shown that S is part of every ideal T, satisfying (1. a) and (1. c); and hence $S \leq L$, proving[7] that L is a radical ideal in R.

---

[3] Cf. Köthe (1), p. 165.

[4] The existence of radical ideals will be assured by Theorem 1.2 below.

[5] This ideal has been considered by Fitting (1), p. 21.

[6] Köthe (1), p. 170.

Remarks. 1. The following construction of the lower radical may be of some interest (in particular in case the transfinite induction involved in it happens to stop after a finite number of steps):

(i) $Q(0) = 0$.

(ii) Suppose that the two-sided ideal $Q(u)$ has been defined for every $u < v$.

Case 1. If $v = w + 1$ is not a limit-ordinal, then $Q(v)$ is the uniquely determined ideal in R which contains $Q(w)$ and which satisfies: $Q(v)/Q(w) = N(R/Q(w))$; that $Q(v)$ is a two-sided ideal in R, is an immediate consequence of Lemma 1. 1.

Case 2. If v is a limit-ordinal, then denote by $Q(v)$ the join of all the ideals $Q(u)$ for $u < v$. It is readily verified that $Q(v)$ is a two-sided ideal in R.

(iii) Since the $Q(v)$ form an ascending chain of ideals in R, there exists a (smallest) ordinal z such that $Q(z) = Q(z+1)$; and we put $Q(z) = Q$.

It is clear from our construction of Q that Q is a two-sided ideal in R and that R/Q does not contain nilpotent right-ideals different from 0. Hence it follows from the definition of the lower radical L of R that $L \leq Q$.

Suppose next that the ideal T in R meets the requirements (1. a) and (1. c). It is clear that $Q(0) \leq T$; and thus we may assume that every $Q(u)$ for $u < v$ is part of T. Then it is readily verified that $Q(v)$ is also part of T; and thus it follows by complete induction that Q is part of T. But this shows that $Q \leq L$; and thus we have shown that the ideal Q just constructed is the lower radical L of R.

2.[8] If T is a two-sided ideal between L and U, then the upper radical of R/T is U/T; and if T is a radical ideal, then the lower radical of R/T is 0.

---

[7] The author is indebted to the referee for this proof, which is much simpler than the author's original proof, of the fact that L is a radical ideal.

[8] This remark is due to the referee.

3. It has been pointed out before that every radical ideal is situated between the upper and lower radical; and it is a consequence of Theorem 1. 2 that every subideal of the upper radical is a nilideal. It may happen that there exist two-sided ideals between the upper and the lower radical which are not radical ideals; an example for this phenomenon will be constructed in section **2**.

4. It is a consequence of Theorem 1. 2 that *the upper and lower radicals are equal, if U/L is a nilpotent ideal in R/L.* In general, however, the upper and lower radicals may be different, as will be seen from the example constructed in section **2**.

If the upper and the lower radical are equal, then this ideal may be termed *the radical $K = K(R)$ of the ring R*; and we say then that *the radical of R exists.* It should be noted that the radical need not be nilpotent, as may be seen from an example due to Köthe.[9]

2. Existence of different radical ideals. In this section we construct a ring with the following properties:

(i) Every element in the ring is a nilelement so that the ring is its own upper radical U.

(ii) The ring does not contain nilpotent right-ideals different from 0 so that its lower radical is 0.

(iii) The ring contains a two-sided ideal which is not a radical ideal.

Denote by G an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $b(0), b(1), b(-1), \cdots, b(i), b(-i), \cdots$ a basis of G. Then there exists one and only one endomorphism ($=$ homomorphism) $u(i)$, for $i = 1, 2, \cdots$, of G such that

$$b(j)u(i) = \begin{cases} 0, & \text{if } j \equiv 0 \bmod 2^i \\ b(j-1), & \text{if } j \not\equiv 0 \bmod 2^i. \end{cases}$$

The following statement may be easily verified by complete induction:

$$(*) \quad b(j)u(i_0) \cdots u(i_m) = \begin{cases} b(j-m-1), \text{ if } j \not\equiv n \bmod 2^{i_n} \text{ for } 0 \leq n \leq m \\ 0, \text{ if } j \equiv n \bmod 2^{i_n} \text{ for at least one } n \text{ with } 0 \leq n \leq m. \end{cases}$$

Denote by U the ring of endomorphisms of G which is generated by the endomorphisms $u(1), u(2), \cdots$.

Every element $x \neq 0$ in U has the form: $x = \sum_{i=1}^{k} x_i$ where $x_i = \pm u((i, 0))$

---

[9] Köthe (1), p. 165.

$\cdots u((i, m_i))$, $0 < (i, j)$, $0 \leq m_i$. If h is the maximum of all the numbers $2^{(i.n)}$, then it is an immediate consequence of (*) that $b(j)x_{i_1} \cdots x_{i_h} = 0$ for every j, since the product of endomorphisms contains at least h factors $u(s)$ with $2^s \leq h$. Consequently this product is 0 and this implies that $x^h = 0$. The ring U has therefore property (i).

For a proof of property (ii) we need a closer analysis of the structure of the endomorphisms in U. If x is an endomorphism different from 0 in U, then $x = \sum_{i=0}^{m} y(i)$, $y(i) = \sum_{j=1}^{k(i)} e(i,j)y(i,j)$, $e(i,j) = \pm 1$, $y(i,j) = u((i,j;0)) \cdots u((i,j;i))$, $0 < (i,j;n)$, $0 \leq k(i)$, and $y(n) \neq 0$. Denote by r some integer satisfying: $m + 1 < 2^r$ and $(i, j; n) \leq r$ for $i = 0, \cdots, m$; $j = 1, \cdots, k(i)$; $n = 0, \cdots, i$.

If h is some preassigned positive integer, then denote by s an integer satisfying $2^{r+1}h \leq 2^s$. We proceed to prove that $(xu(s)^{2^r-m-1})^h \neq 0$; a fact that implies the impossibility of nilpotence of right-ideals different from 0.

It will be convenient to put $z(i,j) = y(i,j)u(s)^{2^r-m-1}$.

Since $y(m) \neq 0$, there exists an integer t such that $b(t)y(m) \neq 0$. Hence we may assume that the positive integer k has been determined in such a way that

$$b(t)y(m,j) = \begin{cases} b(t-m-1) & \text{for } j = 1, \cdots, k \\ 0 & \text{for } k < j. \end{cases}$$

Then $b(t)y(m) = \sum_{j=1}^{k} e(m, j)b(t - m - 1) \neq 0$; and this shows that $\sum_{j=1}^{k} e(m,j) \neq 0$, a fact we shall have to use later on.

If $0 \leq i_v \leq m$, $1 \leq j_v \leq k(i_v)$ for $1 \leq v \leq h$, and if e is 0 or 1, then it follows from (*) that $b(t + eh2^r)z(i_1, j_1) \cdots z(i_h, j_h)$ is either 0 or $b(t + eh2^r - \sum_{v=1}^{h} (i_v + 1 + 2^r - m - 1)) = b(t + (e-1)h2^r + mh - \sum_{v=1}^{h} i_v)$ and this latter element is equal to $b(t + (e-1)h2^r)$ if, and only if, $i_1 = \cdots = i_h = m$.

If $k < j_v$ for some v, then $b(t)y(m, j_v) = 0$; and thus there follows from (*) the existence of an integer n such that $0 \leq n \leq m$ and $t \equiv n \bmod 2^{(m, j_v; n)}$. But then we find that $t + eh2^r \equiv t \equiv n \equiv n + (v-1)2^r \bmod 2^{(m, j_v; n)}$ and hence we may deduce from (*) that $b(t + eh2^r)z(m, j_1) \cdots z(m, j_h) = 0$. The results obtained in the last two paragraphs may be stated as follows:

If $b(t + eh2^r)z(i_1, j_1) \cdots z(i_h, j_h) = b(t + (e-1)h2^r)$, then $i_1 = \cdots = i_h = m$ and $1 \leq j_v \leq k$.

Assume now the existence of integers j_v, j'_v such that $1 \leq j_v, j'_v \leq k$ and

$b(t)z(m,j_1)\cdots z(m,j_h)=0$, $b(t+h2^r)z(m,j'_1)\cdots z(m,j'_h)=0$. Then we infer from (*) the existence of integers n, v satisfying: $1\leq v\leq h$ and

$$t\equiv\begin{cases} n+(v-1)2^r \bmod 2^{(m,j_v;\,n)} & \text{for } 0\leq n\leq m\\ n+(v-1)2^r \bmod 2^s & \text{for } m<n<2^r.\end{cases}$$

If $0\leq n\leq m$, then $t\equiv n \bmod 2^{(m,j_v;\,n)}$; and we could deduce from (*) that $b(t)y(m,j_v)=0$ which is impossible, since $j_v\leq k$. Thus there exist integers n, v such that $m<n<2^r$, $1\leq v\leq h$ and $t\equiv n+(v-1)2^r \bmod 2^s$; and likewise there exist integers n', v' such that $m<n'<2^r$, $1\leq v'\leq h$ and $t+h2^r\equiv n'+(v'-1)2^r \bmod 2^s$. Since $r<s$, it follows from these two congruences that $n\equiv n' \bmod 2^r$; and since $m<n$, $n'<2^r$, we deduce that $n=n'$. Sonsequently $(v-1+h)2^r\equiv(v'-1)2^r \bmod 2^s$ or $h+v-v'\equiv 0 \bmod 2^{s-r}$. Since $1\leq v$, $v'\leq h$, we have $0<h+v-v'<2h\leq 2^{s-r}$, a contradiction. Thus we have shown that there exists an integer t' such that

$$b(t')z(i_1,j_1)\cdots z(i_h,j_h)=b(t'-h2^r)$$

if, and only if, $i_1=\cdots=i_h=m$ and $1\leq j_v\leq k$ for $v=1,\cdots,h$.

Now it is readily verified that

$$b(t')(xu(s)^{2^r-m-1})^h=\Big(\sum_{j=1}^{k}e(m,j)\Big)^h b(t'-h2^r)+\sum_{i\neq t'-h2^r}c(i)b(i)$$
$$\neq 0,$$

since the factor of $b(t'-h2^r)$ is, by a previous remark, different from 0.

This completes the proof of the fact that (ii) is satisfied by the ring U.

To prove (iii) let us consider the two-sided ideal T in U which is generated by $u(2)$, $u(3),\cdots$. Every element in T is a linear combination of products $u(i_0)\cdots u(i_m)$ with the restriction that none of these products is a power of $u(1)$.

We note that $b(j)u(1)=\begin{cases}0 & \text{for even } j\\ b(j-1) & \text{for odd } j\end{cases}$. Hence $u(1)^2=0$; and it is readily verified that $U^2\leq T$. Thus all we have to show is that $u(1)$ is not an element in T.

Every element $x\neq 0$ in T has the form: $x=\sum_{i=2}^{k}c(i)u(i)+x'$ for x' in U^2 so that $b(t)x'=\sum_{j\neq t-1}d(j)b(j)$. If $x=u(1)$, then it follows from $b(1)u(i)=b(0)$ that $\sum_{i=2}^{k}c(i)=1$; and it follows from $b(2)u(i)=\begin{cases}0 & \text{for } i=1\\ b(1) & \text{for } i\neq 1\end{cases}$ that $\sum_{i=2}^{k}c(i)=0$. This contradiction shows that $u(1)$ is not in T so that $U^2\leq T<U$, as was required.

If we adjoin to the ring U of endomorphisms of the abelian group G the identity-endomorphism, we obtain a ring R whose upper radical is U, whose lower radical is 0 and which contains the two-sided nilideal T such that R/T contains nilpotent right-ideals different from 0.

3. Characterization of the upper radical. The right-ideal J in the ring R is termed *a minimal right-ideal,* if $0 < J$, and if there does not exist a right-ideal J' such that $0 < J' < J$.

THEOREM 3. 1. *If T is a radical ideal in the ring R, and if every right-ideal different from 0 in the quotient ring R/T contains a minimal right-ideal, then T is the upper radical U of R and the upper radical of R contains every nilideal*[10] *in R.*

Proof. If the nilideal J in R is not part of T, then there exists an ideal V between T and $T + J$ such that V/T is a minimal right-ideal in R/T. Since V/T is part of $(T + J)/T$, and since J is a nilideal, V/T is a nilideal. A minimal right-ideal is either nilpotent or idempotent; and since T is a radical ideal in R, it follows that V/T is idempotent. But it is well known[11] that idempotent minimal right-ideals contain idempotent elements not 0. Thus the nilideal V/T contains an idempotent element not 0, a contradiction. Hence it follows that T contains every nilideal in R. Since T is a radical ideal, it is a two-sided nilideal and therefore part of the upper radical. Since the upper radical is the sum of all the two-sided nilideals, it follows from what we have shown just now that U is part of T, i. e. $U = T$ contains every nilideal.

The right-ideal J is termed *a maximal right-ideal in the ring R,* if $J < R$, and if there does not exist a right-ideal J' such that $J < J' < R$. The element e in R is a *left-identity element,* if $ex = x$ for every element x in the ring R.

LEMMA 3. 2. *If the ring R contains a left-identity element e, then each nilideal is part of every maximal right-ideal.*

Proof. If the nilideal V were not part of the maximal right-ideal J, then $R = V + J$ and there would exist elements v and j in V and J respectively such that $e = v + j$. By complete induction we may show the existence of elements $j(i)$ in J such that $e = v^i + j(i)$, since

$$e = e^2 = (j + v)v^i + ej(i) = v^{i+1} + jv^i + j(i) = v^{i+1} + j(i + 1).$$

---

[10] This shows that under the hypotheses of Theorem 3. 1 the upper radical meets all the requirements, concerning right-ideals, imposed upon the radical by Köthe (1), p. 169.

[11] Cf. e. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

But v is a nilelement; and hence it follows that e is in J and that therefore $J = R$, an impossibility which proves our contention.

THEOREM 3. 3.[12] *If the ring R contains a left-identity element, and if every right-ideal different from 0 in the quotient ring R/U contains a minimal right-ideal, then the upper radical U of R is the cross-cut of all the maximal right-ideals in R.*

Remark. The example in section **2** shows the impossibility of omitting the hypothesis concerning R/U; and Theorem 9. 6 below shows the need for assuming the existence of a left-identity element in R.

Proof. If the intersection J of all the maximal right-ideals in R/U were different from 0, then J would contain a minimal right-ideal J'; and consequently we could infer[13] the existence of an idempotent $e \neq 0$ in J'. It is well known that R/U is the direct sum of the right-ideals J' and Z where J' consists of all the elements $v = ev$ and where Z consists of the elements z satisfying $ez = 0$. Thus the crosscut of Z and J' is 0. Since J' is a minimal right-ideal in R/U, Z is a maximal right-ideal in R/U. Hence $J' \leq J \leq Z$, a contradiction showing that $J = 0$. Thus we have proved that U is the intersection of all the maximal right-ideals in R which contain U; and it follows from Lemma 3. 2 that U is part of every maximal right-ideal in R, since U is, by Theorem 1. 2, a nilideal.

4. The anti-radical.

We state the following well known fact[14] for future reference.

LEMMA 4. 1. *If N is a sum of (a finite or infinite number of) minimal right-ideals in the ring R, then every right-ideal contained in N is a direct summand of N and is itself a direct sum of minimal right-ideals in R.*

The sum[15] $M = M(R)$ of all the minimal right-ideals in the ring R shall be called *the anti-radical of R*; and we put $M(R) = 0$, if there are no minimal right-ideals in R. This definition may be justified by the fact that the upper radical is, under not too narrow assumptions, just the cross-cut of all the maximal right-ideals in R (see Theorem 3.3); and by the fact which we shall prove immediately that radical ideals and the anti-radical annihilate each other.

---

[12] The author proved this theorem originally, using a stronger hypothesis. He is indebted to the referee for supplying him with a proof for the theorem in its present form.

[13] Cf. e. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[14] Cf. e. g. MacLane (1), p. 458, Theorems 3 and 7.

[15] This ideal has been investigated by Hopkins (1) under the hypothesis that the minimum condition be satisfied by the right-ideals in the ring.

If J is a minimal right-ideal in the ring R, x an element in R, then xJ is either 0 or a minimal right-ideal in R; and this shows that *the anti-radical is a two-sided ideal.*

THEOREM 4.2. *If M is the anti-radical of the ring R and J a nilideal in R, then $MJ = 0$.*

Proof. If b is an element in the minimal right-ideal B, j an element in the nilideal J, and if bj were different from 0, then bj would be an element not 0 in the minimal right-ideal B so that B would be the smallest right-ideal containing bj. Consequently there exists an element r in R such that[16] $b = bjr \pm bj \pm \cdots \pm bj = b(jr \pm j \pm \cdots \pm j) = bj'$ where j' is an element in J, since j is an element in the right-ideal J. There exists an integer i such that $j'^i = 0$ and this leads to the contradiction: $0 \neq bj = bj'j = \cdots = bj'^ij = 0$. We have shown, therefore, that $BJ = 0$ for every minimal right-ideal B in R; and this fact clearly implies $MJ = 0$.

The following condition will be imposed frequently upon the rings under consideration.

(4.A) *If x is an element in the ring R, then x is contained in the right-ideal xR.*

This requirement is met e.g. by all the rings which contain a right-identity element; and (4.A) is satisfied by the ring R if, and only if, $J = JR$ for every right-ideal J in R.

THEOREM 4.3. *If condition (4.A) is satisfied by the ring R, if T is a two-sided ideal in R, and if R/T is a sum of minimal right-ideals, then the anti-radical M contains every element x, satisfying: $xT = 0$.*

Proof. If $xT = 0$ is satisfied by the element x in R, and if the right-ideal J in R contains T and is a minimal right-ideal modulo T, then either $xJ = 0$ or xJ is a minimal right-ideal in R. The right-ideal xR is consequently a sum of minimal right-ideals in R, since R/T is a sum of minimal right-ideals in R/T; and as a sum of minimal right-ideals xR is part of the sum M of all the minimal right-ideals in R. Applying condition (4.A) to $xR \leq M$ we find finally that x is in M whenever $xT = 0$.

---

[16] This more complicated form for the most general element in the right-ideal generated by bj is due to the fact that no identity element need exist in R.

Remark. The impossibility of omitting condition (4. A) in Theorem 4. 3 may be seen from the following example: Denote by R the ring of all the multiples of the prime number p considered modulo p^3 and denote by T the ideal of all the multiples of p^2 in R. It is obvious that T is a two-sided ideal in R such that R/T is a minimal ideal and such that $RT = 0$, though R is not a sum of minimal ideals.

The following statement is an immediate consequence of Theorems 1. 2, 4. 2 and 4. 3.

THEOREM 4. 4. (a) $MU = 0$.

(b) *If condition* (4. A) *is satisfied by the ring R, and if R/U is a sum of minimal right-ideals, then the anti-radical is exactly the set of all the the elements x which satisfy:* $xU = 0$.

Remark. The impossibility of omitting the second hypothesis in (b) may be seen from the example of the ring of all the integers whose radical and anti-radical are both 0.

The following lemma will prove useful later on.

LEMMA 4. 5. *If the right-ideal J in the ring R is contained in the anti-radical M of R, then* $J^2 = J^3$.

Proof. It is a consequence of Lemma 4. 1 that J is a sum of minimal right-ideals. If Z is a minimal right-ideal in R, then Z^2 is a subideal of Z and consequently either $Z^2 = Z$ or $Z^2 = 0$. If Z is a minimal right ideal contained in J, then $Z^2 = Z$ implies that $Z \leq J^2$, and $Z^2 = 0$ implies that Z is part of the cross-cut C of J and the upper radical U. From these facts we deduce that $J = J^2 + C$. From Theorem 4. 2 we infer that $JC \leq MU = 0$. Consequently $J^2 = J(J^2 + C) = J^3 + JC = J^3$.

M. Hall has shown [17] that every algebra may be decomposed in one and only one way into the sum of a semi-simple and a "bound" algebra. The following concepts will be needed for an extension of his theorem.

We denote by $A = A(R)$ the set of all the elements x in R which satisfy: $xU = Ux = 0$; and we denote by $B = B(R)$ the set of all the elements b in R which satisfy: $bA^2 = A^2b = 0$. It is obvious that A and B are two-sided ideals in R, and that $U \leq B$.

THEOREM 4. 6. *Suppose that condition* (4. A) *is satisfied by the ring R.*

(a) *If R is the direct sum of the two-sided ideals S and T, if $U \leq T$, and if the cross-cut of A and T is part of U, then $S = A^2$ and $T = B$.*

---

[17] Hall (2), Theorem 2. 2.

(b) *If R/U is a sum of minimal right-ideals, then the cross-cut of A^2 and B is* 0; *and the cross-cut of A and B is part of U.*

(c) *If R/U is a sum of minimal right-ideals, and if the cross-cut of A^2 and the anti-radical M of R is the sum of a finite number of minimal right-ideals,*[18] *then R is the direct sum of A^2 and B.*

Proof. If the two-sided ideals S and T meet the requirements of (a), then $ST = TS = 0$ and consequently $S \leq A$. If D is the cross-cut of A and T, then A is the direct sum of S and D, and D is part of U. Thus $D^2 \leq AU = 0$ and $A^2 = S^2 + D^2 = S$; and this implies $B = T$.

If R/U is the sum of minimal right-ideals, then it follows from Theorem 4. 4, (b) that A is part of the anti-radical M of R; and hence it follows from Lemma 4. 5 that $A^2 = A^3$. If W is the cross-cut of A^2 and B, then we deduce from Lemma 4. 1 the existence of a right-ideal V such that A^2 is the direct sum of W and V. Clearly $A^2W = WA^2 = 0$ and hence we have $A^2 = A^4 = (V + W)^2 = V^2$, proving that $W = 0$. The cross-cut of A and B is part of M and is therefore the sum of minimal right-ideals. If Z is a minimal right-ideal, then either $Z^2 = Z$ or $Z^2 = 0$; and if $Z = Z^2$, then Z is part of A^2 and is therefore not contained in B. If $Z^2 = 0$, then Z is nilpotent and is therefore part of U; and this completes the proof of (b).

If the requirements of (c) are met by the ring R, then A^2 is the sum of a finite number of minimal right-ideals; and since the cross-cut of A^2 and B is 0, it follows from $U \leq B$ that the minimal right-ideals contained in A^2 are idempotent. Now we may deduce by the customary arguments [19] the existence of an idempotent e such that $A^2 = eR$. Denote by E the set of elements x in A^2 such that $xe = 0$. Clearly E is a left-ideal and the two-sided ideal ER satisfies: $(ER)^2 = ERER \leq E^2R \leq EA^2 = EeR = 0$, showing that $ER \leq U$. Applying (4. A) we infer now that $E \leq ER \leq U$; and since the cross-cut of A^2 and U is part of the cross-cut 0 of A^2 and B, it follows that $E = 0$; and from this fact we deduce that $A^2 = eR = eA^2 = Re = A^2e$; and this shows that the element b in R belongs to B if, and only if, $be = eb = 0$. If r is any element in R, then $e(r - ere)$ and $(r - ere)e$ belong both to $eR = Re$ and satisfy therefore: $e(r - ere) = e(r - ere)e = 0$ and $(r - ere)e = e(r - ere)e = 0$. Consequently ere belongs to $A^2 = eRe$ and $r - ere$ belongs to B, i. e. R is the direct sum of A^2 and B.

---

[18] It is not known to the author whether or not this hypothesis is needed for the validity of the proposition (c).

[19] E. g. v. d. Waerden (1), pp. 156-158.

5. The anti-radical series. An ascending chain of two-sided ideals $M_v = M_v(R)$ in the ring R may be defined by complete (transfinite) induction as follows:

(i) M_1 is the anti-radical M of R.

(ii) M_{v+1} is the uniquely determined two-sided ideal in R which contains M_v and which satisfies: $M(R/M_v) = M_{v+1}/M_v$.

(iii) If v is a limit-ordinal, then M_v is the join (and therefore the sum) of all the ideals M_u for $u < v$.

(iv) There exists a smallest ordinal $m = m(R)$ such that $M_m = M_{m+1}$.

THEOREM 5.1. *$R = M_m$ for some (finite or infinite) ordinal m if, and only if, the following condition is satisfied by every quotient-ring of R:*

(5.B) *Every right-ideal different from 0 contains a minimal right-ideal.*

Proof. If condition (5.B) is satisfied by every quotient ring of R, and if $M_v < R$, then R/M_v contains a minimal right-ideal so that $M_v < M_{v+1}$. Since $M_m = M_{m+1}$ for some ordinal m, this shows the sufficiency of the condition.

If $R = M_m$ for some ordinal m, and if T is a two-sided ideal different from R, then it is easy to construct a well-ordered ascending chain of right-ideals $J(v)$ with the following properties:

(a) $J(0) = T$.

(b) $J(v) < J(v+1)$ and there does not exist a right-ideal J satisfying: $J(v) < J < J(v+1)$.

(c) If v is a limit-ordinal, then $J(v)$ is the join of all the right-ideals $J(u)$ for $u < v$.

(d) $J(w) = R$ for some (finite or infinite) ordinal w.

If J is some right-ideal in R such that $T < J$, then there exists a smallest ordinal z such that the cross-cut of J and $J(z)$ is different from T, since the cross-cut J of J and $J(w) = R$ is different from T. It is an obvious consequence of (c) that z cannot be a limit-ordinal; and we may readily verify that the cross-cut V of J and $J(z)$ is a minimal right-ideal modulo T, as was to be shown.

COROLLARY 5.2. *If $R = M_m$ for some ordinal m, then the radical K of the ring R exists.*

Proof. If $R = M_m$, then it follows from Theorem 5.1 that (5.B) is satisfied by R/L. Hence it follows from Theorems 1.2 and 3.1 that the lower radical L and the upper radical U of the ring R are equal, i.e. $U = L$ is the radical K of R.

We note that under the hypotheses of Corollary 5.2 every nilideal is contained in the radical K of R.

It should be mentioned finally that (5.B) is satisfied by every quotient ring of R, whenever the minimum condition is satisfied by the right-ideals in R.

6. The powers of the radical.

If J is a right-ideal in the ring R, then the (finite and transfinite) powers J^v of J are defined by transfinite induction as follows:

(i) $J^1 = J$.

(ii) $J^{v+1} = JJ^v$.

(iii) If v is a limit-ordinal, then J^v is the cross-cut of all the J^u for $u < v$.

(iv) There exists a (smallest) ordinal $s = s(J)$ such that $J^s = J^{s+1}$.

It is clear that the powers of a right-ideal are right-ideals; and that the powers of a two-sided ideal are two-sided ideals.

If J is a right-ideal in R, and if $u \leq v$, then $J^v \leq J^u$.

Proof. We prove this contention by complete induction with regard to v. It is certainly true for $v = 1$; and thus we may assume that $J^w \leq J^u$ for $u \leq w < v$.

Case 1. v is a limit-ordinal.

Then J^v is the cross-cut of all the J^u for $u < v$ so that $u < v$ implies $J^v \leq J^u$.

Case 2. $v = w + 1$ for w a limit-ordinal.

Then $J^v = JJ^w \leq JJ^u = J^{u+1}$ for every $u \leq w$. Thus J^v is part of the cross-cut of all the J^{u+1} for $u < w$; and this cross-cut is just J^w, since w is a limit-ordinal.

Case 3. $v = z + 2$ for some ordinal z.

Then $J^v = JJ^{z+1} \leq JJ^z = J^{z+1} \leq J^w$ for $w \leq z + 1 < v$; and this completes the proof.

THEOREM 6.1. *If J is a nilideal, then $M_v J^v = 0$ for every ordinal v.*

Proof. It is a consequence of Theorem 4.2 that $M_1J^1 = MJ = 0$. Thus we may assume that our assertion holds for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then M_v/M_w is the anti-radical of R/M_w and $(J + M_w)/M_w$ is a nil-ideal in the quotient ring R/M_w. Hence it follows from Theorem 4.2 that their product is 0; and from this fact we deduce $M_vJ \leq M_w$. From the induction-hypothesis we infer $M_wJ^w = 0$. Consequently we find that $M_vJ^v = M_vJJ^w \leq M_wJ^w = 0$.

Case 2. v is a limit-ordinal.

If x is any element in M_v, then we deduce from the definition of M_v as the join of the M_u with $u < v$ the existence of an ordinal $u < v$ such that x is an element in M_u. It is a consequence of the definition of J^v that $J^v \leq J^u$; and hence it follows from the induction-hypothesis that $xJ^v \leq M_uJ^u = 0$; and thus we have shown that $M_vJ^v = 0$.

COROLLARY 6.2. *If $R = M_m$ for some ordinal m, then $K^{m+1} = 0$.*

Remark. The existence of the radical K of R is assured by Corollary 5.2.

Proof. It is an immediate consequence of Theorem 6.1 that

$$K^{m+1} = KK^m \leq RK^m = M_mK^m = 0.$$

Note that we could infer $K^m = 0$ from $R = M_m$, if 0 were the only element x in R satisfying $Rx = 0$.

That $K^k = 0$ need not imply $R = M_m$, may be seen from the example of the ring of all the integers where $M_v = K^v = 0$ for every v.

THEOREM 6.3. *If $R = M_m$ for some ordinal m, then each of the following properties of the right-ideal J in R implies all the others:*

(i) *J is a nilideal.*

(ii) *J is a part of the radical K of R (whose existence is assured by Corollary 5.2).*

(iii) *$J^k = 0$ for some ordinal k.*

Proof. It is a consequence of Theorems 5.1 and 3.1 that every nilideal is part of the radical. If the right-ideal J is part of the radical K of R, then we deduce from Corollary 6.2 that $J^{m+1} \leq K^{m+1} = 0$. If finally $J^k = 0$ for some ordinal k, then we denote by J_v the cross-cut of J and of M_v, and we put $J_0 = 0$. We proceed to prove by complete (transfinite) induction that every J_v is a nilideal, a fact that is clearly true for $v = 0$. Hence we may assume that J_u is a nilideal for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

The right-ideal $J^* = (M_w + J_v)/M_w$ is part of the anti-radical M_v/M_w of the ring R/M_w; it is, therefore, the sum of minimal right ideals in R/M_w. If Z^* is a minimal right-ideal in R/M_w, then either $Z^{*2} = 0$ or $Z^* = Z^{*2}$. If the idempotent minimal right-ideal Z^* were part of J^*, then there would exist an element e in J_v which is not contained in J_w, though $e - e^2$ is an element in J_w, since every idempotent minimal right-ideal contains an idempotent not 0.[20] It is a consequence of the induction-hypothesis that J_w is a nilideal and that therefore $e - e^2$ is a nilelement. Consequently we are able to deduce from a theorem of G. Köthe [21] the existence of an idempotent $j \neq 0$ in $J_v \leq J$, a fact that is clearly incompatible with $J^k = 0$. Thus we have shown that J^* is the sum of minimal right-ideals whose squares are 0; and from this result it is easily deduced that $J^{*2} = 0$. But $J_v^2 \leq M_w$ is an immediate consequence of $J^{*2} = 0$. Since M_w is a nilideal, the square of every element in J_v is a nilelement, i. e. J_v is a nilideal.

Case 2. v is a limit-ordinal.

Then every element in J_v is contained in some J_u for $u < v$; and it follows from the induction-hypothesis that every element in J_v is a nilelement, i. e. that J_v is a nilideal.

This shows that J is a nilideal, since $J = J_m$ is a consequence of $R = M_m$.

Remark. If J is an ideal, neither 0 nor 1, in the ring of natural integers, then $J^\omega = 0$, though J is not a nilideal; and this shows that the hypothesis $R = M_m$ cannot be omitted in Theorem 6. 3.

THEOREM 6. 4. *If condition* (4. A) *is satisfied by the ring* R, *and if* R/U *is a sum of minimal right-ideals, then* M_i *is, for every positive integer* i, *exactly the set of all the elements* x *in* R *which satisfy:* $xU^i = 0$.

Proof. The validity of our contention for $i = 1$ is an immediate consequence of Theorem 4. 4, (b). Thus we may assume that M_{i-1} is exactly the set of all the elements x in R which satisfy: $xU^{i-1} = 0$. The ideal M_i/M_{i-1} is the anti-radical of the ring R/M_{i-1}; and $(U + M_{i-1})/M_{i-1}$ is, by Theorem 1. 2, a two-sided nilideal in R/M_{i-1} modulo which this ring is a sum of minimal right-ideals. Hence it follows from Theorems 4.2 and 4.3 that M_i is exactly the set of all the elements x in R which satisfy: $xU \leq M_{i-1}$. But it is a consequence of the induction-hypothesis that xU is part of M_{i-1} if, and only if, $0 = xUU^{i-1} = xU^i$; and x is therefore an element in M_i if, and only if, $xU^i = 0$, as was to be shown.

---

[20] E. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[21] Köthe (1), p. 168, Hilfssatz 3.

THEOREM 6.5. *If the right-ideal J in R is part of M_i for i a positive integer, then $J^{2^i} = J^{2^i+1}$.*

Proof. It is a consequence of Lemma 4.5 that our theorem is true for $i = 1$. Thus we may assume the validity of the theorem for subideals of M_i and we have to derive it from this induction hypothesis for the subideals of M_{i+1}.

Assume now that the right-ideal J in R is part of M_{i+1}. The right-ideal $(M_i + J)/M_i$ in R/M_i is part of the anti-radical $M(R/M_i) = M_{i+1}/M_i$ of the quotient ring R/M_i; and hence we deduce from Lemma 4.5 that $M_i + J^2 = M_i + J^3$. The cross-cut C of M_i and J^2 is a subideal of M_i and consequently we may infer from the induction hypothesis that $C^{2^i} = C^{2^i+1}$; and the above equation may be restated as $J^2 = C + J^3$, since $J^3 \leq J^2$ and since we may apply the modular (Dedekind's) law. Expanding $(C + J^3)^{2^i-1}$ we obtain

$$J^{2^{i+1}-2} = (C + J^3)^{2^i-1} = C^{2^i-1} + \sum_{j=1}^{2^i-1} V_j$$

where every summand V_j is the sum of products of $2^i - 1$ ideals of which j are equal to J^3 and the remaining $2^i - 1 - j$ factors are equal to C. Since $C \leq J^2$, it follows that $V_j \leq J^{3j+2(2^i-1-j)} = J^{2^{i+1}+j-2} \leq J^{2^{i+1}-1}$ or

$$CV_j \leq J^{2+2^{i+1}-1} = J^{2^{i+1}+1}.$$

Consequently we find that

$$\begin{aligned} J^{2^{i+1}} = J^2 J^{2^{i+1}-2} = (C + J^3) J^{2^{i+1}-2} &= C\Big(C^{2^i-1} + \sum_{j=1}^{2^i-1} V_j\Big) + J^{2^{i+1}+1} \\ &= C^{2^i} + J^{2^{i+1}+1} = C^{2^i+1} + J^{2^{i+1}-1} \\ &= J^{2^{i+1}+1}, \end{aligned}$$

since $C^{2^i+1} \leq J^{2^{i+1}+2} \leq J^{2^{i+1}+1}$; and this completes the proof.

7. Nilpotent ideals. It is an immediate consequence of the definition of radical ideals (cf. condition (1.c)!) that they contain every nilpotent right-ideal. On the other hand it has been pointed out that radical ideals need not be nilpotent. Thus we shall give in this section several criteria for a nilideal, in particular a radical ideal, to be nilpotent.

THEOREM 7. 1. *Every nilideal contained in M_i for i a positive integer is nilpotent.*

Proof. It is a consequence of Theorem 6. 1 that a nilideal J contained in M_i satisfies: $J^{i+1} = JJ^i \leq M_iJ^i = 0$, and J is therefore nilpotent.

THEOREM 7. 2. *If the maximum-condition*[22] *is satisfied by the nilpotent right-ideals in the ring R, then the lower radical of R is nilpotent.*

Remark. That the hypothesis of this theorem is not sufficient for proving that the upper radical is nilpotent, may be seen from the example in section **2**. It seems to be an open question whether or not the maximum condition for right-ideals is sufficient for nilpotence of the upper radical.

Proof. There exists a greatest nilpotent right-ideal G in R. If J is a nilpotent right-ideal in R, then $G + J$ is a nilpotent right-ideal so that J is part of G. Consequently G is the sum of all the nilpotent right-ideals in R. It is a consequence of Lemma 1. 1 that G is a two-sided ideal in R. There cannot exist nilpotent right-ideals different from 0 in R/G, since $Z^i \leq G$ and the nilpotence of G imply the nilpotence of Z. This shows that the nilpotent ideal G is a radical ideal; and it is readily verified that G is the lower radical of the ring R.

THEOREM 7. 3. *If $R = M_m$ for some ordinal m, and if at least one of the two chains M_v and K^v is finite, then the radical K of R is nilpotent.*

Remark. The existence of the radical K is a consequence of Corollary 5. 2.

Proof. If $R = M_m$ for some finite ordinal m, then the nilpotence of the radical is a consequence of Theorem 7. 1, since the radical is by Theorem 1. 2 a nilideal.

If $R = M_m$ for some ordinal m, then it follows from Corollary 6. 2 that $K^{m+1} = 0$. If the chain K^v is finite, then there exists a finite ordinal i such that $K^i = K^{i+1} = \cdots = K^{m+1} = 0$.

The next theorem is a partial converse of Theorem 7. 3.

THEOREM 7. 4. *If condition* (4. A) *is satisfied by the ring R, if the radical K of R exists and is nilpotent, and if R/K is a sum of minimal right-ideals, then $R = M_i$ for some positive integer i.*

Proof. There exists a positive integer i such that $K^i = 0$; and it is a consequence of Theorem 6. 4 that M_i is exactly the set of all the elements x in R which satisfy: $xK^i = 0$. Since $RK^i = R0 = 0$, it follows that $M_i = R$.

---

[22] It states that every not vacuous class of nilpotent right-ideals contains at least one greatest ideal.

The third condition in Theorem 7. 4 cannot be omitted, since in the ring of all the integers radical and anti-radical are both 0.

COROLLARY 7. 5. *Suppose that condition* (4. A) *is satisfied by the ring R, that R/U is a sum of minimal right-ideals, and that* 0 *is the only element x in R that satisfies $Rx = 0$. Then*

$$U^i = 0 \text{ if, and only if, } M_i = R \text{ (for } i \text{ a positive integer).}$$

Proof. That $U^i = 0$ implies $M_i = R$, is an immediate consequence of Theorem 6. 4; and that $M_i = R$ implies $U^i = 0$, may be inferred from Theorem 6. 1, since $0 = M_iU^i = RU^i$, and since $S = 0$ is a consequence of $RS = 0$.

8. Maximum and minimum conditions.[23]

In the results of the previous sections there occurred hypotheses that are connected in various ways with maximum and minimum conditions. In this section we investigate the relations between these properties.

THEOREM 8. 1. *Suppose that the maximum condition is satisfied by the right-ideals in the ring R. Then the minimum condition is satisfied by the right-ideals in R if (and only if) condition* (5. B) *is satisfied by every quotient-ring of R.*

Proof. If condition (5. B) is satisfied by every quotient ring R, then there exists, by Theorem 5. 1, an ordinal m such that $R = M_m$. Since the ideals M_v form an ascending chain, and since the maximum condition is satisfied by the right-ideals in R, there exists a positive integer i such that $M_i = M_{i+1}$. Since $R = M_m$ for some ordinal m, it follows that $R = M_i$ for i a finite ordinal. It is a consequence of the definition of the series M_v that M_{v+1}/M_v is a sum of minimal right-ideals in R/M_v; and we infer from the maximum condition for right-ideals that M_{v+1}/M_v is a sum of a finite number of minimal right-ideals. Consequently there exists a finite composition series[24] of right-ideals in R; and it is well known that the minimum condition for right-ideals is a consequence of this fact.

---

[23] The maximum (minimum) condition is satisfied by the right-ideals in the ring R, if in every not vacuous set of right-ideals there exists a right-ideal which is not smaller (greater) than any other right-ideal in the set.

[24] A composition series is a densest finite ascending chain of right-ideals.

If there exists in the ring R an infinite independent set[25] of right-ideals, then there exists in R a countably infinite, independent set of right-ideals $J(i) \neq 0$ for $i = 1, 2, \cdots$. The chain of right-ideals $\sum_{j<i} J(j)$ is an infinite ascending chain of right-ideals; and the chain $\sum_{i<j} J(j)$ is an infinite descending chain of right-ideals. Thus the maximum as well as the minimum condition for right-ideals implies the following property of rings R:

(8. C) *There does not exist an infinite independent set of right-ideals.*

If (8. C) is satisfied by the ring R, then it follows from Lemma 4. 1 that the anti-radical M of R is the (direct) sum of a finite number of minimal right-ideals. The following statement is a partial converse of this fact.

THEOREM 8. 2. *If condition* (5. B) *is satisfied by the ring R, and if the anti-radical M is a sum of a finite number of minimal right-ideals in R, then* (8. C) *is satisfied by R.*

Proof. Suppose that S is an independent set of right-ideals different from 0 in R. If J is a right-ideal in the set S, then J contains a minimal right-ideal J'. The set S' of these minimal right-ideals J' is independent and contains as many elements as S. Since there does not exist an infinite independent set of minimal right-ideals in R, it follows that S is a finite set and that therefore (8. C) is satisfied by R.

LEMMA 8. 3. *If conditions* (5. B) *and* (8. C) *are satisfied by the ring R, and if R does not contain nilpotent right-ideals different from* 0, *then R is the sum of a finite number of minimal right-ideals.*

Remark. This is a generalization of the so-called "Fundamental Theorem on Semi-simple Rings."[26]

Proof. Suppose that we have constructed idempotents $e_1, \cdots, e_n$ meeting the following requirements:

(a) e_iR is a minimal right-ideal.

(b) $e_ie_j = 0$ for $i \neq j$.

This is certainly possible for $n = 0$.

We note that the right-ideals e_iR form an independent set of right-ideals.

---

[25] The set S of ideals is said to be *independent*, if the cross-cut of any ideal J in S with the sum of the other ideals in S is 0.

[26] Cf. e. g. v. d. Waerden (1), p. 156.

Let $W(n)$ be the right-ideal of all the elements x satisfying: $\sum_{i=1}^{n} e_i x = 0$ ($W(0) = R$). Then it is readily seen that $R = W(n) + \sum_{i=1}^{n} e_i R$; and we have effected the proof of our lemma, if $W(n) = 0$. Should $W(n)$ be different from 0, then it contains a minimal right-ideal J; and J contains, by a well known theorem,[27] an idempotent $e \neq 0$, since $J = J^2$. Put $e_{n+1} = e - e\sum_{i=1}^{n} e_i$. It is readily verified that e_{n+1} is an idempotent, satisfying $e_i e_{n+1} = e_{n+1} e_i = 0$ for $i = 1, \cdots, n$, and that $J = e_{n+1}R$; and thus the idempotents $e_1, \cdots, e_n, e_{n+1}$ meet the requirements (a) and (b).

If this construction would never stop, i. e. if the right-ideal $W(n)$ were different from 0 for every positive n, then we would be led to the infinite independent set of minimal right-ideals $e_1R, \cdots, e_iR, \cdots$, contradicting condition (8. C). This completes the proof.

THEOREM 8. 4. *If condition* (4. A) *is satisfied by the ring R, then each of the following properties implies all the others:*

(1) *The maximum and the minimum condition are satisfied by the right-ideals in R.*

(2) *The minimum condition is satisfied by the right-ideals in R.*

(3) *Conditions* (5. B) *and* (8. C) *are satisfied by every quotient-ring of the ring R; and the descending chain of the powers of the radical*[28] *K of R is finite.*

Proof. It is obvious that (1) implies (2); and that (3) is a consequence of (2) is a consequence of facts we mentioned when introducing condition (8. C).

If (3) is satisfied by the ring R, then it follows from Theorem 5. 1 that $R = M_m$ for some ordinal m; and from Lemma 8. 3 that R/K is the sum of a finite number of right-ideals. There exists a positive integer i such that $K^i = K^{i+1}$; and it is a consequence of Theorem 6. 4 that M_n is, for finite n, exactly the set of all the elements x, satisfying $xK^n = 0$. This shows the equality of $M_i = M_{i+1}$. Hence we have $M_i = M_{i+1} = \cdots = M_m = R$. Since there do not exist infinite independent sets of right-ideals in R/M_n, it follows from Lemma 3. 3 that M_{n+1}/M_n is the sum of a finite number of minimal right-ideals. Consequently there exists a finite composition series of right-ideals in R, a fact which is equivalent to our property (1).

---

[27] Cf. e. g. v. d. Waerden (1), p. 157.

[28] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

Remarks. 1. In proving that (1) is a consequence of (3) we did not use condition (3) in its entirety. That condition (5. B) is satisfied by every quotient ring of R, and that the powers of the radical form a finite chain, are hypotheses indispensable for the above proof. But it is not necessary to assume that (8. C) is satisfied by every quotient ring of R. It would have been sufficient to make sure that the quotient rings R/M_i for finite i and R/K meet the requirement (8. C); and for the latter assumption we could have substituted the weaker hypothesis that R/K is a sum of minimal right-ideals.

2. It has been shown elsewhere[29] that a ring with minimum-condition for right-ideals possesses a right-identity element if, and only if, it satisfies condition (4. A); and Ch. Hopkins[30] has shown that the maximum condition for right-ideals is a consequence of the minimum condition for right-ideals, provided there exists a right-identity element. It should be noted, however, that condition (4. A) is indispensable for the validity of this Theorem 8. 4, as may be seen from the following example: denote by R any infinite abelian group without elements of infinite order which contains only a finite number of elements of order a prime; such a group is the direct sum of a finite abelian group and of a finite number of groups of type[31] p^∞. If we define $xy = 0$ for every pair of elements x and y in R, then R is a commutative ring, satisfying $0 = R^2$. The ideals in R are just the subgroups of the additive group R; and thus it becomes apparent that the minimum condition is satisfied by the ideals in R, but not the maximum condition.

9. Existence of the identity. The following statement is basic for the considerations of this section.

LEMMA 9. 1. *If conditions* (5. B) *and* (8. C) *are satisfied by*[32] R/K, *then there exists an idempotent e in R such that* $x \equiv ex \equiv xe$ modulo K *for every element x in R.*

Proof. R/K is a ring without nilpotent right-ideals different from 0, by Theorem 1. 2; and R/K is, by Lemma 8. 3, the sum of a finite number of minimal right-ideals. Hence there exists, by a well known theorem,[33] an identity element in R/K, i. e. there exists an element f in R such that

---

[29] Baer (1), Corollary to Theorem 6.

[30] Hopkins (1), p. 726, Theorem 6. 4.

[31] The groups of type $p\infty$ have been discovered by H. Prüfer; they are generated by a countable number of elements g_i subject to the relations: g_0 is an element of order p; $g_{i-1} = g_i p$.

[32] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

[33] Cf. e. g. v. d. Waerden (1), p. 156.

$x \equiv xf \equiv fx$ modulo K for every x in R. If f is any element meeting this requirement, then $f^2 - f = f'$ is an element in K; and we deduce from Theorem 1.2 the existence of a positive integer $n = n(f)$ such that $f'^n = 0$. Thus there exists among the elements which represent an identity element modulo K one, say e, with minimal $n(e)$. It is readily verified that e is an idempotent (which clearly meets all our requirements), since otherwise[34] $e_1 = e - 2ee' + e'$ [for $e' = e^2 - e$] would be an element which represents the identity element modulo K, though $n(e_1)$ were smaller than $n(e)$.

THEOREM 9.2. *If* $R = M_m$ *for some (finite or infinite) ordinal* m, *and if condition* (8. C) *is satisfied by the ring*[35] R/K, *then the property that every element* x *in* R *is contained in* Rx *is a necessary and sufficient condition for the existence of a left-identity-element in* R.

Proof. If e is a left-identity-element in R, then $x = ex$ belongs to Rx, showing the necessity of the condition. We assume now that the condition: "x belongs to Rx" is satisfied by the ring R. It is a consequence of Theorem 5.1 that condition (5. B) is satisfied by every quotient ring of R; and hence we may deduce from Lemma 9.1 the existence of an idempotent e such that $x \equiv ex \equiv xe$ modulo K for every element x in R. We denote by W the set of all the elements x in R such that $ex = 0$. Since $x \equiv ex$ modulo K, it follows that W is part of K. Finally we have $R = Re + K$.

We proceed to prove by complete (transfinite) induction that $W \leq K^v$ for every v. This is certainly true for $v = 1$; and thus we may assume that $W \leq K^u$ for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then we deduce from the hypothesis that x is an element in Rx and from the induction hypothesis the inequality:

$$W \leq RW = ReW + KW = KW \leq KK^w = K^v.$$

Case 2. v is a limit-ordinal.

Then K^v is the cross-cut of all the K^u for $u < v$; and $W \leq K^v$ is an immediate inference from the induction-hypothesis.

It is a consequence of $R = M_m$ and Corollary 6.2 that $K^{m+1} = 0$. Thus W, as a part of K^{m+1}, is 0.

---

[34] This construction is due to Köthe (1), p. 169 and Dickson (1), p. 123.

[35] The existence of the radical K is assured by Theorem 5.1 and Corollary 5.2.

Since $x-ex$ belongs to W for every x in R, it follows that $x=ex$ for every x in R, i. e. that e is a left-identity-element in R.

Remark. The condition: $R=M_m$ was only needed in the proof to assure that condition (5. B) is satisfied by R/K and that $K^{m+1}=0$; and these two apparently weaker conditions may be substituted for $R=M_m$.

The *identity* is an element e in R that satisfies $ex=xe=x$.

COROLLARY 9. 3. *If $R=M_m$ for some (finite or infinite) ordinal m, and if condition* (8. C) *is satisfied by the ring*[35] *R/K, then the following two conditions are necessary and sufficient for the existence of the identity* 1 *in the ring R:*

(i) *$xR=0$ implies $x=0$; and*

(ii) *Rx contains x.*

Proof. The necessity of the conditions (i) and (ii) is obvious. If these conditions are satisfied by the ring R, then we deduce from Theorem 9. 2 the existence of a left-identity-element e in R; and e is the identity 1 in R, since

$$(x-xe)R=(x-xe)eR=(xe-xe)R=0 \text{ implies } x=xe \text{ by (i)}.$$

THEOREM 9. 4. *If $R=M_m$ for some (finite or infinite) ordinal m, and if condition* (8. C) *is satisfied by the ring*[36] *R/K, then condition* (4. A) *is a necessary and sufficient condition for the existence of a right-identity-element in R.*

Proof. If e is a right-identity-element in R, then $x=xe$ is contained in the right-ideal xR, showing the necessity of condition (4. A). If condition (4. A) is satisfied by the ring R, then we deduce from Theorem 5. 1 the validity of condition (5. B) in every quotient ring of R; and hence we may infer from Lemma 9. 1 the existence of an idempotent e satisfying $x\equiv ex\equiv xe \bmod K$ for every element x in R. Clearly $R=eR+K$.

It will be convenient to put $0=M_0$. Then we prove by complete (transfinite) induction that $M_v=M_ve$ and that 0 is the only element x in M_v satisfying $xe=0$. This fact is patently true for $v=0$ and thus we assume it to be true for every $u<v$.

Case 1. $v=w+1$ is not a limit-ordinal.

Then $M_w=M_we$ and 0 is the only element x in M_w such that $xe=0$.

---

[36] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

If y is an element in M_v such that $ye = 0$, then $yR = yeR + yK = yK$; and since M_v/M_w is the anti-radical of the ring R/M_w, it follows from Theorem 4.4, (a) that $yR = yK \leq M_w$. But it is a consequence of (4.A) that y is an element in yR and therefore in M_w; and now we deduce from the induction-hypothesis that $y = 0$. Consequently 0 is the only element x in M_v such that $xe = 0$; and $M_v = M_ve$ is an obvious consequence of this fact.

Case 2. v is a limit-ordinal.

If x is an element in M_v, then there exists an ordinal $u < v$ such that x is an element in M_u. Hence it follows from the induction-hypothesis that $x = xe$ showing that $M_v = M_ve$ and that therefore 0 is the only element z in M_v such that $ze = 0$.

Now it is evident that $R = M_m = M_me = Re$ and that consequently e is a right-identity-element for R.

By essentially the same arguments as the ones used in the proof of Corollary 9.3 we deduce the following statement from Theorem 9.4.

COROLLARY 9.5. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition* (8.C) *is satisfied by the ring*[36] *R/K, then the following two conditions are necessary and sufficient for the existence of the identity* 1 *in the ring R:*

(i) *$Rx = 0$ implies $x = 0$; and*

(ii) *xR contains x.*

It has been assumed in Theorems 9.2 and 9.4 and in Corollaries 9.3 and 9.5 that condition (8.C) is satisfied by the ring R/K. The impossibility of omitting this hypothesis may be seen from the following example: Denote by F a field, by G an abelian group which admits the elements in F as operators and whose rank over F is infinite, and by R the ring of all the (proper and improper) automorphisms of the group G over F which map G upon a subgroup of finite rank. It is readily seen that R contains neither a left-identity-element nor a right-identity-element, that $R = M$ and that $K = 0$. If r is any element in R, then denote by G_r the set of all the elements in G which are mapped upon 0 by r. Clearly G^r and G/G_r are isomorphic groups of finite rank over R; and hence there exist idempotents e, f in R such that every element in G^r is left invariant by e, G_r is mapped upon 0 by f and every coset of G/G_r is mapped by f upon an element in itself, since both G^r and G_r are direct summands of G. Clearly $r = re = fr$ showing that every subset S of R is contained in both SR and RS.

THEOREM 9.6. *If conditions* (5.B) *and* (8.C) *are satisfied by*[36] R/K, *then the following pair of properties is a necessary and sufficient condition for the existence of a left-identity-element in* R:

(a) *Every right-ideal different from* R *is contained in a maximal right-ideal in* R.

(b)[37] *The radical* K *of* R *is the cross-cut of all the maximal right-ideals in* R.

Proof. Suppose that there exists a left-identity-element e in R. If the right-ideal J in R is different from R, then there exists a greatest right-ideal G in R which contains J, but which does not contain e. Clearly G is a maximal right-ideal in R, since $R = eR$. It is a consequence of Theorem 1.2 and of Lemma 8.3 that R/K is a sum of minimal right-ideals; and hence it may be inferred from Theorem 3.4 that K is the cross-cut of all the maximal right-ideals in R.

Suppose now, conversely, that the conditions (a) and (b) are satisfied by the ring R. There exists by Lemma 9.1 an idempotent e in R such that $x \equiv ex \equiv xe$ modulo K for every element x in R. Since the elements v satisfying $ev = 0$ are certainly contained in K, it follows that $R = eR + K$. If the right-ideal eR in R were different from R, then there would exist a maximal right-ideal G in R which contains eR. It is a consequence of (b) that $K \leq G$; and thus we are led to the contradiction: $R = eR + K \leq G < R$. Hence $R = eR$ and e is a left-identity-element in R.

Remarks. 1. If R is the ring of all the even rational integers, then the radical of R is 0; and it is readily seen that conditions (a) and (b) are satisfied by R. Condition (8.C) is satisfied too; but R does not contain an identity. This shows the impossibility of omitting the hypothesis that condition (5.B) be satisfied by R/K.

2. Suppose that the abelian group R is the direct sum of a group K of type 2^{∞} and of a cyclic group of order 2 which is generated by an element e. In R we define a commutative multiplication by the rules: $uv = 0$, if at least one of the factors u and v is in K; and $e = e^2$. It is readily seen that K is the only maximal ideal in R, and that K is the radical in R. Thus conditions (5.B) and (8.C) are satisfied by R/K and condition (b) is satisfied by R. But there does not exist an identity element in R; and this shows the impossibility of omitting condition (a) in Theorem 9.6.

---

[37] Considering this condition (b) and Theorem 4.4 it may be shown that Theorem 9.6 is a generalization of a theorem due to M. Hall; cf. Hall (1), p. 362, Theorem 3.2.

10. The quasi-regular elements. S. Perlis[38] has recently discovered a characterization of the radical of an algebra which is rather different from the usual criteria. We devote this section to a generalization of his theory.

An element x in the ring R is termed *quasi-regular*,[39] if there exists an element y in R satisfying: $x + y + xy = 0$. If the ring R possesses an identity element 1, then a necessary and sufficient condition for quasi-regularity of the element x in R is the existence of a solution z of the equation: $(1 + x)z = 1$.

The *right-ideal* J in the ring R shall be termed *quasi-regular*, if every element in J is quasi-regular.

THEOREM 10. 1. *If z is an element in the quasi-regular right-ideal J, then there exists one and only one solution $x = z'$ of the equation $z + x + zx = 0$; and such a solution satisfies $zz' = z'z$.*

Proof. If z is an element in the quasi-regular right-ideal J, then there exists an element y in R such that $z + y + zy = 0$. If y is some solution of this equation, then $y = -(z + zy)$ is an element in J, since z is an element in the right-ideal J. Since J is quasi-regular, so is y; and hence there exists an element w in R such that $y + w + yw = 0$. Consequently we find that

$$\begin{aligned} z &= z + y + w + yw + z(y + w + yw) \\ &= z + y + zy + (y + z + zy)w + w = w; \end{aligned}$$

and this shows that $0 = y + w + yw = y + z + yz$; and we deduce $zy = yz$ from $y + z + zy = 0$.

Suppose now that $x = y$ and $x = y'$ are solutions of the equation: $z + x + zx = 0$. From what we have shown in the previous paragraph of the proof, it follows that $0 = z + y + yz$; and thus we find that

$$\begin{aligned} y' &= y' + z + y + yz + (z + y + yz)y' \\ &= z + y' + zy' + y(z + y' + zy') + y = y; \end{aligned}$$

and this completes the proof.

We denote by $S = S(R)$ *the sum of all the quasi-regular right-ideals in* R.

THEOREM 10. 2. *$S(R)$ is a quasi-regular right-ideal in the ring R.*

Proof. It is readily seen that it suffices to prove the following statement: If u is contained in a quasi-regular right-ideal, and if v is contained in a quasi-regular right-ideal, then $u + v$ is an element of some quasi-regular right-ideal.

---

[38] Perlis (1).

[39] Perlis (1), p. 129.

We note that (in the absence of identity elements) the right-ideal generated by the element w consists of all the elements of the form: $wi + wr$ for r in R and i an ordinary rational integer (i need not be an element in R). Thus we have to show that $(u+v)i + (u+v)r$ is quasi-regular for every r in R and for every rational integer i, whenever both u and v are contained in quasi-regular right-ideals.

Since $ur + ui$ belongs to the right-ideal generated by u, it is quasi-regular; and hence there exists an element s such that

$$(ur + ui) + s + (ur + ui)s = 0;$$

and since $v(r + rs + ri) + vi$ belongs to the right-ideal generated by v, it is quasi-regular too; and this assures the existence of an element t satisfying:

$$v(r + rs + si) + vi + t + (v(r + rs + si) + vi)t = 0.$$

Hence

$$\begin{aligned}((u+v)r + (u+v)i) + (s + t + st) + ((u+v)r + (u+v)i)(s + t + st)\\ = ur + ui + s + (ur + ui)s + (ur + ui + s + (ur + ui)s)t\\ + v(r + rs + si) + vi + t + (v(r + rs + si) + vi)t = 0;\end{aligned}$$

and this proves that $(u+v)r + (u+v)i$ is quasi-regular for every r in R and for every natural integer i; i. e. $u+v$ is contained in a quasi-regular right-ideal in R, as was to be shown.

THEOREM 10. 3. *If the right-ideal J in R consists of elements that are quasi-regular modulo*[40] *$S(R)$, then $J \leq S(R)$.*

Proof. If the element x in R is quasi-regular modulo $S(R)$, then there exists an element y such that $x + y + xy$ is in $S(R)$; and it follows from Theorem 10. 2 that $x + y + xy$ is quasi-regular. Consequently there exists an element z in R such that

$$0 = x + y + xy + z + (x + y + xy)z = x + (y + z + yz) + x(y + z + yz);$$

and x is therefore a quasi-regular element. The right-ideal J is thus quasi-regular, if each of its elements is quasi-regular modulo $S(R)$.

COROLLARY 10. 4. *If the right-ideal J in R is a nilideal modulo*[41] *$S(R)$, then $J \leq S(R)$.*

---

[40] I. e. to every element j in J there exists an element h in R such that $j + h + jh$ belongs to $S(R)$.

[41] I. e. to every element j in J there exists a positive integer n such that j^n belongs to $S(R)$.

Proof. If the element w is a nilelement modulo $S(R)$, then there exists a positive integer n such that w^{2n+1} is an element in $S(R)$. Since

$$w + \sum_{i=1}^{2n} (-1)^i w^i + w \sum_{i=1}^{2n} (-1)^i w^i = w^{2n+1},$$

it follows that w is quasi-regular modulo $S(R)$; and this shows that our contention is an immediate consequence of Theorem 10. 3.

LEMMA 10. 5. *If the element e in $S(R)$ is an idempotent modulo the two-sided ideal T in R, then e belongs to T.*

Proof. If e is an element in $S(R)$, then $-e$ is quasi-regular. Hence there exists an element f in R such that $-e+f-ef=0$. If e is furthermore an idempotent modulo the two-sided ideal T, then

$$e \equiv e^2 \equiv e(f-ef) \equiv ef - e^2 f \equiv ef - ef \equiv 0 \bmod T,$$

i. e. e belongs to T.

THEOREM 10. 6. *If the two-sided ideal T in R is part of $S(R)$, if every right-ideal, not 0, in R/T contains a minimal right-ideal, and if 0 is the only nilpotent right-ideal in R/T, then $T = S(R)$.*

Proof. If T were different from $S(R)$, then there would exist a right-ideal J between T and $S(R)$ such that J/T is a minimal right-ideal in R/T. Since 0 is the only nilpotent right-ideal in R/T, it follows that $J/T = (J/T)^2$; and hence we may deduce from known theorems[42] that J/T contains an idempotent different from 0. But it follows from Lemma 10. 5 that this is impossible; and this shows that $T = S(R)$.

COROLLARY 10. 7. *If every right-ideal, not 0, in R/U contains a minimal right-ideal, then $S(R)$ is the upper radical U of R.*[43]

Proof. It is a consequence of Theorem 1. 2 that U is a nilideal and that R/U does not contain nilpotent right-ideals different from 0; it is a consequence of Corollary 10. 4 that U is part of $S(R)$; and hence we may deduce from Theorem 10. 6 that $U = S(R)$.

We denote by $S^* = S^*(R)$ *the sum of all the quasi-regular two-sided ideals in R.* Clearly S^* is a two-sided ideal which is part of $S(R)$; and hence it follows from Theorem 10. 2 that $S^*(R)$ is quasi-regular. It is a consequence of Theorem 10. 3 that $S^*(R/S^*(R)) = 0$.

---

[42] Cf. e. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[43] This is readily seen to be an extension of Perlis' (1) Theorems 1 and 2.

COROLLARY 10. 8. *If every right-ideal, not* 0, *in* $R/S^*(R)$ *contains a minimal right-ideal, then* $S^*(R) = S(R)$.

Proof. Denote by H the two-sided ideal which contains $S^*(R)$ and which satisfies $H/S^* = N(R/S^*)$ (= sum of all the nilpotent right-ideals in R/S^*). That H is a two-sided ideal, is a consequence of Lemma 1. 1; and it follows from Lemma 1. 1 that H is a nilideal modulo S^*. Hence we deduce from Corollary 10. 4 that H is part of $S(R)$; and we infer from Theorem 10. 2 that the two-sided ideal H is quasi-regular. Hence $H = S^*$ and 0 is the only nilpotent right-ideal in R/S^*. Applying Theorem 10. 6 we see that $S^* = S$.

THEOREM 10. 9. *If the ring* R *possesses an identity element* 1, *then* $S(R)$ *is part of every maximal right-ideal in* R.

Proof. If $S(R)$ were not contained in the maximal right-ideal G in R, then $R = G + S$ and there exist elements g and s in G and S respectively such that $1 = g + s$. It is a consequence of Theorem 10. 2 that $-s$ is quasi-regular and that therefore $-s + t - st = 0$ for some t in R. Hence $1 + t = (g + s)(1 + t) = g + s + gt + st$ or $1 = g + gt$ so that 1 is an element in the right-ideal $G < R$, a contradiction.

COROLLARY 10. 10. *If the ring* R *possesses an identity element* 1, *and if* R/S^* *is a sum of minimal right-ideals, then* $S(R) = S^*(R)$ *is the cross-cut of all the maximal right-ideals in* R.

This is an immediate consequence of Lemma 3. 3, Corollary 10. 8 and Theorem 10.9.

Remark. The upper radical U of the ring R is by Theorem 1. 2 a two-sided nilideal; and hence it is a consequence of Theorem 10. 2 and of Corollary 10. 4 that $U \leq S^*(R)$. That the ideals U and S^* need not be equal may be seen from the following *example*: R is the ring of all the (formal) power series $\sum_{i=0}^{\infty} c_i t^i$ in one indeterminate t with coefficients c_i from some commutative field. An element in R possesses an inverse in R if, and only if, "its absolute term" $c_0 \neq 0$; and an element in R is quasi-regular if, and only if, $1 + c_0 \neq 0$. This shows that $S^*(R) = S(R) = tR$ whereas $U(R) = 0$.

THEOREM 10. 11. *If* J *is a quasi-regular right-ideal in the ring* R, *then* $M_v J^v = 0$ *for every ordinal* v.

Remark. It is a consequence of Corollary 10.4 that Theorem 6.1 is a special case of this Theorem 10.11.

Proof. A. If Z is a minimal right-ideal in R, and if z is any element in Z, then zJ is a right-ideal in R which is part of Z. Thus zJ is either 0 or Z. If zJ were equal to Z, then there would exist an element w in J such that $zw = z$. Since w is an element in the quasi-regular right-ideal J, it follows that $-w$ is quasi-regular; and hence there exists an element v in R such that $w = v - wv$. Consequently

$$z = zw = z(v - wv) = zv - zv = 0,$$

an impossibility. Thus we have shown that $zJ = 0$ for every z in Z; and this shows that $ZJ = 0$ for every minimal right-ideal Z in R. Since the anti-radical $M = M(R)$ is the sum of all the minimal right-ideals in R, it follows that

$$MJ = 0 \textit{ for every quasi-regular right-ideal } J \textit{ in } R.$$

B. We proceed to prove the theorem by complete induction with regard to v. That the theorem holds true for $v = 1$, has been shown under A.; and thus we may assume that our assertion is valid for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then M_v/M_w is the anti-radical of R/M_w and $(J + M_w)/M_w$ is a quasi-regular right-ideal in R/M_w. Hence it follows from A. that their product is 0; and from this fact we deduce that $M_vJ \leq M_w$. From the induction-hypothesis we infer $M_wJ^w = 0$; and thus we find that

$$M_vJ^v = M_vJJ^w \leq M_wJ^w = 0.$$

Case 2. v is a limit-ordinal.

If x is any element in M_v, then we deduce from the definition of M_v as the join of the M_u with $u < v$ the existence of an ordinal $d < v$ such that x is an element in M_d. It is a consequence of the definition of J^v that it is part of J^d; and hence it follows from the induction-hypothesis that $xJ^v \leq M_dJ^d = 0$; and thus we have shown that $M_vJ^v = 0$.

11. Rings admitting operators. The ring R is said to admit the elements in the system $\boldsymbol{V}$ as operators, if to every element r in R and to every element $\boldsymbol{v}$ in $\boldsymbol{V}$ there exists a uniquely determined element $r\boldsymbol{v}$ in R meeting the following requirements:

(i) $(r \pm s)v = rv \pm sv$;
(ii) $(rs)v = r(sv) = (rv)s$.

It is readily verified that Jv is a right-(left-) ideal in R whenever J is a right-(left-) ideal in R; that $r^i = 0$ implies $(rv)^i = 0$; and that therefore Jv is a nilideal (a nilpotent right-ideal) whenever J is a nilideal (a nilpotent right-ideal). Saying that an ideal J in R is *V-admissible*, if $JV \leq J$, it is now easy to prove the following statement:

If the ring R admits the elements in the system V as operators, then the upper and the lower radical and all the ideals M_v in the anti-radical series are V-admissible.[44]

If the ring R satisfies condition (4. A), then there exists to every element x in R an element x^* in R such that $x = xx^*$; and this shows that $xv = x(x^*v)$ for every x in R. Hence we have proved the following theorem:

If condition (4. A) *is satisfied by the ring R, then every right-ideal in R is V-admissible.*

On the basis of this theorem it becomes evident that most of the theorems derived in this paper may be applied to rings admitting operators.

UNIVERSITY OF ILLINOIS,
URBANA, ILL.

BIBLIOGRAPHY.

R. Baer,
(1) "Inverses and zero-divisors in Rings," *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 630-638.
L. E. Dickson,
(1) *Algebren und ihre Zahlentheorie*, Zürich 1927.
H. Fitting,
(1) "Primärkomponentenzerlegung in nichtkommutativen Ringen," *Mathematische Annalen*, vol. 111 (1935), pp. 19-41.
M. Hall,
(1) "A type of algebraic closure," *Annals of Mathematics*, vol. 40 (1939), pp. 360-369.
(2) "The position of the radical in an algebra," *Transactions of the American Mathematical Society*, vol. 48 (1940), pp. 391-404.

---

[44] Cf. Hopkins (1), pp. 729-730.

Ch. Hopkins,
(1) "Rings with minimal conditions for left-ideals," *Annals of Mathematics*, vol. 40 (1939), pp. 712-730.
G. Köthe,
(1) "Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist," *Mathematische Zeitschrift*, vol. 32 (1930), pp. 161-186.
S. MacLane,
(1) "A lattice formulation for transcendence degrees and p-bases," *Duke Mathematical Journal*, vol. 4 (1938), pp. 455-468.
S. Perlis,
(1) "A characterization of the radical of an algebra," *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 128-132.
B. L. v. d. Waerden,
(1) *Moderne Algebra II*, Berlin 1931.

Added in proof: (October, 1943): Since this paper has been submitted for publication, the following investigations which are somewhat related to the present article have appeared in print.

E. Artin-G. Whaples,
(1) "The theory of simple rings," *American Journal of Mathematics*, vol. 65 (1943), pp. 87-107.
R. Brauer,
(1) "On the nilpotency of the radical of a ring," *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 752-758.
N. Jacobson,
(1) *The Theory of Rings*, Mathematical Surveys, No. 2, 1943.
J. Levitzki,
(1) "On the radical of a general ring," *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 462-466.
(2) "Semi-nilpotent ideals," *Duke Mathematical Journal*, vol. 10 (1943), pp. 553-556.

RINGS WITH DUALS.*

By Reinhold Baer.

1. Introduction. If J is a right-ideal in the ring R, then the set $L(J)$ of all the elements x satisfying $xJ = 0$ is a left-ideal in R; in this way there is defined a monotonically decreasing map of the partially ordered set of all the right-ideals in R into the partially ordered set of all the left-ideals in R. A monotonically decreasing and biunique correspondence between the elements of two partially ordered sets has been termed a *duality*; and M. Hall [1] and T. Nakayama [2] have recently started an investigation of those rings in which the operation $L(J)$ defines a duality. Particularly significant is a result, announced by Nakayama,[3] that, under suitable conditions concerning the ring R, the operation $L(J)$ will constitute a duality whenever there exists a duality of the set of right-ideals in R upon the set of left-ideals in R.

Thus the following generalization of these problems seems to be quite natural: the ring R' is said to be *a right-dual of the ring* R, if there exists a duality of the partially ordered set of all the right-ideals in R upon the partially ordered set of all the right-ideals [4] in R'; we propose to investigate the structure of rings possessing a right-dual.

T. Nakayama [5] has given an example of a commutative ring R possessing an identity-element and a duality of the partially ordered set of its ideals upon itself such that neither the maximum nor the minimum condition is satisfied by the ideals in R. Thus the following theorem seems to be noteworthy: The maximum and the minimum condition are satisfied by the right-ideals in the

---

* Received March 15, 1942; Presented to the American Mathematical Society, September, 1942.

[1] Hall (1); the numbers refer to the bibliography at the end of the paper.

[2] Nakayama (1).

[3] Nakayama (2).

[4] It might seem more natural to say that the ring R' is a right-dual of the ring R, if there exists a duality of the partially ordered set of all the right-ideals in R upon the partially ordered set of all the left-ideals in R'. But as far as the ring R is concerned, these two definitions are equivalent, since there exists to every ring R' an anti-isomorphic ring R'', and since, therefore, the existence of a duality of the set of right-ideals in R upon the set of left-ideals in R' implies the existence of a duality of the set of right-ideals in R upon the set of right-ideals in R''. The definition adopted in the text has the great advantage that it concerns itself with right-ideals only, a fact that makes for greater simplicity in notation.

[5] Nakayama (2), p. 55.

ring R, if R possesses a right-identity element, and if there exists to every quotient ring of R a right-dual possessing a left-identity element [Theorem 2. 2]. These rings may furthermore be shown to meet a generalized uniseriality requirement [Theorem 3. 2].[6]

If the right-ideals in R form a finite ordered set, then R is its own right-dual; there exist[7] examples of such rings where the left ideals do not form a finite ordered set so that the operation $L(J)$ certainly is not a duality. If R is a primary ring[8] with identity element all of whose quotient rings possess right-duals with identity elements, and if the right-ideals in R do not form a finite ordered set, then we are able to prove that $L(J)$ is a duality [Theorem 5. 4]; and we give a complete theory of the structure and of the dualities of such rings. [Sections **5** and **6.**]

2. Maximum and minimum conditions for right-ideals. The right-ideal J in the ring R is termed a *minimal* right-ideal, if $0 < J$ and if there does not exist a right-ideal V such that $0 < V < J$. Likewise the right-ideal J in R is said to be a *maximal* right-ideal, if $J < R$ and if there does not exist a right-ideal W such that $J < W < R$.

The set S of right-ideals in R is called *independent*, if the cross-cut of J and of the sum of the right-ideals different from J in S is 0 for every J in S.

The element e in R is a *left-identity element* in R, if $ex = x$ for every element x in R. Right-identity elements are defined analogously and elements that are both right- and left-identity elements are called identity elements (in symbols: 1).

LEMMA 2. 1. *If the ring R possesses a right-dual which contains a left-identity element, then R has the following properties:*

(a) *If J_i for $i = 1, 2, \cdots$ are right-ideals such that $0 < J_i < J_{i-1}$ then their cross-cut is different from* 0.

(b) *Every right-ideal not* 0 *in R contains a minimal right-ideal.*

(c) *There does not exist an infinite independent set of right-ideals in R.*

Proof. Suppose that $\boldsymbol{d}$ is a duality of the system of right-ideals in the ring R upon the system of right-ideals in the ring R' and that R' contains a left-identity element e. If the right-ideals J_i in R satisfy $0 < J_i < J_{i-1}$, and if J is the cross-cut of the right-ideals J_i, then the right-ideals J_i^d in R'

---

[6] This is actually a generalization of the concept of generalized uniserial ring, as defined by Nakayama (1), p. 19, and our concept differs from the one introduced by Nakayama mainly in this that it does not impose any requirements on the left-ideals.

[7] Baer (1).

[8] I. e. a ring which is simple modulo its radical.

satisfy: $J_{i-1}{}^d < J_i{}^d < R'$; and the system of all the elements contained in at least one $J_i{}^d$ is precisely the right-ideal J^d. Since none of the $J_i{}^d$ contains e (as a consequence of $R' = eR'$), neither does J^d contain e. Hence $J^d < R'$ and therefore $0 < J$, showing the validity of (a).

If V is a right-ideal different from 0 in R, then $V^d < R'$. Hence there exists a greatest right-ideal G in R' which contains V^d, but does not contain e. Since $R' = eR'$, it follows that G is a maximal right-ideal in R'. The right-ideal $G^{d^{-1}}$ in R is consequently a minimal right-ideal in R and it is clearly contained in J, i. e. (b) is satisfied by R.

If an infinite independent set of right-ideals in R existed, then there would exist a countably infinite, independent set of right-ideals $J_i \neq 0$ in R. But then the right-ideals $T_i = \sum_{i<j} J_j$ would form a properly descending chain of right-ideals whose cross-cut were 0, contradicting property (a). This completes the proof.

THEOREM 2. 2. *If every element x in the ring R is contained in the right-ideal xR, and if there exists to every quotient-ring of R a right-dual which contains a left-identity element, then both the maximum and the minimum condition are satisfied by the right-ideals in the ring R; and R contains a right-identity element.*

Proof. It is immediately evident that conditions (b) and (c) of Lemma 2. 1 are satisfied by every quotient ring of R. If the two-sided ideals T_i in R would form a properly descending chain ($T_i < T_{i-1}$), then condition (a) would not be satisfied by the right-ideals in the ring R/T for T the cross-cut of the two-sided ideals T_i; and hence there would not exist properly descending chains of two-sided ideals in R. Now Theorem 2. 2 is an immediate consequence of two theorems which we have proved elsewhere.[9]

3. Uniseriality. If the minimum condition is satisfied by the right-ideals in the ring R, then there exists[10] the *radical* $P = P(R)$ of R. This ideal is characterized by the following properties: (1) P is a two-sided ideal; (2) P is a nilpotent ideal, i. e. there exists an integer i such that $P^i = 0$; (3) P contains every nilideal, i. e. every right-ideal J such that there exists to every element x in J an integer n satisfying: $x^n = 0$. If there exists a left-identity element in R, then it can be shown[11] that (4) P is the cross-cut of all the maximal right-ideals in R. A right-ideal in R is termed *regular*, if it is not part of the radical P of R. If the minimum condition is satisfied by the right-ideals in R, then we may deduce from a theorem due to Köthe[12] that every regular right-ideal contains an idempotent not 0. This implies in particular that

---

[9] Baer (3), Theorems 8. 4 and 9. 4.

[10] Hopkins (1), p. 714, Theorem 1. 4.

[11] Baer (3), Theorem 3. 4. If one considers Baer (3), Theorem 9. 6, then one sees that this condition (4) and the existence of a left-identity element are almost equivalent requirements.

minimal regular right-ideals are direct summands of R;

here we term a regular right-ideal minimal, if no proper subideal is regular.

Consideration of property (4) of the radical of R leads to the definition of the *anti-radical* $M = M(R)$ of the ring R as the sum of all the minimal right-ideals in R. It is a two-sided ideal in R; if the right-ideal J is part of M, then J is a direct summand of M and J is the direct sum of minimal right-ideals.[13]

LEMMA 3.1. *Suppose that the minimum condition is satisfied by the right-ideals in the ring R, that R contains a left-identity element and that there exists a right-dual of R which possesses a left-identity element. Then the regular right-ideal J in R is a minimal regular right-ideal if,*[14] *and only if, J contains one and only one minimal right-ideal in R.*

Proof. Assume first that the regular right-ideal J in R contains one and only one minimal right-ideal in R. Since the minimum condition is satisfied by the right-ideals in R, there exists a minimal regular right-ideal J^* which is part of J. It has been pointed out before that J^* is a direct summand of the ring R, and hence there exists a right-ideal J^{**} such that J is the direct sum of J^* and J^{**}. Since J^* is regular, it contains a minimal right-ideal in R. Since J contains only one minimal right-ideal, it follows that J^{**} does not contain any minimal right-ideal; and consequently we may deduce from the minimum condition that $J^{**} = 0$, i. e. that $J = J^*$ is a minimal regular right-ideal.

Assume now, conversely, that J is a minimal regular right-ideal in R. Then there exists a right-ideal V in R such that R is the direct sum of J and V. There exist furthermore a ring R' which contains a left-identity element and a duality $\boldsymbol{d}$ of the system of all the right-ideals in R upon the system of all the right-ideals in R'. It is a consequence of a theorem of Ch. Hopkins[15] that the maximum condition is satisfied by the right-ideals in R;

---

[12] Köthe (1), p. 168, Satz 7.

[13] Cp. e. g. Baer (3), Section 4, or Hopkins (1).

[14] It should be noted that the existence of a dual is not needed for proving the sufficiency of this condition.

[15] Hopkins (1), p. 726, Theorem 6.4.

and thus it follows that both the maximum and the minimum condition are satisfied by the right-ideals in R'. Using the property (4) of the radical, mentioned before, we see now that $P(R)^d = M(R')$, $M(R)^d = P(R')$. It is readily seen that $(P(R) + J)/P(R)$ is a minimal right-ideal in the ring $R/P(R)$ and that consequently $P(R) + V$ is a maximal right-ideal in R. The duality $\boldsymbol{d}$ maps therefore $P(R) + V$ upon a minimal right-ideal in R'; and since $(P(R) + V)^d$ is just the cross-cut of $M(R')$ and V^d, it follows that V^d contains one and only one minimal right-ideal in R'. Hence it follows from the first part of this lemma that V^d is a minimal regular right-ideal in R'; and R' is the direct sum of J^d and V^d, since R is the direct sum of J and V. The arguments used in proving that V^d contains one and only one minimal right-ideal may now be applied on J^d to show that J contains one and only one minimal right-ideal. This completes the proof of the lemma.

The element c in the partially ordered set S is said to be a *cycle of order* n, if there exist exactly $n + 1$ elements x in S which satisfy: $x \leqq c$, and if the parts of c in S form an ordered set. Thus the minimal right-ideals in the ring R are just the cycles of order 1 in the partially ordered set of all the right-ideals in R.

The two-sided ideal $M_i = M_i(R)$ in the ring R is defined inductively by the following rules: $M_0 = 0$, $M_{i+1}/M_i = M(R/M_i)$. If the minimum-condition is satisfied by the right-ideals in R, then $M_i < R$ implies $M_i < M_{i+1}$; and if both the maximum and the minimum condition are satisfied by the right-ideals in R, then there exists a positive integer m such that $R = M_m$.

THEOREM 3.2. *Suppose that the minimum condition is satisfied by the right-ideals in the ring R, that R contains a left-identity element, and that there exists to every quotient-ring of R a right-dual which possesses a left-identity element. If the minimal regular right-ideal J in R is contained in M_i, but is not contained in M_{i-1}, then J is a cycle of order i in the partially ordered set of right-ideals in R.*

Remarks. 1. The hypothesis that the minimum condition be satisfied by the right-ideals in R cannot be deduced from the other hypotheses by means of Theorem 2.2, since we had to impose there a hypothesis which amounted to the existence of a right-identity element, whereas here we assume the existence of a left-identity element.

2. It is a consequence of a theorem of Ch. Hopkins[15] that the maximum condition is satisfied by the right-ideals in the ring R. Hence there exists, as has been remarked before, a positive integer m such that $R = M_m$; and

consequently there exists to every right-ideal J in R an integer i such that $J \leq M_i$, $J \neq M_{i-1}$.

Proof. If $0 < j \leq i$, then $(J + M_{i-j})/M_{i-j}$ is a minimal regular right-ideal in the quotient-ring R/M_{i-j} which is contained in $M_j(R/M_{i-j})$, but not in $M_{j-1}(R/M_{i-j})$. We proceed to prove by complete induction with regard to j that $(J + M_{i-j})/M_{i-j}$ is a cycle of order j in the partially ordered set of all the right-ideals in R/M_{i-j}. It follows from $M = M_1$ and from Lemma 3.1 that $(J + M_{i-1})/M_{i-1}$ is a minimal right-ideal in R/M_{i-1}, i.e. a cycle of order 1 in the partially ordered set of all the right-ideals in R/M_{i-1}. Thus we may assume that our contention holds true for $j - 1$, i.e. that $(J + M_{i-j+1})/M_{i-j+1}$ is a cycle of order $j - 1$ in the partially ordered set of all the right-ideals in R/M_{i-j+1}. We note that $M_{i-j+1}/M_{i-j} = M(R/M_{i-j})$; and hence we may infer from Lemma 3.1 that the cross-cut of $(J + M_{i-j})/M_{i-j}$ and M_{i-j+1}/M_{i-j} is a cycle of order 1 (= a minimal right-ideal in R/M_{i-j}); and now it is readily seen that $(J + M_{i-j})/M_{i-j}$ is a cycle of order j in the partially ordered set of all the right-ideals in R/M_{i-j}. This completes the inductive proof. Hence it follows, in particular, that J is a cycle of order i in the partially ordered set of all the right-ideals in R, since $M_0 = 0$.

4. Duals of primary abelian operator groups. The abelian group G admitting the elements in the ring E as operators is said to be a *primary abelian operator group over E*, if

(i) E contains an identity element 1 satisfying $g1 = g$ for every g in G and $x1 = 1x = x$ for every x in E, and if

(ii) E is a cycle of order $m = m(E)$ in the partially ordered set of all the right-ideals in E.

We note that E is a (not necessarily commutative) field if, and only if, $m(E) = 1$; and one may verify readily that a ring E satisfying condition (ii) has the following properties: each right-ideal in E is a two-sided ideal and is furthermore a power of the radical $P = P(E)$.

If g is an element in the primary abelian operator group G over E, then gE is the E-admissible subgroup of G generated by g; and it is a consequence of condition (ii) that gE is a cycle of an order not exceeding $m(E)$ in the partially ordered set of all the E-admissible subgroups of G. We note finally that all the admissible subgroups and quotient groups of primary abelian operator groups over E are themselves primary abelian operator groups over E.

THEOREM 4.1. *If the primary abelian operator group G over E contains* *at least two independent cycles of order $m(E)$, and if there exists a primary abelian operator group G' over E' such that the partially ordered set of all the E'-admissible subgroups of G' is the dual of the partially ordered set of all the E-admissible subgroups of G, then every left-ideal in E is a two-sided ideal in E.*

Proof. Denote by $\boldsymbol{d}$ a duality of the set of all the E-admissible subgroups of G upon the set of all the E'-admissible subgroups of G'. Suppose furthermore that U and V are E-admissible subgroups of G such that $U < V$ and such that there exist at most two different cycles of order 1 in the partially ordered set of the E-admissible subgroups of V/U. It is easily seen that there exists one and only one E-admissible subgroup W of G such that $U < W \leq V$ and such that W/U is a cycle of order 1 in the partially ordered set of all the E-admissible subgroups of V/U. Since $W^d < U^d$, there exists an element z in U^d which is not contained in W^d. There exists one and only one E-admissible subgroup Z of G such that $Z^d = V^d + zE'$. Then $V^d \leq Z^d \leq U^d$, but $Z^d \nleq W^d$; and hence $U \leq Z \leq V$, but $W \nleq Z$. Since W/U is the only cycle of order 1 in V/U, it is part of every subgroup not 0 of V/U; and thus it follows that $Z = U$ or $U^d = V^d + zE'$, showing that U^d/V^d is a cycle. But the dual of a cycle is a cycle, i. e. V/U is a cycle in the partially ordered set of the E-admissible subgroups of G/U. But it has been shown elsewhere[16] that this condition and the hypotheses of the Theorem assure the two-sidedness of the left-ideals in E.

5. Primary rings with duals. A ring R is said to be *primary*, if it meets the following requirements:

(a) R contains an identity element 1.

(b) The minimum condition is satisfied by the right-ideals in R.

(c) There does not exist a two-sided ideal T in R satisfying: $P < T < R$ (where $P = P(R)$ is the radical of R).

A ring R is said to be *completely primary*, if it satisfies the conditions (a), (b) and has the following stronger property:

(c*) There does not exist a right-ideal J in R satisfying: $P < J < R$.

The condition (c*) may be seen to be equivalent to the more symmetric condition:

(c**) R/P is a (not necessarily commutative) field.

It is a consequence of a theorem of Ch. Hopkins[17] that the maximum condition is satisfied by the right-ideals in a primary ring R; and thus the following facts are restatements of results of G. Köthe:[18]

---

[16] Baer (1).

[17] Hopkins (1), p. 726, Theorem 6. 4.

LEMMA 5.1. *The ring R is primary if, and only if, R is the ring E_n of all the n by n matrices with coefficients from the completely primary ring E. The order n and the completely primary ring E are [essentially] uniquely determined by the primary ring $R = E_n$.*

A ring is termed *uni-serial,* if every minimal regular right-ideal is a cycle in the partially ordered set of all its right-ideals.[19] The following statement may be proved by arguments which K. Asano[20] used for the proof of a similar fact:

LEMMA 5.2. *A ring E meeting the requirements* (a) *and* (b) *above is a completely primary, uni-serial ring if, and only if, every right-ideal in E has the form $P(E)^i$ for suitable i; and the completely primary ring E is uni-serial if, and only if, E_n is uni-serial.*

We are now ready to prove that primarity and uni-seriality are invariant under dualities.

THEOREM 5.3. *Suppose that the minimum condition is satisfied by the right-ideals in the rings R and R', that each of these rings contains an identity element and that there exists a duality **d** of the partially ordered set of all the right-ideals in R upon the partially ordered set of all the right-ideals in R'.*

(i) *If R is uni-serial, then R' is uni-serial.*

(ii) *If R is primary, then R' is primary.*

Proof. Suppose first that R is uni-serial and that J' is a minimal regular right-ideal in R'. Then J' is, as has been pointed out in section **3**, a direct summand of R', i.e. R' is the direct sum of J' and of some right-ideal J''. It is a consequence of Lemma 3.1 that J' contains one and only one minimal right-ideal; and J'' is therefore a greatest right-ideal not containing $M(R')$. Using the property (4) of the radical, enunciated at the beginning of section **3**, it follows that $M(R')^{d^{-1}} = P(R)$ and $M(R)^{d} = P(R')$. Hence $J''^{d^{-1}}$ is a minimal regular right-ideal in R and is therefore a cycle in the partially ordered set of all the right-ideals in R. But R is the direct sum of $J'^{d^{-1}}$ and $J''^{d^{-1}}$, since R' is the direct sum of J' and J''. Thus R is a cycle in the partially ordered set of all the right-ideals in R which contain J'^{d-1}; and J' is consequently a cycle in the partially ordered set of the right-ideals in R', i.e. R' is uni-serial.

---

[18] Köthe (1), p. 176, Hauptsatz 12 and p. 179, Zusatz 2.

[19] Note that this definition of uni-seriality does not impose any requirements on the left-ideals; cf. [6].

[20] Asano (1), p. 233/234, Hilfssätze 3, 4.

Suppose now that R is primary and that T' is a two-sided ideal in R' such that $P(R') < T' < R'$. Since $R'/P(R')$ is a semi-simple ring, there exists[21] a two-sided ideal T'' such that $R' = T' + T''$ and such that $P(R')$ is the cross-cut of T' and T''. If $S' = T'^{d-1}$ and S''^{d-1}, then $M(R)$ is the direct sum of S' and S'', since the sum of T' and T'' is R', and since the cross-cut of T' and T'' is just $P(R') = M(R)^d$. Every minimal right-ideal of $R'/P(R')$ is contained in one and only one of the two-sided ideals $T'/P(R')$ and $T''/P(R')$ whose direct sum is the ring $R'/P(R')$; and this implies that every greatest proper subideal of $M(R)$ contains one and only one of the two ideals S' and S''.

It is a consequence of Lemma 5.1 that R is the ring E_n of all the n by n matrices with coefficients from the completely primary ring E. The ring E is isomorphic to a ring eRe where e is a primitive idempotent in R (i.e. an idempotent different from 0 which is not the sum of two orthogonal idempotents both of which are different from 0); and it is a consequence of the fundamental property of minimal regular right-ideals (mentioned in section **3**) that eR is a minimal regular right-ideal. Hence we deduce from Lemma 3.1 that eR contains one and only one minimal right-ideal; and this makes it evident that E itself contains one and only one minimal right-ideal, or $M(E)$ is a minimal right-ideal in E.

If z is any element not 0 in $M(E)$, then $M(E) = zE$, since E is a minimal right-ideal in E. Since $P(E)$ is nilpotent, $M(E)P(E) = 0$. The set Z of all the elements x in E which satisfy: $zx = 0$ is a right-ideal satisfying: $P(E) \leq Z < E$; and since E is a completely primary ring, it follows that $Z = P(E)$.

We note next the facts that $M(R) = M(E_n)$ is the set of all the matrices in E_n whose coefficients are in $M(E)$; and that $P(R) = P(E_n)$ is the set of all the matrices in E_n whose coefficients are in $P(E)$.

Let z be any element not 0 in $M(E)$; and define, for every matrix (a_{ij}) in E_n, the map f by $(a_{ij})^f = (za_{ij})$. Clearly f is a single-valued transformation which preserves addition and which satisfies: $R^f = M(R)$, since $zE = M(E)$. It has been shown that $zx = 0$ if, and only if, x is in $P(E)$. Consequently $(a_{ij})^f = 0$ if, and only if, every a_{ij} is in $P(E)$ i.e. if, and only if, (a_{ij}) is in $P(R)$. This shows that f induces an isomorphism of the additive group of the ring $R^* = R/P(R)$ upon $M(R)$ and that this induced isomorphism maps right-ideals in R^* upon the right-ideals in R which are part of $M(R)$. If V' and V'' are the right-ideals in R^* which are mapped by f upon the right-ideals S' and S'' respectively, then it follows from the properties of S' and S'' that the ring R^* is the direct sum of the right-ideals V' and V''; and every maximal right-ideal in R^* contains one and only one of the ideals V' and V''.

---

[21] Cf. e. g. v. d. Waerden (1), p. 170.

The ring R^* is by its construction essentially the same as the ring of all n by n matrices with coefficients from the field $E^* = E/P(E)$. Thus there exists a system e_{ij} of n^2 matrix units in R^* satisfying: $e_{ij}e_{hk} = \begin{matrix} 0 \text{ for } j \neq h \\ e_{ik} \text{ for } j = h \end{matrix}$; and the elements e_{ii} are a set of n mutually orthogonal primitive idempotents in R^*.

The right-ideal $\sum\limits_{j \neq i} e_{jj}R^*$ is clearly a maximal right-ideal in R^* so that it contains one and only one of the ideals V' and V''. We may assume, therefore, without loss in generality, that

$$V'' \leq \sum_{j \neq i} e_{jj}R^* \text{ for } i = 1, \cdots, h; \quad V' \leq \sum_{j \neq i} e_{jj}R^* \text{ for } i = h+1, \cdots, n;$$

and we deduce from the fact that R^* is the direct sum of V' and V'' as well as the direct sum of the $e_{jj}R^*$ that

$$V' = \sum_{j=1}^{h} e_{jj}R^*, \qquad V'' = \sum_{j=h+1}^{n} e_{jj}R^*.$$

It is readily verified that the $n-1$ elements $e_{11} + e_{n1}, e_{22}, \cdots, e_{n-1,n-1}$ are mutually orthogonal idempotents of which the first is different from 0 whereas the $n-2$ others are primitive idempotents. If e_{nn} were an element in the right-ideal $W = (e_{11} + e_{n1})R^* + \sum\limits_{i=2}^{n-1} e_{ii}R^*$, then $e_{nn} = (e_{11} + e_{n1})r_1 + \sum\limits_{i=2}^{n-1} e_{ii}r_i$. If $0 < i < n$, then $0 = e_{ii}e_{nn} = e_{ii}r_i$ so that $e_{nn} = (e_{11} + e_{n1})r_1 = e_{n1}r_1$ and hence $e_{1n} = e_{1n}e_{nn} = e_{1n}e_{n1}r_1 = e_{11}r_1 = 0$, an impossibility. Thus $W < R^*$, showing that W is a maximal right-ideal in R^* which does not contain e_{nn}. Thus in particular V'' is not part of W. Since one of the ideals V' and V'' is contained in W, we have found that $V' \leq W$. Consequently e_{11} is an element in W and therefore e_{n1} and $e_{nn} = e_{n1}e_{1n}$ are elements in W. This impossibility shows that the original hypothesis of non-primarity of R' has led us to a contradiction; and this completes the proof of the theorem.

THEOREM 5.4. *Suppose that the ring E is completely primary and uni-serial. Then*

(I) E_1 $(=E)$ *is its own right-dual.*

(II) *There exists a right-dual to* E_n *for* $n > 1$ *which contains an identity element if, and only if, every left-ideal in* E *is a right-ideal.*

Proof. The assertion (I) is an obvious consequence of the fact that the right-ideals in E form a finite ordered set.

If every left-ideal in E is a right-ideal, then every ideal in E (left-ideals as well as right-ideals) is a power of the radical $P(E)$ and it is known[22] that a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in E_n is effected by mapping the right-ideal J in E_n upon the set $L(J)$ of all the elements x in E_n which satisfy $xJ = 0$.

Suppose finally that $1 < n$ and that there exists a duality $\boldsymbol{d}$ of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the right-ideals in the ring R' which contains an identity element. It is a consequence of Lemmas 5. 1 and 5. 2 that the ring E_n is primary and uni-serial; and thus we deduce from Theorem 5. 3 that R' is primary and uni-serial. There exist by Lemmas 5. 1 and 5. 2 an integer k and a completely primary, uni-serial ring F such that $R' = F_k$.

Denote by (E, n) the abelian group of all the n-tuplets of elements in E; and by (F, k) the abelian group of all the k-tuplets of elements in F. The group (E, n) admits the elements in E as operators (multiplication from the right), is a primary abelian operator group over E (in the sense of section **4**) and is the direct sum of n cycles of order $m(E)$; similar statements may be made concerning (F, k).

If B is an E-admissible subgroup of (E, n), then denote by $B^{\boldsymbol{p}}$ the set of all those matrices in E_n all of whose columns are n-tuplets in B; and if J is a right-ideal in F_k, then denote by $J^{\boldsymbol{q}}$ the set of all the k-tuplets in (F, k) which are columns of matrices in J. It is readily verified that $\boldsymbol{p}$ is a projectivity[23] of the partially ordered set of all the E-admissible subgroups of (E, n) upon the partially ordered set of all the right-ideals in E_n; and that $\boldsymbol{q}$ is a projectivity of the partially ordered set of all the right-ideals in F_k upon the partially ordered set of all the F-admissible subgroups of (F, k). The transformation $\boldsymbol{pdq}$ is therefore a duality of the partially ordered set of all the E-admissible subgroups of (E, n) upon the partially ordered set of all the F-admissible subgroups of (F, k); and hence it follows from Theorem 4. 1 that every left-ideal in E is a two-sided ideal in E, as was to be shown.

---

[22] Cf. e. g. Baer (2), Theorem 7. 5, p. 210 or Nakayama (1), Theorem 6, p. 9.

[23] A projectivity between two partially ordered sets is a biunique and monotonically increasing correspondence between the elements of these partially ordered sets.

It will be convenient to term a ring R *quasi-primary*, if R contains an identity-element 1, and if there exists in R a two-sided ideal $T < R$ such that every two-sided ideal different from R in R is part of T. That every primary ring is quasi-primary, is a consequence of the following slightly more general statement.

LEMMA 5. 5. *If the ring R contains an identity element* 1, *and if there exists in R a two-sided ideal T, all of whose elements are nil-elements,*[24] *such that R/T is a simple ring,*[25] *then the ring R is quasi-primary.*

Proof. Suppose that the two-sided ideal S in R is not part of T. Then $T < S + T$ and $S + T$ is a two-sided ideal so that $R = S + T$. Hence there exist elements s and t in S and T, respectively, such that $1 = s + t$. Consequently $1 = (s + t)^n = s_n + t^n$ for every n and s_n an element in the two-sided ideal S. Since t is a nil-element, $t^i = 0$ for some i. Thus 1 is in S and $S = R$, as was to be shown.

THEOREM 5. 6. *The ring R is quasi-primary and there exists to every quotient-ring of R a right-dual possessing an identity element* 1 *if, and only if, R satisfies the following two conditions:*

(i) *R is the ring E_n of all the n by n matrices with coefficients from the completely primary and uni-serial ring E.*

(ii) *Either $n = 1$ or else every left-ideal in E is a right-ideal in E.*

Proof. If R is a quasi-primary ring whose quotient-rings possess duals with identity elements, then we deduce from Theorems 2. 2 and 3. 2 that the minimum condition is satisfied by the right-ideals in R and that R is uni-serial. Consequently the radical $P(R)$ of R exists. Since R is quasi-primary, there exists one and only one greatest two-sided ideal T different from R. Clearly $P(R) \leq T$. Since $R/P(R)$ is semi-simple, there exists a two-sided ideal T' such that $R = T + T'$ and such that $P(R)$ is the cross-cut of T and T'. It is impossible that T' is part of T and hence we have $R = T'$. Thus the cross-cut of R and T is the radical $P(R)$ of R showing that $R/P(R)$ is simple, i. e. R is a primary ring. It is a consequence of Lemmas 5. 1 and 5. 2 that there exist a positive integer n and a completely primary, uni-serial ring E such that $R = E_n$; and the necessity of conditions (i) and (ii) is now a consequence of Theorem 5. 4.

If the ring R satisfies conditions (i) and (ii), then we infer from Lemma

---

[24] The element x is a nilelement, if $x^i = 0$ for some i.

[25] I. e. if there does not exist a two-sided ideal T' such that $T < T' < R$.

5.5 that R is quasi-primary. It is easily seen that every quotient-ring of R meets the requirements (i) and (ii); and the sufficiency of (i) and (ii) is an immediate consequence of Theorem 5.4.

We note that we have proved the following fact:

If the maximum condition is satisfied by the right-ideals in the quasi-primary ring R, then R is a primary ring.

6. The dualities of primary, uni-serial rings. If S is a subset of the ring R, then we denote by $L(S)$ the set of all the elements x in R which satisfy: $xS = 0$; and we denote by $K(S)$ the set of all the elements y in R such that $Sy = 0$. Clearly $L(S)$ is a left-ideal and $K(S)$ a right-ideal in R; and both these functions are monotonically decreasing.

If E is a completely primary and uni-serial ring all of whose left-ideals are two-sided ideals, then the following statements hold:[26]

(1) $K(L(J)) = J$ for every right-ideal J in E.

(2) $L(K(J)) = J$ for every left-ideal J in E_n.

(3) The functions K and L define dualities between the partially ordered set of all the right-ideals in E_n and the partially ordered set of all the left-ideals in E_n.

THEOREM 6.1. *If the ring E is uni-serial and completely primary, if $2 < n$, and if $\boldsymbol{d}$ is a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in the ring R with identity element 1, then there exists an isomorphism f of the ring E_n upon the ring R such that $J^d = L(J^f)$ for every right-ideal J in E_n.*

Proof. It is a consequence of Theorem 5.4, (II), of $2 < n$, and of the existence of the duality $\boldsymbol{d}$ that every left-ideal in E is a two-sided ideal in E. It is a consequence of Lemmas 5.1 and 5.2 that the ring E_n is primary and uni-serial; and hence we deduce from Theorem 5.3 that the ring R is primary and that every minimal regular left-ideal in R is a cycle in the partially ordered set of all the left-ideals in R, i.e. R is left-sided uni-serial. Now we infer from Lemmas 5.1 and 5.2 and from Theorem 5.4, (II) the existence of a completely primary and uni-serial ring F, all of whose left-ideals are two-sided, such that $R = F_k$ for some positive integer k (note that certainly $1 < k$). It is a consequence of the remark (3) above that a projectivity of the set of all the right-ideals in E_n upon the set of all the right-ideals in $R = F_k$ is obtained by mapping the right-ideal J in E_n upon the right-ideal $K(J^d)$ in F_k.

---

[26] Cf. footnote [22] above.

Denote as before by (E, n) the primary abelian operator group over E which consists of all the n-tuplets of elements in E, and where the elements in E operate upon the group (E, n) by multiplication from the right. If $a = (a_1, \cdots, a_n)$ is an element in (E, n), and if $b = (b_{ij})$ is an element in E_n, then $ba = (\cdots, \sum_{i=1}^{n} b_{ji}a_i, \cdots)$ is a well-determined element in (E, n). If J is a right-ideal in E_n, then $J(E, n)$ is an E-admissible subgroup of (E, n) which consists exactly of all the columns of matrices in J; and it is readily verified that a projectivity of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the E-admissible subgroups of (E, n) is obtained by mapping the right-ideal J in E_n upon $J(E, n)$. Analogous statements may be made concerning F_k and (F, k).

If J is a right-ideal in E_n, then we put $(J(E, n))^{\boldsymbol{p}} = K(J^{\boldsymbol{d}})(F, k)$; and it follows, from the results stated at the end of the first and second paragraph of the proof, that $\boldsymbol{p}$ is a projectivity of the partially ordered set of all the E-admissible subgroups of (E, n) upon the partially ordered set of all the F-admissible subgroups of (F, k). Since $2 < n$, we may apply a fundamental theorem on the projectivities of primary abelian operator groups;[27] and hence there exists an isomorphism $\boldsymbol{g}$ of E upon F, an isomorphism $\boldsymbol{g}$ of (E, n) upon (F, k) such that $(ae)^{\boldsymbol{g}} = a^{\boldsymbol{g}}e^{\boldsymbol{g}}$ for a in (E, n) and e in E and such that $S^{\boldsymbol{g}} = S^{\boldsymbol{p}}$ for every E-admissible subgroup S of (E, n). There exists an isomorphism $\boldsymbol{f}$ of E_n upon F_k such that $(ba)^{\boldsymbol{g}} = b^{\boldsymbol{f}}a^{\boldsymbol{g}}$ for every b in E_n and every a in (E, n), since E_n is anti-isomorphic to the ring of linear transformations of (E, n) and F_k is anti-isomorphic to the ring of linear transformations of (F, k). Now we find that

$$K(J^{\boldsymbol{d}})(F, k) = (J(E, n))^{\boldsymbol{p}} = (J(E, n))^{\boldsymbol{g}} = J^{\boldsymbol{f}}(F, k) \quad \text{or} \quad K(J^{\boldsymbol{d}}) = J^{\boldsymbol{f}};$$

and we infer from (1) above that $L(J^{\boldsymbol{f}}) = J^{\boldsymbol{d}}$ for every right-ideal J in E_n.

Corollary 6.2. *If the ring E is uni-serial and completely primary, if $2 < n$, and if $\boldsymbol{d}$ is a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the right-ideals in the ring R with identity element 1, then there exists an anti-isomorphism $\boldsymbol{f}$ of E_n upon R such that $J^{\boldsymbol{d}} = K(J^{\boldsymbol{f}})$ for every right-ideal J in E.*

Proof. There exists an anti-isomorphism $\boldsymbol{h}$ of the ring R upon a suitable ring R^*. If J is a right-ideal in E_n, then $J^{\boldsymbol{dh}}$ is a left-ideal in R^*; and $\boldsymbol{dh}$ is a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in R^*. Hence there exists by Theorem 6.1 an isomorphism $\boldsymbol{r}$ of the ring E_n upon the ring R^* such that

---

[27] Baer (1), Theorem II, 3.1, p. 311.

$J^{dh} = L(J^r)$ for every right-ideal J in E_n. It is clear that $\boldsymbol{f} = \boldsymbol{rh}^{-1}$ is an anti-isomorphism of E_n upon R which satisfies:

$$J^d = (J^{dh})^{h^{-1}} = (L(J^r))^{h^{-1}} = K(J^{rh^{-1}}) = K(J^f).$$

Remark. The hypothesis: $2 < n$ cannot be omitted in Theorem 6.1 and Corollary 6.2. For suppose that E and F are two countably infinite, commutative fields which are not isomorphic. Then every proper right-ideal in E_2 or F_2 is a minimal right-ideal; and the same holds for the left-ideals. Thus there exists a duality of the partially ordered set of all the right-ideals in E_2 upon the partially ordered set of all the left-(right-)ideals in F_2, though the rings E_2 and F_2 are neither isomorphic nor anti-isomorphic.

COROLLARY 6.3. *Suppose that E is a completely primary and uni-serial ring, that $2 < n$; and that the ring R contains an identity element* 1. *Then there exists a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-(right-)ideals in R if, and only if, the rings E_n and R are isomorphic (anti-isomorphic) rings and every left-ideal in E is a right-ideal in E.*

Proof. The necessity of the conditions is an immediate consequence of Theorems 5.4 and 6.1 and of Corollary 6.2. If, conversely, every left-ideal in E is a right-ideal in E, then it follows from the proposition (3), stated at the beginning of this section **6**, that a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in E_n is obtained by mapping the right-ideal J in E_n upon the left-ideal $L(J)$ in E_n. If $\boldsymbol{f}$ is an isomorphism of the ring E_n upon the ring $E_n^{\boldsymbol{f}}$, then a duality of the set of all the right-ideals in E_n upon the set of all the left-ideals in $E_n^{\boldsymbol{f}}$ is effected by mapping the right-ideal J in E_n upon $L(J)^{\boldsymbol{f}}$; and if $\boldsymbol{g}$ is an anti-isomorphism of the ring E_n upon the ring $E_n^{\boldsymbol{g}}$, then a duality of the set of all the right-ideals in E_n upon the set of the right-ideals in $E_n^{\boldsymbol{g}}$ is obtained by mapping the right-ideal J in E_n upon the right-ideal $L(J)^{\boldsymbol{g}}$ in $E_n^{\boldsymbol{g}}$.

COROLLARY 6.4. *Suppose that E is a completely primary and uni-serial ring and that $2 < n$.*

(a) *There exists a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in E_n if, and only if, every left-ideal in E is two-sided.*

(b) *There exists a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the right-ideals in E_n if, and only if, there exists an anti-automorphism of E.*

Proof. The statement (a) is an immediate consequence of Corollary 6. 3, since every ring is isomorphic to itself. If there exists an anti-automorphism of E, then every left-ideal in E is a right-ideal, since every right-ideal in the completely primary and uni-serial ring E is a left-ideal; and there exists clearly an anti-automorphism of E_n. Thus the sufficiency of the condition in (b) is an immediate consequence of Corollary 6. 3. If there exists a duality of the partially ordered set of all the right-ideals in E_n upon the set of the left-ideals in E_n, then we infer from Corollary 6. 3 the existence of an anti-automorphism of E_n; and we deduce from Lemma 5. 1 the existence of an anti-automorphism of E.

Remark. It has been pointed out that the hypothesis: $2 < n$ is needed for the proof of the necessity of the conditions enumerated in Corollaries 6. 3 and 6. 4; but it is clear from the proofs that this hypothesis is not needed for proving the sufficiency of these conditions.

UNIVERSITY OF ILLINOIS,
URBANA, ILL.

BIBLIOGRAPHY.

K. Asano,
(1) "Über verallgemeinerte abelsche Gruppen mit hyperkomplexem Operatorenring und ihre Anwendungen," *Jap. Journ. of Math.*, vol. 15 (1939), pp. 231-253.

R. Baer,
(1) "A unified theory of finite abelian groups and of finite-dimensional projective spaces," *Transactions of the American Mathematical Society*, vol. 52 (1942), pp. 283-343.
(2) "Automorphism rings of primary abelian operator groups," *Annals of Mathematics*, vol. 44 (1943), pp. 192-227.
(3) "Radical ideals," *American Journal of Mathematics*, vol. 65 (1943), pp. 537-568.

M. Hall,
(1) "A type of algebraic closure," *Annals of Mathematics*, vol. 40 (1939), pp. 360-369.

Ch. Hopkins,
(1) "Rings with minimal conditions for left ideals," *Annals of Mathematics*, vol. 40 (1939), pp. 712-730.

G. Köthe,
(1) "Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist," *Mathematische Zeitschrift*, vol. 32 (1930), pp. 161-186.

T. Nakayama,
(1) "On Frobeniusean algebras; I and II," *Annals of Mathematics*, vol. 40 (1939), pp. 611-633 and vol. 42 (1941), pp. 1-21.
(2) "Algebras with anti-isomorphic left and right ideal lattices," *Proc. Imp. Acad. Tokyo*, vol. 17 (1941), pp. 53-56.

B. L. v. d. Waerden,
(1) *Moderne Algebra II*, Leipzig, 1931.

THE LINEAR q-DIFFERENCE EQUATION OF THE SECOND ORDER.* [1]

By Jeanne Le Caine.

1. The general matrix equation

$$Y\left(\frac{ax+b}{cx+d}\right)=R(x)Y(x)$$

in which Y and R are matrices of order n, leads to a variety of interesting functions. If the linear transformation $(ax+b)/(cx+d)$ has one fixed point, there exists a linear transformation on the independent variable by which this matrix equation can be reduced to the difference equation

$$Y(x+1)=R(x)Y(x).$$

If the transformation $(ax+b)/(cx+d)$ has two distinct fixed points, the general matrix equation can be reduced to a q-difference equation

$$Y(qx)=R(x)Y(x).$$

If the limiting forms of the equation are considered, it contains also the differential equation

$$Y'(x)=R(x)Y(x).$$

The simplest case of the general matrix equation is the case $n=1$. The difference equation in this case leads to the theory of the gamma function, which is related to the sine function by the equation $\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$. The q-difference equation leads to the theory of what may be called the q-gamma function. This can be considered the logical "half" of the Weierstrass sigma function on account of the relation

$$\Gamma_q{}^0(1-x)=\Gamma_q{}^\infty(1-x)\,c\,e^{-(\eta/2)t^2-\pi i t}\sigma(t).$$

The differential equation leads to the exponential function.

From the point of view of simplicity the next case to consider is the case

---

* Received February 7, 1942.

[1] This paper is part of a dissertation written at Radcliffe College under the direction of Professor G. D. Birkhoff, while the author was holding the Travelling Fellowship given by the Canadian Federation of University Women. The author wishes to express her sincere thanks to Professor Birkhoff for his aid in the preparation of this paper.

where $n=2$, and the matrix $R(x)$ takes the simplified form $(1/p(x))(Ax+B)$ which can be immediately reduced, by means of the case $n=1$, to the form

$$Y\left(\frac{ax+b}{cx+d}\right)=(Ax+B)Y(x).$$

The linear differential system obtained from this by a limiting process is essentially equivalent to the hypergeometric equation of Gauss. The linear difference system is the hypergeometric difference equation. The q-difference equation in the general form leads to the theory of Heine Series.

The importance of the matrix as a starting point is obvious for differential equations from the work of Schlesinger. A detailed treatment of the hypergeometric difference equation from the matrix point of view is given by Batchelder.[2] The analogous case for q-difference equations has not been treated.

In the present paper, the aim is to treat certain questions of importance in the classification of the q-gamma matrices of the second order, which are the solutions of the q-difference equation

(1) $Y(qx)=(Ax+B)Y(x)$, where A, B are constant matrices of order 2,

$|Ax+B|\not\equiv 0$, and $|q|$ is assumed greater than one. Some of the known facts about Heine Series[3] are used as are also the theoretical results which were obtained by Grévy and Leau, Carmichael, Adams, Birkhoff, and Trjitzinsky.[4]

In order to make clear the advantages of the matrix method in this discussion, the paper is divided into two parts. The first develops some

---

[2] P. M. Batchelder, *An Introduction to Linear Difference Equations*, Cambridge, 1927.

[3] H. E. Heine, "Untersuchungen über die Reihe

$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}x^{2}+\cdots;"$$

Journal für Mathematik, vol. 34 (1847), pp. 285-328. E. R. Smith, "Zür Theorie der Heineschen Reihe und ihrer Verallgemeinerung," *Inaugural Dissertation*, München, (1911).

[4] A comprehensive bibliography on the field of difference equations is given by N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924. This has been brought up to the year 1931 by C. R. Adams in an article entitled "Linear q-difference equations," *Bulletin of the American Mathematical Society*, vol. 37 (1931), pp. 361-400. A few references of importance in the following are: C. R. Adams, "On the linear ordinary q-difference equation," *Annals of Mathematics*, (2), vol. 30 (1929), pp. 195-205; G. D. Birkhoff, "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations," *Proceedings of the American Academy of Arts and Sciences*, vol. 49 (1913), pp. 521-568; W. J. Trjitzinsky, "Analytic theory of linear q-difference equations," *Acta Mathematica*, vol. 61 (1933), pp. 1-38.

elementary algebraic theorems regarding the effect of certain types of transformations on equation (1). The second deals with the explicit form of the q-gamma matrices and the relations between them which can be obtained by application of the algebraic theorems of part A.

Among the new results obtained are the following. In **2** a simple normal form for equation (1) is set up and the conditions are obtained under which (1) can be reduced to this form by the transformation $Y(x) = C\bar{Y}(x)$, C a non-singular constant matrix. The following section contains a discussion of necessary and sufficient conditions for the existence of a rational transformation, $Y(x) = L(x)\bar{Y}(x)$, by means of which equation (1) is transformed into a linear equation of the same form.

The explicit forms of the q-gamma matrices which are the fundamental solutions of equation (1) are listed in **6**. In **7** it is shown that if two equations are related by a rational transformation, a rational relation exists between the corresponding fundamental solutions of the equations, and that this is the only case in which such a relation can exist. The position of the family of rationally related equations in the solution of the Riemann Problem for equations in which $|A| \neq 0$ and $|B| \neq 0$ is stated in **7**. In the concluding section it is shown that, by making use of the rational relations between equations, a fundamental solution of an equation for which $|A|$ and $|B|$ are not zero can be represented as a convergent matrix product in which the polynomial part of each factor can be taken to be linear.

A: Transformations on the Matrix Equation.

2. Linear transformations. In the equation

$$Y(qx) = (Ax + B)Y(x) \tag{1}$$

let a_1, a_2 represent the characteristic roots of the matrix A; b_1, b_2 the characteristic roots of the matrix B; α_1, α_2, the zeros of the determinant

$$|Ax + B| = a_1a_2x^2 + (a_{11}b_{22} + b_{11}a_{22} - a_{12}b_{21} - a_{21}b_{12})x + b_1b_2.$$

These six constants will be called the *characteristic constants* of the matrix $Ax + B$, or of equation (1). A matrix (or equation) will be said to be of *type* I if all its characteristic constants are finite and different from zero. The discussion in this and in the following section deals with matrices of type I. The modifications necessary for matrices not of type I will be discussed later.

The characteristic constants of a matrix satisfy the relation $b_1b_2 = a_1a_2\alpha_1\alpha_2$, and so not more than five can be assigned arbitrarily. If a_{12} and b_{12} are simultaneously zero, the matrix (or equation) is said to be reducible, and the constants satisfy the additional conditions $b_1 = -a_1\alpha_1$, $b_2 = -a_2\alpha_2$.

If to equation (1) is applied the linear transformation

(2) $Y(x) = C\bar{Y}(x)$, C a matrix of constants, $|C| \neq 0$, the equation becomes

$$\bar{Y}(qx) = C^{-1}(Ax + B)C\bar{Y}(x) = (C^{-1}ACx + C^{-1}BC)\bar{Y}(x)$$

which is a linear matrix equation of the same form as (1), and with the same characteristic constants. Thus *if two equations are related by a linear transformation* (2) *they have the same characteristic constants.*

Under certain conditions the converse of this is true. This can be stated as follows:

THEOREM I. *If a_1, a_2, b_1, b_2, α_1, α_2, are the characteristic constants of equation* (1) *and A and B do not commute, a non-singular matrix C exists such that the transformation* (2) *applied to equation* (1) *gives*

$$(3)\quad \bar{Y}(qx) = \begin{pmatrix} 0 & a_1(x-\alpha_1) \\ -a_2(x-\alpha_2) & (a_1+a_2)x+b_1+b_2 \end{pmatrix}\bar{Y}(x) = (\bar{A}x+\bar{B})\bar{Y}(x)$$

or

$$(4)\qquad \bar{Y}(qx) = \begin{pmatrix} a_1x+b_1 & 0 \\ 1 & k(a_1x+b_1) \end{pmatrix}\bar{Y}(x).$$

In proving this theorem, there is no loss of generality if $Ax + B$ is assumed to have the form $\begin{pmatrix} a_1x+b_{11} & b_{12} \\ a_{21}x+b_{21} & a_2x+b_{22} \end{pmatrix}$ with $a_{21} = 0$ if $a_1 \neq a_2$, and $a_{21} = 0$ or 1 if $a_1 = a_2$, since there always exists a non-singular matrix C such that $C^{-1}AC = \begin{pmatrix} a_1 & 0 \\ a_{21} & a_2 \end{pmatrix}$. The possibility $a_{21} = 0$, $a_1 = a_2$, is excluded because A cannot be a multiple of the unit matrix since A and B do not commute.

If $b_{12} \neq 0$, the matrix

$$C = \begin{pmatrix} 1, & -\dfrac{a_1}{a_2} \\ -\dfrac{b_{11}+a_1\alpha_2}{b_{12}}, & \dfrac{b_{11}+a_1\alpha_2}{b_{12}} - \dfrac{a_{21}}{a_2} \end{pmatrix}, \quad |C| = \{(b_{11}+a_1\alpha_2)(a_2-a_1) - a_{21}b_{12}\}/b_{12}$$

is non-singular, and $C^{-1}(Ax+B)C = \bar{A}x + B$.

If $b_{12} = 0$, then $b_{11} = b_1 = -a_1\alpha_1$ and $b_{22} = b_2 = -a_2\alpha_2$. The diagonal elements of the matrix $Ax + B$ can be interchanged by a non-singular transformation (2), and if $\alpha_1 \neq \alpha_2$ a further transformation with

$$C = \begin{pmatrix} -1 & , & 1 \\ \dfrac{a_1b_{21}-a_{21}b_2}{a_1^2(\alpha_2-\alpha_1)}, & & -\dfrac{a_2b_{21}-a_{21}b_2}{a_2^2(\alpha_2-\alpha_1)} + \dfrac{a_{21}}{a_2} \end{pmatrix}$$

puts $Ax+B$ in the form $\bar{A}x+\bar{B}$. If $\alpha_1=\alpha_2$, then $a_2=ka_1$ and $b_2=kb_1$ and the only case in which A and B do not commute reduces to (4).

That the restrictions made in the above theorem are necessary can be seen by considering the commutativity of $\bar{A}$ and $\bar{B}$. These matrices commute if and only if $b_1=-a_1\alpha_1$, $b_2=-a_2\alpha_2$, $\alpha_1=\alpha_2$. Therefore $C^{-1}AC=\bar{A}$ and $C^{-1}BC=\bar{B}$ are impossible when A and B commute and the conditions $b_1=-a_1\alpha_1$, $b_2=-a_2\alpha_2$, $\alpha_1=\alpha_2$ are not satisfied, and also when these conditions are satisfied and A and B do not commute, as in equation (4).

If the equation takes the form (4), the simple rational transformation

$$Y(x)=\begin{pmatrix} a_1x/q+b_1 & 0 \\ 1 & 1 \end{pmatrix}\bar{Y}(x)$$

yields the equation

$$\bar{Y}(qx)=\begin{pmatrix} a_1x/q+b_1 & 0 \\ k(a_1x+b_1) & k(a_1x+b_1) \end{pmatrix}\bar{Y}(x).$$

This is a linear equation, the characteristic constants of which no longer satisfy the condition $\bar{\alpha}_1=\bar{\alpha}_2$, and can therefore be transformed by a constant transformation into the corresponding form (3).

In this paper the case where A and B commute will be omitted for the sake of brevity as it can be shown that in this case the solution of equation (1) can be expressed completely in terms of q-gamma functions of the first order, and that the discussion of the equation is analogous to that of first order equations.

In the following, only equations which can be put in the form (3) by a linear transformation will be considered. For such equations the characteristic constants determine the family of equations related to each other by linear transformations. The matrix $\bar{A}x+\bar{B}$ will be called the *normal form* of the matrix $Ax+B$, and equation (3) will be called the *normal member* of the family of equations to which it belongs.

3. Rational transformations on equations of Type I. If the transformation

(5) $\quad Y(x)=L(x)\bar{Y}(x)$, $L(x)$ a rational matrix, $|L(x)|\not\equiv 0$,

is applied to equation (1) the equation becomes

$$\bar{Y}(qx)=L^{-1}(qx)(Ax+B)L(x)\bar{Y}(x).$$

In general, the matrix of this equation is not linear, and the new equation does not fall into the class under consideration. For certain choices of $L(x)$, however, the matrix $L^{-1}(qx)(Ax+B)L(x)$ is again a linear matrix of the form $Cx+D$.

THEOREM II. *If* $Ax + B$ *is the normal matrix with characteristic constants* $a_1, a_2, b_1, b_2, \alpha_1, \alpha_2$ *and if* $c_1, c_2, d_1, d_2, \beta_1, \beta_2$ *are such that*

(6) $c_1 = q^{n_1}a_1,\ c_2 = q^{n_2}a_2,\ d_1 = q^{m_1}b_1,\ d_2 = q^{m_2}b_2,\ \alpha_2 = q^{k_2}\beta_2,\ d_1d_2 = c_1c_2\beta_1\beta_2$ *where* m_1, m_2, n_1, n_2, k_2 *are arbitrary integers, there exists some non-singular rational matrix* $L(x)$ *such that* $L^{-1}(qx)(Ax + B)L(x)$ *is the normal matrix with characteristic constants* $c_1, c_2, d_1, d_2, \beta_1, \beta_2$.

First consider the cases in which one of the integers m_1, m_2, n_1, n_2, k_2 is ± 1, and the others are all zero. These cases fall into three groups.

(i) $m_1 = m_2 = n_1 = n_2 = 0,\ k_2 = \pm 1$.

If $L(x) = \begin{pmatrix} \dfrac{x/q - \alpha_1}{x - \alpha_2} & 0 \\ 0 & 1 \end{pmatrix}$, $L^{-1}(qx)(Ax + B)L(x)$ is the normal matrix with characteristic constants $a_1, a_2, b_1, b_2, q\alpha_2, \alpha_1/q$, that is $k_2 = 1$. A simple modification of this matrix gives the rational matrix required for the case $k_2 = -1$.

(ii) $m_1 = m_2 = k_2 = 0;\ n_1 = \pm 1,\ n_2 = 0$ or $n_1 = 0,\ n_2 = \pm 1$.

If $L(x) = \begin{pmatrix} -(a_1x + a_2\alpha_1 + b_1 + b_2)/(x - \alpha_1), & a_1 \\ -a_2(x - \alpha_2)/(x - \alpha_1), & a_2 \end{pmatrix}$, and is non-singular, $L^{-1}(qx)(Ax + B)L(x)$ is the normal matrix with characteristic constants $qa_1, a_2, b_1, b_2, \alpha_1/q, \alpha_2$. This is the case $n_1 = 1,\ n_2 = 0$, and an analogous rational matrix can be found for each of the other cases in this group.

(iii) $n_1 = n_2 = k_2 = 0;\ m_1 = \pm 1,\ m_2 = 0$ or $m_1 = 0,\ m_2 = \pm 1$. To obtain characteristic constants $a_1, a_2, qb_1, b_2, \alpha_2, q\alpha_1$, the case $m_1 = 1$,

$$L(x) = \begin{pmatrix} -(x - q\alpha_1)/\alpha_2x, & -a_1(x - q\alpha_1)/(b_2x) \\ a_2(x - q\alpha_1)/b_2x, & -[qb_1\alpha_2 + b_2x + (a_1 + a_2)x_2x]/b_2\alpha_2x \end{pmatrix}$$

and similar matrices can be found for the other cases.

The matrix $L(x)$ in (i) is always non-singular. The matrices $L(x)$ in (ii) and (iii) will be non-singular unless $Ax + B$ is reducible, that is $b_1 = -a_1\alpha_1$ and $b_2 = -a_2\alpha_2$. In this case a satisfactory non-singular matrix $L(x)$ can be found as the product $L_1(x)L_2(x)L_3(x)$ where $L_1(x)$ is chosen so that $L_1^{-1}(qx)(Ax + B)L_1(x) = A_1x + B_1$ has characteristic constants $a_1, a_2, b_1, b_2, \alpha_1/q, \alpha_2q$ and is irreducible; $L_2(x)$ is the appropriate matrix of group (ii) or (iii) for $A_1x + B_1$ so that for the matrix $L_2^{-1}(qx)(A_1x + B_1)L_2(x) = A_2x + B_2$ the integers m_1, m_2, n_1, n_2 have the required values, but $k_2 = 1$; $L_3(x)$ is a matrix of group (i) such that for $L_3^{-1}(qx)(A_2x + B_2)L_3(x)$ the value of k_2 is zero.

It is now clear that the normal matrix with characteristic constants $c_1, c_2, d_1, d_2, \beta_1, \beta_2$ satisfying conditions (6) can be obtained from the normal matrix $Ax + B$ by a finite sequence of transformations of the forms discussed under (i), (ii) and (iii) above. Thus the $L(x)$ of the theorem can be obtained as the product of a finite number of non-singular rational matrices.

It can also be shown that if a linear equation $\bar{Y}(qx) = (Cx + D)\bar{Y}(x)$ can be derived from $Y(qx) = (Ax + B)Y(x)$ by a transformation of the form (5), then the characteristic constants of these equations must satisfy conditions (6).

Theorem III. *If for some $L(x)$, $L^{-1}(qx)(Ax + B)L(x) = Cx + D$, then*

$$(6)\quad c_1 = q^{n_1}a_1,\ c_2 = q^{n_2}a_2,\ d_1 = q^{m_1}b_1,\ d_2 = q^{m_2}b_2,\ \beta_2 = q^{k_2}\alpha_2,\ d_1d_2 = c_1c_2\beta_1\beta_2$$

where m_1, m_2, n_1, n_2, k_2 are integers, $a_1, a_2, b_1, b_2, \alpha_1, \alpha_2$ are the characteristic constants of $Ax + B$ and $c_1, c_2, d_1, d_2, \beta_1, \beta_2$ are the characteristic constants of $Cx + D$.

In order to establish this it is more convenient to write

$$L^{-1}(qx)(Ax + B)L(x) = Cx + D$$

in the form

$$(Ax + B)P(x)r(qx) = P(qx)(Cx + D)r(x),$$

or

$$(7)\quad (Ax + B)\{P_nq^mx^{n+m} + (P_nr_{m-1} + qP_{n-1})q^{m-1}x^{n+m+1} + \cdots + (r_0P_1 + r_1P_0q)x + r_0P_0\}$$
$$= \{P_nq^nx^{n+m} + (P_nr_{m-1}q + P_{n-1})q^{n-1}x^{n+m-1} + \cdots + (r_0qP_1 + r_1P_0)x + r_0P_0\}(Cx + D)$$

where the rational matrix $L(x)$ is replaced by $P(x)/r(x)$, $P(x)$ a polynomial matrix $P_0 + P_1x + P_2x^2 + \cdots + P_nx^n$, and $r(x)$ a polynomial $r_0 + r_1x + \cdots + r_{m-1}x^{m-1} + x^m$.

Let the zeros of $P(x)$ be $p_1, \cdots, p_{2n}$ and the zeros of $r(x)$ be $\gamma_1, \cdots, \gamma_m$. The determinant of the left hand side of (7) is a polynomial with zeros $\alpha_1, \alpha_2, p_1, \cdots, p_{2n}, \gamma_1/q, \cdots, \gamma_m/q$ and the determinant of the right hand side is a polynomial with zeros $\beta_1, \beta_2, p_1/q, \cdots, p_{2n}/q, \gamma_1, \cdots, \gamma_m$. There is no loss of generality if all the constants p_i, γ_i are assumed not equal to zero. These two sets must be identical. That is, α_1 must equal $\beta_1, \beta_2, p_i/q$ for some $i = 1, \cdots, 2n$ or γ_i for some $i = 1, \cdots, m$. If $\alpha_1 = \beta_1$, the required relation between α_1 and β_1 is established. If $\alpha_1 = p_1/q$, then since p_1 cannot equal p_1/q, $p_1 = \beta_1, \beta_2$, or p_j/q for some $j = 2, \cdots, 2n$ or γ_j for some $j = 1, \cdots, m$. If $p_1 = \beta_1$, then $\alpha_1 = \beta_1/q$ and the required relation is established. Continuing in this way it is evident that for some k_1 and k_2, $\alpha_1 = q^{k_1}\beta_1$ and $\alpha_2 = q^{k_2}\beta_2$.

The remaining relationships can be established from the set of matrix equations obtained by equating coefficients of like powers of x in (6), that is

$$Br_0P_0 = r_0P_0D$$

$$B(r_0P_1 + P_0qr_1) + AP_0r_0 = r_0P_0C + (r_0qP_1 + r_1P_0)D$$
$$B(r_1P_2 + P_1qr_2) + A(r_0P_1 + r_1P_0q) = (r_0P_1q + r_1P_0)C + (r_1qP_2 + r_2P_1)D.$$
$$\cdots\cdots$$

It can be assumed that P_0 is not the zero matrix. If $|P_0| \neq 0$, the first equation implies that $b_1 = d_1$, and $b_2 = d_2$. If $|P_0| = 0$, by multiplying each side of equation (7) on the right by a suitable constant matrix E and on the left by its inverse, P_0 can be reduced to one of the forms $\begin{pmatrix} p_0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $p_0 \neq 0$. In the first case the first equation gives $\bar{b}_{11}p_0 = p_0\bar{d}_{11}$, $\bar{b}_{21} = 0$, and $\bar{d}_{12} = 0$, whence $b_1 = d_1$. The second equation now gives $r_0b_2p_{22} = r_0qp_{22}d_2$, that is, if $p_{22} \neq 0$, $b_2 = qd_2$. Otherwise the investigation must be carried further, but a continuation of a similar argument gives the desired result.

According to these results, there is associated with each equation (1) a family of equations, which consists of all those equations which can be obtained from (1) by transformations of the form (4). Such a family of rationally related equations is characterized by a set of characteristic constants $q^{n_1}a_1$, $q^{n_2}a_2$, $q^{m_1}b_1$, $q^{m_2}b_2$, $q^{k_1}\alpha_1$, $q^{k_2}\alpha_2$ where $b_1b_2 = a_1a_2\alpha_1\alpha_2$, and m_1, m_2, n_1, n_2, k_1, k_2 are arbitrary integers subject to the relation $m_1 + m_2 = n_1 + n_2 + k_1 + k_2$.

4. Transformations on the independent variable. Another way in which the equation

$$(1) \qquad Y(qx) = (Ax + B)Y(x)$$

can be transformed into an equation of the same form is by the transformations on the independent variable, $x = kx'$, $k \neq 0$, and $x = 1/x'$. If $x = kx'$, equation (1) becomes

$$Y(qkx') = (Akx' + B)Y(kx') \quad \text{or} \quad Z(qx') = (Akx' + B)Z(x'),$$

where $Z(x') = Y(kx')$. This is an equation with characteristic constants ka_1, ka_2, b_1, b_2, α_1/k, α_2/k.

If $x = 1/x'$ equation (1) becomes $Y(q/x') = (A/x' + B)Y(1/x')$ or $Y(1/x') = (x'/|Ax' + B|)(B'x + A')Y(q/x')$, where B' is the adjoint of B and A' is the adjoint of A. If $Y(q/x') = \Gamma_q\{x'/|Ax' + B|\}Z(x')$ where $\Gamma_q\{x'/|Ax' + B|\}$ is a solution of the first order equation $y(qx')$

$=\{x'/|Ax'+B|\}y(x')$, equation (1) becomes $Z(qx')=(B'x'+A')Z(x')$. This is an equation of the same form as (1), with characteristic constants b_1, b_2, a_1, a_2, $1/\alpha_1$, $1/\alpha_2$. It should be noted that this discussion is not restricted to equations of type I.

5. Equations of Type II. An equation (or matrix) is said to be of type II if $a_1a_2b_1b_2=0$. Equations of this type can be divided into three groups: Type IIA, with $b_1b_2=0$, $a_1a_2\neq 0$, and hence α_1, α_2 are finite and $\alpha_1\alpha_2=0$; Type IIB, $a_1a_2=0$, $b_1b_2\neq 0$, and $\alpha_1\alpha_2=\infty$; Type IIC, $b_1b_2=0$, $a_1a_2=0$, so that $\alpha_1=0$, and $\alpha_2=\infty$. For equations of type IIA it can be shown exactly as in **2** that if A and B do not commute a non-singular matrix C exists such that the transformation (2) applied to the equation puts it in the normal form (3). The form (4) is not possible under the condition of type IIA.

For equations of type IIB a modified form of equations (3) must be used as normal form since some of the elements in the matrix are indeterminate. It is unnecessary to consider separately matrices of type IIB since by a transformation on the independent variable of the form $x=1/x'$ as discussed in the preceding section, such an equation can be transformed into an equation of type IIA.

For equations of type IIC ($a_2=0$, $b_2=0$) the matrix $Ax+B$ can be put into the form $\begin{pmatrix} a_1x+b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ if $a_1\neq 0$, and $|Ax+B|=a_1b_{22}x$. The normal form in this case is $\begin{pmatrix} 0 & a_1x \\ -b_{22} & a_1x+b_1 \end{pmatrix}$. If $a_1=0$, b_1 cannot be zero, and by a transformation of the form $x=1/x'$ this can be transformed into the preceding case.

Families of rationally related equations can be set up for equations of type II as they were in **3**. The second theorem of **3** applies without modification. The first theorem of **3** can be established by exhibiting matrices $L(x)$ similar to those for type I. Since multiplying the characteristic constants by powers of q cannot affect the type of the equation, rational relations can exist only between equations of the same type.

B: Relations Between Solutions.

6. Fundamental solutions. The matrix equation (3) is equivalent to the second order equation with linear coefficients

$$(8)\quad a_1(x-\alpha_1)y(q^2x)-[(a_1+a_2)x+b_1+b_2]y(qx)+a_2(x-\alpha_2)y(x)=0,$$

in the sense that each can be derived from the other. A matrix solution of

(3) can be built up from an independent set of solutions of (8), that is a pair of solutions $y_1(x)$, $y_2(x)$ such that $y_1(x)y_2(qx)-y_2(x)y_1(qx)\neq 0$. The definition of type given for equations of the form (3) can also be applied to equation (8). The forms of the solutions of (8) and therefore of (3) are essentially different for equations of type I and type II.

Heine[5] showed that the general series

$$\phi(\alpha,\beta,\gamma,q,x)=1+\sum_{\nu=1}^{\infty}\frac{(1-q^{\alpha})(1-q^{\alpha+1})\cdots(1-q^{\alpha+\nu-1})(1-q^{\beta})\cdots(1-q^{\beta+\nu-1})}{(1-q)(1-q^{2})\cdots(1-q^{\nu})(1-q^{\gamma})\cdots(1-q^{\gamma+\nu-1})}x$$

satisfies a particular second order equation. Any second order equation (8) of type I with the additional condition that $a_1\neq q^{n_1}a_2$, $b_1\neq q^{n_2}b_2$; n_1, n_2 integers, can be reduced to this equation. An equation of type I for which $a_1\neq q^{n_1}a_2$, $b_1\neq q^{n_2}b_2$, n_1, n_2 integers, will be said to be of type Ia, otherwise of type Ib. Smith[6] obtained, in terms of Heine series, two fundamental systems of solutions for an equation of type Ia, one valid in the neighborhood of zero, the other valid in the neighborhood of infinity. He also discussed the solutions of an equation (8) of type Ib. His results can be obtained more simply by using the method given by Adams[7] for equations of order n. Using the notation

$$\alpha=\frac{\log a_2-\log b_2}{\log q}+\frac{\log(-\alpha_2)}{\log q},\qquad \beta=\frac{\log a_1-\log b_2}{\log q}+\frac{\log(-\alpha_2)}{\log q}$$

$$\gamma=\frac{\log b_1-\log b_2}{\log q}+1,\qquad t=\frac{\log x}{\log q},$$

the fundamental matrix solutions for equations of type Ia are:

$$\Gamma_q{}^{0}(x;A,B)=E(x)\begin{pmatrix} b_1{}^{t} & 0\\ 0 & b_2{}^{t}\end{pmatrix}\begin{pmatrix} k_1 & 0\\ 0 & k_2\end{pmatrix}e^{\pi i t}$$

and

$$\Gamma_q{}^{\infty}(x;A,B)=G(x)\begin{pmatrix} a_1{}^{t} & 0\\ 0 & a_2{}^{t}\end{pmatrix}\begin{pmatrix} k_1 & 0\\ 0 & k_2\end{pmatrix}.$$

Here k_1, k_2 are arbitrary constants. The matrix $E(x)$ is equal to

$$\Gamma_q{}^{0}(1-x/\alpha_1)$$
$$\times\begin{pmatrix} \phi(\alpha,\beta,\gamma,q,x/\alpha_2), & \phi(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,q,x/\alpha_2)\\ -\dfrac{b_1}{a_1\alpha_1}\,\phi(\alpha,\beta,\gamma,q,qx/\alpha_2), & -\dfrac{b_2}{a_1\alpha_1}\,\phi(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,q,qx/\alpha_2)\end{pmatrix}$$

---

[5] H. E. Heine, *loc. cit.*

[6] E. R. Smith, *loc. cit.*

[7] C. R. Adams, *loc. cit.*

and is a matrix of entire functions. The matrix $G(x)$ is equal to

$$\Gamma_q^\infty(1-\alpha_1/x)$$
$$\times\begin{pmatrix}\phi(\alpha,\alpha-\gamma+1,\beta-\gamma+1,q,\alpha_1 q^2/x), & \phi(\beta,\beta-\gamma+1,\beta-\alpha+1,q,\alpha_1 q^2/x)\\ \phi(\alpha,\alpha-\gamma+1,\beta-\gamma+1,q,\alpha_1 q/x), & \phi(\beta,\beta-\gamma+1,\beta-\alpha+1,q,\alpha_1 q/x)\end{pmatrix}.$$

It is a matrix of meromorphic functions and is analytic in the neighborhood of infinity.

For an equation (3) of type Ib $\Gamma_q^0(x;A,B)$ is as above unless $b_1=q^{n_2}b_2$, in which case $\Gamma_q^0(x;A,B)=b_1^t e^{\pi i t}S_0(x)\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ where $S_0(x)$ is a matrix of power series in x analytic at $x=0$; $\Gamma_q^\infty(x;A,B)$ is as above unless $a_1=q^{n_1}a_2$, when $\Gamma_q^\infty(x;A,B)=a_1^t q^{\frac{1}{2}(t^2-t)}S_\infty(x)\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ where $S_\infty(x)$ is a matrix of power series in $1/x$, analytic at $x=\infty$.

Formal matrix solutions for equations of type II can be obtained by the same method. For type IIA, $\Gamma_q^\infty(x;A,B)=q^{\frac{1}{2}(t^2-t)}G(x)\begin{pmatrix}a_1^t, & 0\\ 0, & a_2^t\end{pmatrix}$ where $G(x)$ is a matrix of power series in $1/x$ which converge everywhere except for $x=0$. The fundamental solutions about zero have the following forms:

1) If $b_2=0, b_1\neq 0, \alpha_1\neq 0: \Gamma_q^0(x;A;B)=S_1(x)\begin{pmatrix}b_1^t & 0\\ 0 & (a_1a_2\alpha_1/b_1)^t q^{\frac{1}{2}(t^2-t)}\end{pmatrix}e^{\pi i t}.$

2) If $b_2=0, b_1\neq 0, \alpha_1=0: \Gamma_q^0(x;A,B)=S_2(x)\begin{pmatrix}b_1^t & 0\\ 0 & (a_1a_2/b_1)^t q^{(t^2-t)}\end{pmatrix}.$

3) If $b_1=0, b_2=0, \alpha_1\neq 0: \Gamma_q^0(x;A,B)=q^{\frac{1}{4}(t^2-t)}S_3(x)\begin{pmatrix}(a_1a_2\alpha_1)^{\frac{1}{2}t} & 0\\ 0 & [-(a_1a_2\alpha_1)^{\frac{1}{2}}]^t\end{pmatrix}e^{\pi i}$

or type IIC, $\Gamma_q^\infty(x;A,B)=S(x)\begin{pmatrix}a_1^t q^{\frac{1}{2}(t^2-t)} & 0\\ 0 & (-b_{22})^t\end{pmatrix}.$

1) If $b_1\neq 0, \Gamma_q^0(x;A,B)=S_4(x)\begin{pmatrix}b_1^t & 0\\ 0 & (-b_{22}a_1/b_1)^t q^{\frac{1}{2}(t^2-t)}\end{pmatrix}.$

2) If $b_1=0, \Gamma_q^0(x;A,B)=S_5(x)\begin{pmatrix}(b_{22}a_1)^{\frac{1}{2}t} & 0\\ 0 & [-(b_{22}a_1)^{\frac{1}{2}}]^t\end{pmatrix}.$

The matrices $S_1(x)$, $S_2(x)$, $S_4(x)$ are matrices of power series in x; the elements of one column are convergent series, those of the other, divergent series. The elements of $S_3(x)$ and $S_5(x)$ are convergent power series in $x^{\frac{1}{2}}$.

If $\bar{Y}(x)$ is any matrix solution of (1) the general solution of (1) is $\bar{Y}(x)P(x)$, where $P(x)$ is a matrix of q-periodic functions, that is, $P(qx)=P(x)$.[8]

---

[8] R. D. Carmichael, "The general theory of linear q-difference equations," *American Journal of Mathematics*, vol. 34 (1912), pp. 147-168.

7. Rational relations between fundamental solutions. The rational transformations (5) by which an equation (3) is transformed into another equation of the same form, set up rational relations between the fundamental solutions of these equations. This is because these transformations have the property stated in the following theorem.

THEOREM IV. *If the transformation* $Y(x) = L(x)\bar{Y}(x)$ *transforms the equation* $Y(qx) = (Ax + B)Y(x)$ *into* $\bar{Y}(qx) = (Cx + D)\bar{Y}(x)$, *then*

$$\Gamma_q{}^0(x; A, B) = L(x)\Gamma_q{}^0(x; C, D)\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

and

$$\Gamma_q{}^\infty(x; A, B) = L(x)\Gamma_q{}^\infty(x; C, D)\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

where r_1, r_2, s_1, s_2 *are constants.*

Since the two equations belong to the same family of rationally related equations, they must be of the same type and their characteristic values must have the property that $a_1 = q^{n_1}c_1$, $a_2 = q^{n_2}c_2$, $b_1 = q^{m_1}d_1$, $b_2 = q^{m_2}d_2$, $\alpha_2 = q^{k_2}\beta_2$, where a_1, a_2, b_1, b_2, α_1, α_2 are the characteristic constants of $Ax + B$, c_1, c_2, d_1, d_2, β_1, β_2 are the characteristic constants of $Cx + D$, and m_1, m_2, n_1, n_2, k_2 are integers. The relations imply that $b_1{}^t = x^{m_1}d_1{}^t$, $b_2{}^t = x^{m_2}d_2{}^t$, $a_1{}^t = x^{n_1}c_1{}^t$, $a_2{}^t = x^{n_2}c_2{}^t$. From the relationship which exists between the two equations, it follows that $\Gamma_q{}^0(x; A, B)$ and $L(x)\Gamma_q{}^0(x; C, D)$ are solutions of the same equation, and therefore $\Gamma_q{}^0(x; A, B) = L(x)\Gamma_q{}^0(x; C, D)P(x)$, where $P(x)$ is a q-periodic matrix. If the equations are of type Ia, this relation is $S(x)\begin{pmatrix} x^{m_1} & 0 \\ 0 & x^{m_2} \end{pmatrix}\begin{pmatrix} d_1{}^t & 0 \\ 0 & d_2{}^t \end{pmatrix} = L(x)\bar{S}(x)\begin{pmatrix} d_1{}^t & 0 \\ 0 & d_2{}^t \end{pmatrix}P(x)$, and therefore $P(x)$ must be a diagonal matrix of constants, $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. The theorem can be established by a similar argument for equations of other types, and for the fundamental solutions about infinity.

The rational relations which exist between solutions of equations of the same family are the only rational relationships which can exist. If $\Gamma_q{}^0(x; A, B) = L(x)\Gamma^0{}_q(x; C, D)$ this value can be substituted for $\Gamma_q{}^0(x; A, B)$ in the relation $\Gamma_q{}^0(qx; A, B) = (Ax + B)\Gamma_q{}^0(x; A, B)$, giving

$$L(qx)\Gamma_q{}^0(qx; C, D) = (Ax + B)L(x)\Gamma_q{}^0(x; C, D)$$

or, since $\Gamma_q{}^0(qx; C, D)\Gamma_q{}^{0-1}(x; C, D) = Cx + D$,

$$L(qx)(Cx + D) = (Ax + B)L(x).$$

Therefore the two equations belong to the same family.

In this way there is set up a characterization of all situations in which a rational relationship exists between two fundamental solutions of a second order equation, where it is assumed that both are about zero, or both about infinity. By looking at individual elements of the matrices involved, interesting rational relationships are obtained which must be satisfied by the series involved in the matrices $S(x;A,B)$ and $S(x;C,D)$. No transcendental term occurs, since the relation between them takes the form $S(x;A,B)=L(x)S(x;C,D)\begin{pmatrix} x^{s_1} & 0 \\ 0 & x^{s_2} \end{pmatrix}$ where s_1, s_2 are integers.

For matrices of type Ia, the series involved are Heine Series and the relations obtained in this way include the known relations satisfied by Heine Series. These were obtained by Heine by formal manipulation suggested by analogy with the similar relationships which exist between hypergeometric series. The approach based on the q-difference equation gives a logical basis for these relationships, and a characterization of all the relations of this form which can exist.

8. The Riemann problem. The q-periodic matrix which has the property that $\Gamma_q{}^0(x;A,B)=\Gamma_q{}^\infty(x;A,B)P(x)$ is called the fundamental periodic matrix for equation (1). For the equation of the first order, $y(qx)=(1-x)y(x)$, the fundamental periodic function has been explicitly determined.[9] This function is

$$p(x)=\Gamma_q{}^0(1-x)/\Gamma_q{}^\infty(1-x)=ce^{-(\eta/2)t^2-\pi i t}\sigma(t)$$

where $\sigma(t)$ is the Weierstrass sigma function belonging to the periods $\omega=1$, $\omega'=2\pi i/\log q$, and $c=(1-1/q)^2(1-1/q^2)^2\cdots$.

If (1) is of type Ia the explicit form of its fundamental matrix can be obtained by making use of the asymptotic properties of the Heine Series.[10] The result, expressed in terms of the fundamental periodic function for equations of order one is $P(x;A,B)=(p(-a_ix/b_j)\cdot c_{ij})$, $(i,j=1,2)$, where

$$c_{11}=\prod_{\nu=0}^{\infty}\frac{(1+a_1\alpha_1q^\nu/b_2)(1+b_1q^{\nu+1}/a_2\alpha_2)}{(1-b_1q^{\nu+1}/b_2)(1-a_1q^\nu/a_2)},\quad c_{21}=\prod_{\nu=0}^{\infty}\frac{(1+a_2\alpha_2q^\nu/b_2)(1+b_1q^{\nu+1}/a_1\alpha_2)}{(1-b_1q^{\nu+1}/b_2)(1-a_2q^\nu/a_1)}$$

$$c_{12}=\prod_{\nu=0}^{\infty}\frac{(1+a_1\alpha_2q^\nu/b_1)(1+b_2q^{\nu+1}/a_2\alpha_2)}{(1-b_2q^{\nu+1}/b_1)(1-a_1q^\nu/a_2)},\quad c_{22}=\prod_{\nu=0}^{\infty}\frac{(1+a_2\alpha_2/b_1q^\nu)(1+b_2q^{\nu+1}/a_1\alpha_2)}{(1-b_2q^{\nu+1}/b_1)(1-a_2q^\nu/a_1)}$$

The Riemann problem has been stated and solved for q-difference equations of order n, with properties corresponding to those of type Ia as defined here.[11] For q-difference equations of the second order the results can be obtained in a more precise form, and without using the limiting process employed in the general case. The method by which this is done is indicated below.

---

[9] Birkhoff, *loc. cit.*, p. 564.

[10] Smith, *loc. cit.*

[11] Birkhoff, *loc. cit.*, pp. 559-568.

Assume that two matrices $Y_0(x)$ and $Y_\infty(x)$ are defined as follows:

$$Y_0(x) = B(x)\begin{pmatrix} b_1{}^t & 0 \\ 0 & b_2{}^t \end{pmatrix} e^{\pi i t}$$

$$Y_\infty(x) = A(x)\begin{pmatrix} a_1{}^t & 0 \\ 0 & a_2{}^t \end{pmatrix} q^{\frac{1}{2}(t^2-t)}$$

where $B(x)$ is analytic throughout the finite plane, $|B(0)| \neq 0$, so that $Y_0(x)$ is analytic for $x \neq 0, \infty$; $A(x)$ is analytic at $x = \infty$, $|A(\infty)| \neq 0$, and $A(x)$ is analytic except for poles throughout the finite plane. Let $P(x)$ be defined by the equation $Y_0(x) = Y_\infty(x)P(x)$. Assume that $P(x)$ has the property that $P(qx) = P(x)$, and that there exists some loop L about $x = 0$ which cuts the spirals $\theta = c + \arg q/\log|q|$ only once, and is such that $Y_\infty(x)$ is analytic outside this loop $(x \neq \infty)$ and $|Y_\infty(x)| \neq 0$ outside this loop. Then $P(x)$ is analytic outside the loop L, and therefore because of the relation $P(qx) = P(x)$, $P(x)$ is analytic throughout the plane, $x \neq 0, \infty$. From the definition of $P(x)$ it follows that

$$p_{ij}(x) = e^{(\log b_j - \log a_i)t} q^{\frac{1}{2}(t^2-t)} e^{-\pi i t} \phi_{ij}(x)$$

where $\phi_{ij}(x)$ is analytic, and therefore $p_{ij}(x) = c_{ij}p(-a_i x/b_j)$ where $p(x)$ is the fundamental periodic function for equations of order one.

Since $Y_0(x)$ is unaltered in form if it is replaced by $Y_0(x)\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ and similarly $Y_\infty(x)$ may be replaced by $Y_\infty(x)\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, the coefficients c_{ij} are not uniquely determined, but can be replaced by $c_j c_{ij}/d_i$. This does not alter the zeros of $|P(x)|$ which are the solutions of the equation

$$\frac{p(-a_1x/b_1)p(-a_2x/b_2)}{p(-a_1x/b_2)p(-a_2x/b_1)} = \frac{c_{12}c_{21}}{c_{11}c_{22}}.$$

The zeros of $|P(x)|$ occur in two sets of the form $q^n\alpha$ $(n = 0, \pm 1, \pm 2, \cdots)$. If α_1 is a member of one set of zeros, a member of the second set is given by $\alpha_2 = b_1b_2/a_1a_2\alpha_1$. This follows from the property $p(x) = p(1/x)$. Thus the equation $|P(x)| = 0$ determines transcendentally two sets $q^n\alpha_1$, and $q^n\alpha_2$, $(n = 0, \pm 1, \pm 2, \cdots)$. These lie on the two spirals

$$S_1\colon\ \theta - \arg\alpha_1 = \frac{\arg q}{\log|q|}(\log r - \log|\alpha_1|)$$

and

$$S_2 : \theta - \arg \alpha_2 = \frac{\arg q}{\log |q|} (\log r - \log |\alpha_2|).$$

If $\log b_1$, $\log b_2$, $\log a_1$, $\log a_2$, and $P(x)$ are assigned in advance, where $P(x)$ is subject to the restrictions $p_{ij}(x) = c_{ij} p(-a_i x/b_j)$, there exists a matrix equation with $Y_0(x)$, $Y_\infty(x)$ as matrix solutions. For any loop about $x = 0$, not passing through $|P(x)| = 0$, which has the property that it cuts the spirals S_1 and S_2 only once, there exist matrices $Y_0(x)$ and $Y_\infty(x)$ with the property that $|Y_0(x)| \neq 0$ within and along the loop, and the elements of $Y_\infty(x)$ are analytic and $|Y_\infty(x)| \neq 0$ without the loop. Matrices having these properties are given by the fundamental matrix solutions of some equation belonging to the family with characteristic constants $q^{n_1}a_1$, $q^{n_2}a_2$, $q^{m_1}b_1$, $q^{m_2}b_2$, $q^{k_1}\alpha_1$, $q^{k_2}\alpha_2$, $m_1 + m_2 = n_1 + n_2 + k_1 + k_2$. If the loop has the additional property that if it cuts the spiral S_1 between the points α_1 and $q\alpha_1$, it cuts S_2 between α_2 and $q\alpha_2$, the matrices having these properties are the fundamental solutions of the equation with characteristic constants a_1, a_2, b_1, b_2, α_1, α_2.

9. Expression of the fundamental solutions as matrix products. The q-gamma matrices $\Gamma_q{}^0(x; A, B)$ and $\Gamma_q{}^\infty(x; A, B)$ which are solutions of equations of type I can be expressed as convergent infinite products of matrices. This infinite product representation is analogous to that for gamma functions of the first order. The method is indicated below for $\Gamma_q{}^0(x; A, B)$, which will be denoted by $\Gamma_0(x)$ for simplicity.

The equation $Y(qx) = (Ax + B)Y(x)$ is equivalent to

$$Y(x) = (Ax/q + B)(Ax/q^2 + B) \cdots (Ax/q^n + B)Y(x/q^n).$$

In particular,

$$\Gamma_0(x) = (Ax/q + B)(Ax/q^2 + B) \cdots (Ax/q^n + B)\Gamma_0(x/q^n).$$

The matrix $\Gamma_0(x) = S_0(x)T_0(x)$, where $S_0(x)$ is a power series and $T_0(x)$ is a matrix the elements of which are transcendental. Replace $\Gamma_0(x/q^n)$ by $S_{0k}(x/q^n)T_0(x/q^n)$ where $S_{0k}(x/q^n)$ is the matrix obtained by including only the first $k + 1$ terms of $S_0(x)$. Consider the limit

$$\lim_{n\to\infty} \{ \prod_{m=1}^{n} (Ax/q^m + B) S_{0k}(x/q^n) T_0(x/q^n) \}. \tag{9}$$

If this limit exists, it is equal to $\Gamma_0(x)$ since $\lim_{n\to\infty} S_0(x/q^n) S_0^{-1}(x/q^n) = I$. The product can be written in the form

$$S_{0k}^{-1}(x) T_0^{-1}(x) \prod_{m=1}^{n} [T_0^{-1}(x/q^{m-1}) S_{0k}^{-1}(x/q^{m-1}) (Ax/q^m + B) S_{0k}(x/q^m) T_0(x/q^m)].$$

Each factor here is $T_0^{-1}(qz)S_{0k}^{-1}(qz)(Az+B)S_{0k}(z)T_0(z)$, $z=x/q^m$, which can be written as $I+z^{k+1}T_0^{-1}(z)\Psi(z)T_0(z)$, $\Psi(z)$ a matrix of power series, since $S_{0k}(z)T(z)$ satisfies the equation up to terms of the k-th degree. Therefore the matrix product for $\Gamma_0(x)$ is

$$\Gamma_0(x)=S_{0k}^{-1}(x)T_0^{-1}(x)\prod_{m=1}^{\infty}[I+(x/q^m)^{k+1}T_0^{-1}(x/q^m)\Psi(x/q^m)T_0(x/q^m)],$$

which will converge uniformly in a region R if k can be chosen so that the series $\sum_{m=1}^{\infty}(x/q^m)^{k+1}T_0^{-1}(x/q^m)\Psi(x/q^m)T_0(x/q^m)$ converges absolutely in R. For x in some region R, and $m>N$ the terms of the matrix $\Psi(x/q^m)$ are less in absolute value than some constant M; therefore the convergence of the series depends on the form of the matrices $T(x)$ and therefore on the type of the equation. If the equation is of type Ia, $T_0(x)=\begin{pmatrix} b_1{}^t & 0 \\ 0 & b_2{}^t \end{pmatrix}$. The general term in the matrix series is then

$$(x/q^m)^{k+1}\begin{pmatrix} \Psi_{11}(x/q^m), & (b_1/b_2)^{\log(x/q^m)/\log q}\Psi_{12}(x/q^m) \\ (b_2/b_1)^{\log(x/q^m)/\log q}\Psi_{21}(x/q^m), & \Psi_{22}(x/q^m) \end{pmatrix}.$$

But $(b_i/b_j)^{\log(x/q^m)/\log q}=(x/q^m)^{(\log b_i-\log b_j)/\log q}$. Thus the series converges if $\sum_{m=1}^{\infty}(x/q^m)^{k+1+(\log b_i-\log b_j)/\log q}$ converges. If b_1, b_2 lie in the circular ring $|q|^{-\frac{1}{2}}<x\leqq|q|^{\frac{1}{2}}$ then $|b_i/b_j|<|q|$ and

$$|q^{-m(k+1+\log(b_i/b_j)/\log q)}|=|q|^{-n(k+1+\log(b_i/b_j)/\log q)}\leqq|q|^{-nk}.$$

Thus in this case the series converges if $k>0$, that is if $k=1$. For any equation of type Ia, there is some $L(x)$ such that a transformation (5) transforms the equation into an equation of the family for which b_1, b_2 lie in the ring $|q|^{-\frac{1}{2}}<x\leqq|q|^{\frac{1}{2}}$. Thus $\Gamma_0(x)$ for any equation of type Ia can be expressed as $L(x)\Pi(x)$ where $\Pi(x)$ is a product of the form (9) with k chosen equal to 1.

For equations of type Ib, k can always be chosen equal to 1.

SMITH COLLEGE,
NORTHAMPTON, MASSACHUSETTS.

A CLASS OF TRANSFORMATION GROUPS IN E_n.*

By DEANE MONTGOMERY[1] and LEO ZIPPIN.

---

1. Compact transformation groups in E_n (euclidean n-space) which have at least one $n-1$ dimensional orbit appear to be of considerable interest, and we study such groups here. It is known [6] that with a single exception all orbits of such a group are $n-1$ dimensional. We show that this one exceptional orbit consists of a single point, that is, that it is a stationary point of the group. Our principal conclusion is that there is a "ray" which forms a cross section for the orbits and that G acts on E_n in a way which is determined by its action on one $n-1$ dimensional orbit and by this ray.

2. We follow the terminology of [4] and [6]. We assume that G is a compact connected topological group satisfying the second countability axiom, and that to every element g of G there is given a homeomorphism $g(x)$ of E_n into itself in such a way that

(1) $g_1[g_2(x)] = (g_1g_2)(x)$
(2) $g(x)$ is simultaneously continuous in g and x
(3) if e is the identity of G, $e(x) = x$ for all x.

We assume furthermore that G is effective which means that two distinct elements of G have associated with them two distinct homeomorphisms.

We shall have occasion to use the decomposition space E^*_n whose points are the orbits of G in E_n. The natural transformation L from E_n to E^*_n is continuous and open. We take it as known [6] that if one orbit is $n-1$ dimensional then every orbit is $n-1$ dimensional except for one orbit of dimension less than $n-1$, which we denote by $G(z)$ where z is some point of this orbit. The space E^*_n is homeomorphic to a closed half line and its end point is $L(z)$. It is not immediately obvious that $G(z)$ is a point but this will be verified in what follows. We also take it as known that G must be a Lie group. This is because G acts effectively and with locally connected orbits on the connected locally Euclidean space $E_n - G(z)$ [4, 6].

3. We begin by proving a lemma on finding cross sections. For this lemma it is not necessary to assume that our decomposition is given by a

---

* Received June 5, 1942.

[1] Guggenheim fellow.

group. In this section, then, we take R to be a compact arc-wise connected metric space which is filled with a continuous collection of compact arc-wise connected sets. In analogy with the situation for groups we denote by $G(x)$ that set of the collection which contains x. The sets of the collection are to be mutually exclusive and by definition the continuity of the collection means that the natural mapping L from R to the decomposition space is continuous and open. We assume that $L(R)$, the decomposition space of R, is homeomorphic to an interval a^*b^* and that the collection enjoys the following property:

(A) For every $\epsilon > 0$ there is a $\delta > 0$ such that if y is in $G(x)$ and $d(x, y)$ is less than δ then x and y may be joined by an arc in $G(x)$ of diameter ϵ.

We have assumed that any two points x and y of R can be joined by an arc and we also assume that this arc can be so chosen that its diameter approaches zero with $d(x, y)$. This assumption is used for the first time in the corollary to Lemma 1.

LEMMA 1. *Assuming the conditions described in the preceding paragraph let p be in $L^{-1}(a^*)$ and q be in $L^{-1}(b^*)$. Then there is a monotonic arc from p to q, that is the homeomorphic image $x(t)$ of an interval such that if $t_1 < t_2$, then $Lx(t_1) \leqq Lx(t_2)$.*

This lemma does not assert that a cross sectioning arc exists but only that there exists an arc which never moves backward. It might of course remain in the same set $G(x)$ throughout an interval of values of t.

In order to prove the lemma we begin by choosing an arc J from p to q. Let $a^*_1, a^*_2, a^*_3, \cdots$ be the interior rational points of the interval a^*b^*. Let p_1 and q_1 be, respectively, the first point and last point in which J meets $L^{-1}(a^*_1)$. Now if p_1 and q_1 are distinct we replace the subarc p_1q_1 of J by an arc (p_1q_1) which lies in the set $L^{-1}(a^*_1)$ and which has diameter less than two times (any fixed multiple greater than one would suffice) the greatest lower bound of the diameters of such arcs. The arc J_1 obtained in this way joins p and q and cuts $L^{-1}(a^*_1)$ in a connected set. If p_1 and q_1 are not distinct we let J_1 be J.

Next let p_2 and q_2 be respectively the first and last points in which J_1 meets $L^{-1}(a^*_2)$. If p_2 and q_2 are distinct we replace the arc p_2q_2 by an arc (p_2q_2) which lies in $L^{-1}(a^*_2)$ and whose diameter is less than two times the greatest lower bound of the diameters of such arcs. We obtain in this way an arc J_2 which joins p and q and meets $L^{-1}(a^*_1)$ and $L^{-1}(a^*_2)$ in connected sets. Proceeding in this way we obtain a sequence of arcs $J_1, J_2, \cdots, J_n, \cdots$ such that J_n meets each of the sets $L^{-1}(a^*_n), L^{-1}(b^*), L^{-1}(a^*_1), \cdots, L^{-1}(a^*_n)$ in a connected set.

The arcs p_nq_n are disjoint subarcs of J and hence the diameter of p_nq_n approaches 0 as n tends to infinity, because an arc can contain at most a finite number of disjoined subarcs whose diameters are greater than a fixed positive number. Because of property (A) and the choice of (p_nq_n) it follows that diameter (p_nq_n) approaches zero.

By definition there is a homeomorphism $h(t)$ taking the interval $(0,1)$ into J. Under this homeomorphism there is a certain interior subinterval I_1 of $(0,1)$ going into the interval p_1q_1. By altering $h(t)$ only on I_1, or not at all if p_1, q_1 are the same point, we obtain a homeomorphism $h_1(t)$ taking (0.1) to J_1. Continuing this process of alterations we obtain a sequence of homeomorphisms $h_1(t), h_2(t), \cdots$ where $h_n(t)$ takes $(0,1)$ to J_n. Because of the remarks in the preceding paragraph the sequence of continuous functions $h_n(t)$ must converge uniformly. Hence they converge to a continuous function $x(t)$. We see also that $x(t)$ is one to one and that it defines a monotonic arc as required in the conclusion of the lemma. Our proof is therefore complete.

COROLLARY. *For each $\epsilon > 0$ there exists a $\delta > 0$ such that if p, q are in R, not necessarily in $L^{-1}(a^*)$ and $L^{-1}(b^*)$, and if $d(p,q) < \delta$, $L(p) \leqq L(q)$, then there exists a monotonic arc from p to q of diameter $< \epsilon$.*

This can be proved by essentially the same argument as the lemma. The first arc J joining p and q can be chosen to have small diameter. By property A the arcs (p_nq_n) can also be chosen with small diameter. Hence the limit arc will have small diameter as we wished to prove.

We shall use the symbol $C(s)$ to denote a finite set of points $a_1, a_2, \cdots, a_k$ in R such that for $i = 1, \cdots, k-1$, we have $L(a_i) < L(a_{i+1})$ and $d(a_i, a_{i+1}) < s$. We speak of this as a chain of mesh less than s joining a_1 and a_k. Note that this finite set is required to be properly monotonic.

LEMMA 2. *If J is a monotonic arc from p to q, $L(p) < L(q)$, and if O is any open set including J then for every $s > 0$ there exists a $C(s)$ joining p and q which is in O.*

We first choose on J a finite set of points $p = b_1, b_2, \cdots, b_k = q$ such that $d(b_i, b_{i+1}) < s/3$, and $L(b_i) \leqq L(b_{i+1})$. With the aid of these points we shall define a chain $C(s) = a_1, \cdots, a_k$. Let $p = a_1$. If $L(b_1) < L(b_2)$ we let $a_2 = b_2$. If not let b_m be the first point b such that $L(a_1) < L(b_m)$. Then $b_1, b_2, \cdots, b_{m-1}$ are all in $G(b_1)$. Now choose $x_2, x_3, \cdots, x_{m-1}$ so that $L(x_2) < L(x_3) \cdots < L(x_{m-1}) < L(b_m)$ and so that the Hausdorff distance from $G(x_2), \cdots, G(x_{m-1})$ to $G(a_1)$ is less than δ where δ is chosen so that the δ neighborhood of J is in O, and so that $\delta < s/3$. We now choose a_i so that a_i is in $G(x_i)$ and so that

$$d(b_i, a_i) < \delta \qquad (i = 2, \cdots, m-1).$$

Choose $a_m = b_m$. The set $a_1, a_2, \cdots, a_m$ is now properly monotonic and has the proper mesh. We continue this process until we come to b_n where b_n is the last b such that $L(b_n) < L(b_k)$. Here we choose a_i's "slanting" from b_n toward b_k entirely analogous to the process already described.

THEOREM 3.1. *Under the hypothesis of Lemma 1 there is an arc from p to q which is a cross section, that is to say an arc from p to q which intersects each set $G(x)$ in one and only one point.*

From the corollary to Lemma 1 and from Lemma 2 it follows that for every positive number ϵ there is a positive number δ such that if p and q are in R, $L(p) < L(q)$, and $d(p, q) < \delta$ then there is for every s a $C(s)$ joining p and q and having diameter less than ϵ.

Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers such that $\epsilon_1 + \epsilon_2 + \cdots$ converges.

Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers such that

3.1. $\delta_i < \epsilon_i/3$.

3.2. If p and q are two points in R such that $L(p) < L(q)$ and $d(p, q) < \delta_i$ then for every s there is a $C(s)$ joining p and q of diameter $< \epsilon_i$.

We choose a $C(\delta_1)$ joining p and q, and let its points be

$$a_{11}, a_{12}, \cdots, a_{1k_1}.$$

Then for each i choose a $C(\delta_2)$ of diameter $< \epsilon_1$ joining a_{1i} to $a_{1\,i+1}$. Adding these points (for all i) to those of $C(\delta_1)$ we obtain a $C(\delta_2)$ joining p and q which has the additional property that the finite set between a_1 and $a_{1\,i+1}$ has diameter $< \epsilon_1$. We shall name the points of the set $C(\delta_2)$ as follows:

$$a_{21}, a_{23}, \cdots, a_{2k_2}.$$

We proceed by induction in a natural manner, the general stage being that we have obtained a $C(\delta_{m+1})$ joining p and q and such that the points between $a_{m\,i}$ and $a_{m\,i+1}$ (of the previous $C(\delta_m)$) have diameter less than ϵ_m. The points of this $C(\delta_{m+1})$ are denoted by

$$a_{m+1\,1}, a_{m+1\,2}, \cdots, a_{m+1\,k_{m+1}}.$$

Let A_m be the points of the $C(\delta_m)$ joining p and q which we find by this process. Clearly A_m is contained in A_{m+1}. We now let A denote the closure of $A_1 + A_2 + \cdots$, and we wish to show that A is an arc forming the desired cross section. In order to prove this it is only necessary to prove that A contains one and only one point on each set $G(x)$.

It is clear that A contains at least one point on each set $G(x)$, that is in each $L^{-1}(x^*)$ for x^* in a^*b^*. Suppose now that y_1 and y_2 are two points of A in a certain set $G(x)$ in R, and let $d(y_1, y_2) = r$. Choose N so large that

$$\epsilon_N + \epsilon_{N+1} + \cdots < r/6.$$

Let a_{Nm_1} be the last point of A_N such that $L(a_{Nm_1})$ precedes $L(x)$ in a^*b^* and let a_{Nm_2} be the first point of A_N such that $L(a_{Nm_2})$ follows $L(x)$ in a^*b^*. Then m_2 is either $m_1 + 1$ or $m_1 + 2$. In any case all points of $A_1 + A_2 + A_3 + \cdots$ between a_{Nm_1} and a_{Nm_2} have a distance from a_{Nm_1} which is at most equal to $2(\epsilon_N + \epsilon_{N+1} + \cdots) < 2r/6$. Therefore these points cannot have both y_1 and y_2 as limit points. Hence A has at most one point on each set $G(x)$ and consequently A is an arc and an arc of the desired kind.

4. We return now to the situation of the introduction and of section **2**, that is to the case where we have a compact connected topological group G acting in E_n and where we assume that some orbit is $n - 1$ dimensional. This will be the hypothesis throughout this entire section and the next. In **2** we have already summarized certain conclusions which we may draw from this hypothesis and we shall now draw further conclusions. If H is any topological group we shall denote by H^1 the identity component of H. The symbol G_x denotes the closed subgroup of G leaving x fixed. From [3] we know that if y is near enough to x then G_y is conjugate to a subgroup of G_x.

THEOREM 4. 1. *If x and y are in $E_n - G(z)$, then $G_x{}^1$ and $G_y{}^1$ are conjugate.*

For all x and y in $E_n - G(z)$ the dimension of G_x equals the dimension of G_y. Hence for all y in some neighborhood of a fixed point x, $G_y{}^1$ is conjugate to $G_x{}^1$ in view of the remark made just before the theorem. Now let p be a definite point of $E_n - G(z)$. We have just shown that the points y such that $G_y{}^1$ is conjugate to $G_p{}^1$ form an open set. But it can also be seen that this set is closed in $E_n - G(z)$ so that it must be all of $E_n - G(z)$ which concludes the proof.

We consider the integral valued function $\alpha(x)$ defined for every x in $E_n - G(z)$ to have as its value the number of components of G_x. This function is constant on each orbit so that we have an associated function $\alpha^*(x^*)$ defined in $E^*_n - z^*$. Both of these functions are lower semi-continuous and this, together with the fact that they are integral valued, implies that they are constant on certain open sets. If $\alpha(x)$ is constant everywhere then G_x and G_y are conjugate for every x and y in $E_n - G(z)$. This may be proved by a method similar to that used in the proof of Theorem 4. 1.

THEOREM 4. 2. *If x and y are in $E_n - G(z)$ then G_x and G_y are conjugate.*

In view of the remark above it is only necessary to prove that $\alpha(x)$ is constant on $E_n - G(z)$. We have already noted that it is constant on certain open subsets of $E_n - G(z)$.

Consider a point of $E^*_n - z^*$ around which $\alpha^*(x^*)$ is constant. If $\alpha^*(x^*)$ is not constant everywhere then as we move along the ray E^*_n in either one direction or the other we must come to a first point b^* where the function has a different value. We assume, to be definite, that b^* is such that for all points x^* of some interval immediately to the left of b^* we have

$$\alpha^*(x^*) = r$$

whereas

$$\alpha^*(b^*) = s.$$

Then $s = tr$ for some positive integer t greater than one so that we may also write

$$\alpha^*(b^*) = tr.$$

This means roughly that the $n - 1$ dimensional manifold $L^{-1}(x^*)$ is wound t times around the $n - 1$ dimensional manifold $L^{-1}(b^*)$. Intuition tells us that such a phenomenon can not occur in E_n and this intuitive conjecture can be verified in several ways. In this connection see Eilenberg [2] where a more general situation of this sort is discussed. We now proceed to carry out the verification in a way reminiscent of Eilenberg's method.

Let b be a point of $L^{-1}(b^*)$ and let V be that part of a neighborhood of b which is inside $G(b)$. We assume V to be so small that for x in V, $\alpha(x) = r$. For any x in V we now define a continuous transformation T_x of $G(x)$ into $G(b)$ as follows: Let g_x be an element of G such that

$$g_x G_x g_x^{-1} \subset G_b$$

and assume (see [3]) that g_x approaches the identity as x approaches b. Then if $y = g(x)$ let

$$T_x(y) = g_x g\, g^{-1}(b).$$

This transformation is a local homeomorphism of order t taking $G(x)$ into $G(b)$. Under it no point moves very far and T_x has degree t for properly chosen orientations of $G(x)$ and $G(b)$.

The point z is inside $G(b)$ and also inside $G(x)$ for x in V. These manifolds are orientable and imbedded topologically in E_n. Hence the order of z with respect to them must have absolute value one. But in view of the results about T_x the order of one of them is t times the order of the other which we see is impossible. Hence $\alpha(x)$ is constant everywhere.

THEOREM 4. 3. *If a^*b^* is any closed interval of $E^*_n - z^*$ then the orbits of $L^{-1}(a^*b^*)$ have property* (A).

Assuming the lemma to be false there exists a positive ϵ and for each n a pair of points x_n and y_n in the same orbit such that $d(x_n, y_n) < 1/n$ and

x_n and y_n cannot be joined by an arc in their orbit of diameter less than ϵ. We may take it that these pairs of points converge to a point p of $L^{-1}(a^*b^*)$. We now define a homeomorphism T_n of $G(x_n)$ into $G(p)$ much as the local homeomorphism T_x was defined above. Choose a g_n such that

$$g_n G_{x_n} g_n^{-1} = G_p$$

and furthermore make this choice so that g_n approaches the identity of G. For any point $y = g(x_n)$ let

$$T_n(y) = g_n g g_n^{-1}(p).$$

That $T_n(z)$ is uniquely defined and a homeomorphism is not difficult to verify. It is a homeomorphism such that

$$d[y, T_n(y)]$$

approaches zero uniformly as n tends toward infinity.

Hence $T_n(x_n)$ and $T_n(y_n)$ may be joined by an arc in $G(p)$ whose diameter approaches zero when n approaches infinity. Then T_n^{-1} will take these arcs to arcs of small diameter in $G(x_n)$ which join x_n and y_n. We have therefore, been led to a contradiction which proves the theorem.

5. If a^*b^* is any closed interval of $E^*_n - z^*$ we have now shown that the orbits filling $L^{-1}(a^*b^*)$ fulfill all the conditions imposed at the beginning of section **3**. Consequently there exists an arc which is a cross section for the orbits of $L^{-1}(a^*b^*)$ and which joins any point of $L^{-1}(a^*)$ with any point of $L^{-1}(b^*)$. If x is any point not in $G(z)$ we may, by a repeated application of the process, construct a ray beginning at x which is a cross section for all orbits not inside $G(x)$. The points on $G(x)$ or outside of $G(x)$ form a set which is homeomorphic to the topological product of $G(x)$ and a ray. Any subset of this set may clearly be deformed in a natural way to a position outside of any preassigned bounded set of E_n. We note that during this deformation the set never comes inside $G(x)$, in particular it never comes into contact with $G(z)$. For this process we may choose x as near to $G(z)$ as we like so long as we choose it outside of $G(z)$.

THEOREM 5. 1. *The orbit $G(z)$ is a point.*

The fact that G is a Lie group (section **2**) implies that if $G(z)$ is not a point it is a manifold of some dimension p, and therefore it carries a cycle modulo 2 of dimension p. This cycle must be linked with a cycle in the complement of $G(z)$ by the Alexander duality theorem. Thus we see that there is a cycle C in $E_n - G(z)$ which does not bound in $E_n - G(z)$. As we have remarked this cycle may be deformed to a position outside any bounded set in E_n, for instance it may be deformed to a position outside some $n-1$ sphere including $G(z)$. In this position the cycle bounds outside of $G(z)$ and hence in its original position it must also have bounded outside of $G(z)$. Hence $G(z)$ is a point, namely the point z.

THEOREM 5.2. *There exists a ray beginning at z which is a cross section for all the orbits in E_n.*

We have already seen that we may construct a ray cross section for the orbit $G(x)$ and all exterior orbits when x is not in $G(z)$. Starting with such a cross section we choose a point x_1 not z but inside $G(x)$. We now add to the first cross section an arc x_1x which cross sections the orbits $G(x_1)$ and $G(x)$ and all orbits in between. Continuing in this way we obtain an increasing sequence of rays whose initial points approach z. This sequence of rays approaches a ray which is a cross section of all the orbits in E_n.

THEOREM 5.3. *If x is not z then any cycle of dimension $< n-1$ in $E_n - G(x)$ bounds in $E_n - G(x)$.*

This is because any cycle in $E_n - G(x)$ may be deformed either to z or toward infinity without touching $G(x)$.

This theorem implies that if x is not z, then $G(x)$ has the same homology properties as an $n-1$ sphere, at least where we use such coefficients in $G(x)$ as for instance the reals mod 1, the integers, or the integers mod m.

Since G_x remains constant up to conjugates we have justified our remark in the introduction that the action of G is determined by its action on one orbit and by the ray cross section.

Our results also enable us to see that for any x the image of a sphere of dimension less than $n-1$ in $G(x)$ can be shrunk to a point in $G(x)$.

INSTITUTE FOR ADVANCED STUDY,
QUEENS COLLEGE.

BIBLIOGRAPHY.

1. Alexandroff and Hopf, *Topologie*, vol. I, Berlin (1935).
2. Eilenberg, " Monotone families of manifolds " (abstract), *Bulletin of the American Mathematical Society*, vol. 47 (1941), p. 53.
3. Montgomery and Zippin, "A theorem on Lie groups," *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 448-452.
4. Montgomery and Zippin, " Topological transformation groups I," *Annals of Mathematics*, vol. 41 (1940), pp. 778-791.
5. Pontrjagin, *Topological Groups*, Princeton (1939).
6. Zippin, " Transformation groups," *Lectures in Topology*, Ann Arbor (1941), pp. 191-221.

ON THE DISTRIBUTION OF VALUES OF TRIGONOMETRIC SUMS WITH LINEARLY INDEPENDENT FREQUENCIES.*

By M. Kac.

1. Let

$$f(t) = \sum_{k=1}^{n} a_k \cos 2\pi\lambda_k t, \tag{1}$$

where the a's are real and the λ's real and linearly independent. Denote by $N_T(a)$ the number of t's in $(-T, T)$ for which $f(t) = a$ and consider the ratio $N_T(a)/2T$. The problem of finding

$$E(a) = \lim_{T\to\infty} \frac{N_T(a)}{2T} \tag{2}$$

was considered by E. R. van Kampen, A. Wintner and the present author[1] in an even more general case than (1). However, with the exception of the case $n = 2$, $a_1 = a_2 = 1$, no explicit formula for $E(a)$ was found and it seemed worthwhile to provide such a formula. In what follows we shall prove that

$$E(a) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi \, \frac{\prod_{k=1}^{n} J_0(|a_k|\,\xi) - \prod_{k=1}^{n} J_0(|a_k|\sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2} \, d\eta\, d\xi. \tag{3}$$

2. Let μ be an arbitrary positive number and put

$$F = F(t; P) = F(t; \phi_1, \cdots, \phi_n) = -a + \sum_{k=1}^{n} a_k \cos 2\pi(\lambda_k t + \phi_k) \quad (0 \leqq \phi_k < 1)$$

$$F' = \partial F/\partial t.$$

Denote by $G_a(\phi_1, \cdots, \phi_n)$ the number of t's in $(0, \mu)$ for which $F = 0$. Then by section 2 of *loc. cit.*[1] if $G_a(\phi_1, \cdots, \phi_n)$ is Riemann integrable the limit (2) exists and

$$E(a) = \frac{1}{\mu} \int_0^1 \cdots \int_0^1 G_a(\phi_1, \cdots, \phi_n)\, d\phi_1 \cdots d\phi_n.^2 \tag{4}$$

We first prove that G_a is Riemann integrable.

---

* Received July 16, 1942.

[1] *American Journal of Mathematics*, vol. 61 (1939), pp. 985-991, in particular, sections 2 and 3. The $(\lambda t + \phi)$'s in formula (12) should be replaced by $2\pi(\lambda t + \phi)$'s.

[2] The right side of (4) does not depend on μ. In *loc. cit.*[1] we put $\mu = 1/\lambda_1$ (assuming $\lambda_1 > 0$) in which case the integral (4) can be reduced to an $(n-1)$-dimensional one.

I. *$G_a(\phi_1, \cdots, \phi_n) = G_a(P)$ $(0 \leq \phi_k < 1)$ is bounded.* If $G_a(P)$ were not bounded we could find such a sequence $\{P_n\}$ of points in the n-dimensional cube that $G(P_n) \to \infty$ as $n \to \infty$. $\{P_n\}$ obviously contains a convergent subsequence which, without loss of generality, may be assumed to be the sequence itself. Let P_0 be the limit of $\{P_n\}$. The functions $F(z; P_n)$ and $F(z; P_0)$ (as functions of the complex variable z) are analytic and $F(z; P_n)$ converges uniformly to $F(z; P_0)$ in every bounded region of the complex z-plane.

Since $F(z; P_0)$ is not identically zero it is a well-known theorem that the number of complex roots of $F(z; P_n)$ in every fixed region approaches the number of complex roots of $F(z; P_0)$ in the same region. This will contradict our assumption that $G_a(P_n) \to \infty$, if we fix the region in such a way as to contain the interval $(0, \mu)$.

II. *The set of points of discontinuity of $G_a(P)$ has n-dimensional Lebesgue measure 0.* It is easily seen that if P_1 is such that $F(t; P_1)$ $(0 \leq t \leq \mu)$ has single roots only, $G_a(P)$ is continuous at $P = P_1$. Hence, the set of points of discontinuity of $G_a(P)$ is the set H of those P's for which there exists at least one t $(0 \leq t \leq \mu)$ such that simultaneously

$$F(t; P) = 0 \quad \text{and} \quad F'(t; P) = 0.$$

Let $\epsilon > 0$; notice that

$$| F'(t; P) | = 2\pi \left| \sum_{k=1}^{n} a_k \lambda_k \sin 2\pi(\lambda_k t + \phi_k) \right| \leq 2\pi \sum_{k=1}^{n} | a_k \lambda_k | = M_1$$

$$| F''(t; P) | = 4\pi^2 \left| \sum_{k=1}^{n} a_k \lambda_k^2 \cos 2\pi(\lambda_k t + \phi_k) \right| \leq 4\pi^2 \sum_{k=1}^{n} | a_k | \lambda_k^2 = M_2$$

and put $M = \operatorname{Max}(M_1, M_2)$. Put furthermore $t_j = j\epsilon/M$ $(j = 0, 1, \cdots, s)$, where $s = [1 + (M\mu/\epsilon)]$ and observe that $0 = t_0 < t_1 < t_2 < \cdots < t_s \leq \mu$. Let now E_j be the set of those P's for which

$$| F(t_j; P) | < \epsilon \quad \text{and} \quad | F'(t_j; P) | < \epsilon,$$

and let $E = \sum_{j=0}^{s} E_j$.

I assert that $H \subset E$. In fact, let $P_0 \epsilon H$. Then there exists a τ $(0 \leq \tau \leq \mu)$ such that $F(\tau; P_0) = F'(\tau; P_0) = 0$ and we can find such a j that $t_j < \tau \leq t_{j+1} (t_{s+1} = \mu)$. Hence,

$$| F(t_j; P_0) | = | F(t_j; P_0) - F(\tau; P_0) | \leq M_1 | \tau - t_j | < \epsilon$$

and

$$| F'(t_j; P_0) | = | F'(t_j; P_0) - F'(\tau; P_0) | \leq M_2 | \tau - t_j | < \epsilon.$$

This means that $P_0 \epsilon E_j$ and therefore $P_0 \epsilon E$. Thus $H \subset E$ and

$$\text{meas. } H \leq \sum_{j=1}^{s} \text{meas. } E_j.$$

In order to find meas. E_j let us recall that

$$\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta} \exp\{i(\gamma\xi + \delta\eta)\} \, d\xi d\eta$$

is equal to 1 if both $|\gamma|$ and $|\delta|$ are less than ϵ, 0 if either $|\gamma|$ or $|\delta|$ is greater than ϵ, 1/2 if either $|\gamma| = \epsilon$, $|\delta| < \epsilon$ or $|\gamma| < \epsilon$, $|\delta| = \epsilon$, and 1/4 if $|\gamma| = |\delta| = \epsilon$. This leads to the formula

$$\text{meas. } E_j = \frac{1}{\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta}$$

$$\times \left[\int_0^1 \cdots \int_0^1 \exp\{i(\xi F(t_j; P) + \eta F'(t_j; P))\} d\phi_1 \cdots d\phi_n\right] d\xi d\eta$$ [3]

if one notices that the measure of the set of those P's for which $F(t_j; P) = \epsilon$ and the measure of the set of P's for which $F'(t_j; P) = \epsilon$ are equal to 0.[4]

Now, we note that for every t

$$\int_0^1 \cdots \int_0^1 \exp\{i(\xi F + \eta F')\} d\phi_1 \cdots d\phi_n$$

(5)

$$= \exp(-ia\xi) \prod_{k=1}^{n} \int_0^1 \exp\{i(a_k\xi \cos 2\pi(\lambda_k t + \phi_k) + 2\pi a_k \lambda_k \eta \sin 2\pi(\lambda_k t + \phi_k))\} d\phi_k$$

$$= \exp(-ia\xi) \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2 \lambda_k^2 \eta^2}),$$

and hence

$$\text{meas. } E_j = \frac{1}{\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta}$$

$$\times \exp(-ia\xi) \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2 \lambda_k^2 \eta^2}) \, d\xi d\eta.$$

---

[3] The interchange of the order of integration can be justified in a way exactly analogous to that followed by Kac and Steinhaus in "Sur les fonctions indépendantes III," *Studia Mathematica*, vol. 6 (1936), pp. 93-94.

[4] This statement can be easily proved. It suffices, however, for our purposes to recall that every measurable function $\phi(P)$ has the property that for "nearly all" ϵ's (i.e. except perhaps a denumerable set) the measure of the set of P's for which $\phi(P) = \epsilon$ is equal to zero. In view of this fact we may restrict ourselves to ϵ's not belonging to the exceptional set and save ourselves the trouble of proving that in our particular case the exceptional set is empty.

J_0 denotes here, as usual, the Bessel function of order 0. Since $J_0(|u|) = O(|u|^{-1/2})$ as $|u| \to \infty$ we see that for $n \geq 3$

$$K(\xi, \eta) = \exp(-ia\xi) \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2}) = O(|\xi|^{-3/4} |\eta|^{-3/4})$$

so that $K(\xi, \eta) \in L^{3/2}$. Hence, by applying the Hölder-Riesz inequality we have

$$\text{meas. } E_j \leq \frac{1}{\pi^2} \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left| \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta} \right|^3 d\xi d\eta \right)^{1/3}$$

$$\times \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |K(\xi, \eta)|^{3/2} d\xi d\eta \right)^{2/3} = \frac{A\epsilon^{4/3}}{\pi^2} \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left| \frac{\sin \xi \sin \eta}{\xi\eta} \right|^3 d\xi d\eta \right)^{1/3}$$

$$= B\epsilon^{4/3}.$$

Finally,

$$\text{meas. } H \leq \text{meas. } E \leq \sum_{j=1}^{s} \text{meas. } E_j = Bs\epsilon^{4/3} = B[1 + (M\mu/\epsilon)]\epsilon^{4/3} = O(\epsilon^{1/3})$$

and since ϵ can be made arbitrarily small we have

$$\text{meas. } H = 0.$$

The above reasoning breaks down for $n = 2$. But in this case the proof that meas. $H = 0$ follows from quite elementary geometric considerations. I and II imply, in view of a well-known theorem, the Riemann integrability of G_a.

3. We shall base our future considerations on the following formula

$$G_a(\phi_1, \cdots, \phi_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_0^{\mu} \cos(\xi F) \, |F'| \, dt \right] d\xi. \tag{6}$$

To prove (6) let the α's, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq \mu$, be the turning points of F so that

$$\int_0^{\mu} \cos(\xi F) \, |F'| \, dt = \sum_{j=0}^{r} \pm ([\sin \beta_{j+1}\xi - \sin \beta_j\xi]/\xi), \tag{7}$$

where $\beta_0 = F(0)$, $\beta_{r+1} = F(\mu)$, $\beta_j = F(\alpha_j)$ $(1 \leq j \leq r)$ and a + or — sign is assigned according as $\beta_{j+1} > \beta_j$ or $\beta_{j+1} < \beta_j$. By a well-known formula of Dirichlet

$$\left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin \beta_{j+1}\xi - \sin \beta_j\xi}{\xi} d\xi \right|$$

is equal to 2, 1 or 0 according as $\beta_j\beta_{j+1} < 0$, $\beta_j\beta_{j+1} = 0$ or $\beta_j\beta_{j+1} > 0$.[5] The signs in (6) are such that

$$(8)\quad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^{\mu}\cos(\xi F)\,|\,F'\,|\,dt\right]d\xi = \frac{1}{2}\sum_{j=0}^{r}\left|\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin\beta_{j+1}\xi-\sin\beta_j\xi}{\xi}\,d\xi\right|$$

and the right side is easily seen to be equal to the number of zeros of F in $(0,\mu)$. This proves (6). Let us notice that r (the number of turning points of F in $(0,\mu)$) although depending on $\phi_1,\cdots,\phi_n$, remains bounded. In fact r is the number of zeros of F' in $(0,\mu)$ and the reasoning given in **2** I is applicable to this case. Thus there exists a number C such that

$$(9)\qquad r < C.$$

4.[6] Let

$$G_a^{(m)}(P) = \frac{1}{2\pi}\int_{-m}^{m}\left[\int_0^{\mu}\cos(\xi F)\,|\,F'\,|\,dt\right]d\xi.$$

It is well known that for every m and b

$$\left|\frac{1}{2\pi}\int_{-m}^{m}\frac{\sin b\xi}{\xi}\,d\xi\right| < 1$$

and hence by (7) and (9) we have

$$G_a^{(m)}(P) < C.$$

Now, for every P, $G_a^{(m)}(P) \to G_a(P)$ as $m \to \infty$ and since the $G_a^{(m)}(P)$ are uniformly bounded we have by a well-known theorem that

$$(10)\quad \lim_{m\to\infty}\int_0^1\cdots\int_0^1 G_a^{(m)}(P)\,d\phi_1\cdots d\phi_n = \int_0^1\cdots\int_0^1 G_a(P)\,d\phi_1\cdots d\phi_n.$$

On the other hand

$$\int_0^1\cdots\int_0^1 G_a^{(m)}(P)\,d\phi_1\cdots d\phi_n = \frac{1}{2\pi}\int_{-m}^{m}\left\{\int_0^1\cdots\int_0^1\left[\int_0^{\mu}\cos(\xi F)\,|\,F'\,|\,dt\right]d\phi_1\cdots d\phi_n d\xi\right.$$

---

[5] It is obvious that $\beta_j \neq \beta_{j+1}$ and therefore β_j and β_{j+1} cannot be simultaneously 0. If 0 or μ is a root of $F = 0$ it is counted as $\frac{1}{2}$ in formula (6). Since the set of those P's for which either 0 or μ is a root, is of measure zero the following considerations are not affected by this peculiarity of formula (6).

[6] This section employs exactly the same reasoning as the one in *loc. cit.*[3]

which together with (10) implies that

$$L_a(\xi) = \int_0^1 \cdots \int_0^1 \left[\int_0^\mu \cos(\xi F) \mid F' \mid dt \right] d\phi_1 \cdots d\phi_n$$

is integrable in $(-\infty, \infty)$ and that

$$\int_0^1 \cdots \int_0^1 G_a(P) d\phi_1 \cdots d\phi_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L_a(\xi) d\xi. \tag{11}$$

5. It remains now to compute $L_a(\xi)$. It is well known that for every real γ,

$$\mid \gamma \mid = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos \gamma\eta}{\eta^2} d\eta$$

and therefore

$$\mid F' \mid = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos(\eta F')}{\eta^2} d\eta.$$

Hence

$$L_a(\xi) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{D(\xi, \eta)}{\eta^2} d\eta,$$

where

$$D(\xi, \eta) = \int_0^\mu \int_0^1 \cdots \int_0^1 [\cos(\xi F) - \cos(\eta F) \cos(\eta F')] d\phi_1 \cdots d\phi_n dt$$

and the interchange of the order of integration is justified by the fact that the only infinite integral occurring in the formula is absolutely convergent. Now,

$$\begin{aligned} \cos(\xi F) \cos(\eta F') = {} & \tfrac{1}{4} \exp\{i(\xi F + \eta F')\} + \tfrac{1}{4} \exp\{i(\xi F - \eta F')\} \\ & + \tfrac{1}{4} \exp\{i(-\xi F + \eta F')\} + \tfrac{1}{4} \exp\{i(-\xi F - \eta F')\} \end{aligned}$$

and it follows (see (5)) that

$$\begin{aligned} \int_0^1 \cdots \int_0^1 & \cos(\xi F) \cos(\eta F') d\phi_1 \cdots d\phi_n \\ & = \cos a\xi \prod_{k=1}^{n} J_0(\mid a_k \mid \sqrt{\xi^2 + 4\pi^2 \lambda_k^2 \eta^2}). \end{aligned}$$

Putting $\eta = 0$ we get

$$\int_0^1 \cdots \int_0^1 \cos(\xi F) d\phi_1 \cdots d\phi_n = \cos a\xi \prod_{k=1}^{n} J_0(\mid a_k \mid \xi).$$

Thus

$$L_a(\xi) = \frac{\mu \cos a\xi}{\pi} \int_{+\infty}^{-\infty} \frac{\prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k|\sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2}\, d\eta$$

and finally by using (4) and (11)

$$E(a) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi \, \frac{\prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k|\sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2}\, d\eta d\xi.$$

6. Since $J_0(|u|) = O(|u|^{-1/2})$ as $|u| \to \infty$ we see that for $n \geqq 5$ the integrand of (12) is absolutely integrable. It may also be noted that if $a_1 = a_2 = \cdots a_n = 1$ a simple asymptotic formula can be obtained if the λ's satisfy certain conditions. In fact, we have in this case

$$E(a) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi \, \frac{J_0^n(\xi) - \prod_{k=1}^{n} J_0(\sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2}\, d\eta d\xi$$

and letting $\xi = u/\sqrt{n}$, $\eta = v/2\pi\sqrt{n}$, $a = b\sqrt{n}$ we obtain

$$E(b\sqrt{n}) = \frac{1}{\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos bv \, \frac{J_0^n(u/\sqrt{n}) - \prod_{k=1}^{n} J_0(\sqrt{u^2 + \lambda_k^2 v^2/n})}{v^2}\, dv du.$$

Assuming now that

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \lambda_k^2 = \lambda, \qquad \lim_{n\to\infty} n^{-2} \sum_{k=1}^{n} \lambda_k^4 = 0$$

we have

$$J_0^n(u/\sqrt{n}) \to \exp(-u^2/2)$$

$$\prod_{k=1}^{n} J_0(\sqrt{u^2 + \lambda_k^2 v^2/n}) \to \exp(-u^2/2)\exp(-\lambda v^2/2)$$

and the convergence is uniform in every bounded region of the (u, v) plane. Thus, since all the integrals involved are absolutely convergent,

$$E(b\sqrt{n}) \sim \frac{1}{\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos bu \exp(-u^2/2)\, \frac{1 - \exp(-\lambda v^2/2)}{v^2}\, dv du.$$

The last integral can be easily evaluated and we obtain the formula

$$E(b\sqrt{n}) \sim \frac{2}{\pi} \sqrt{\lambda} \exp(-b^2/2).$$

CORNELL UNIVERSITY,
ITHACA, N. Y.

ON DOUBLE STURM-LIOUVILLE SERIES.*

By JOSEPHINE MITCHELL.

Introduction. In this paper we consider the *double Sturm-Liouville series* (SL series) of a function, Lebesgue integrable [1] on the square Q $(0 \leqq x \leqq \pi,$ $0 \leqq y \leqq \pi)$, that is, the Fourier development of this function with respect to the double orthonormal system, $\{\phi_m(x)\phi_n(y)\}$, $(m, n = 0, 1, 2, \cdots)$ (1). Our main method is the comparison of the double SL series of a given function $f(x, y)$ with its Fourier cosine-cosine series (FCC series). A. Haar [9] introduced this idea in the one-dimensional case, proving that the difference of the partial sums of the SL and Fourier cosine developments of an integrable function $f(x)$ converges to 0 uniformly on $(0, \pi)$; briefly, the two series are said to be uniformly *equiconvergent*.[2] The proof depends mainly upon the fact that the difference between the SL and cosine kernels (the *difference kernel*) is bounded [9].

It is natural to generalize the concepts of "equiconvergence" and "equisummability" to the double SL and FCC series. However the "cross product" terms of the corresponding difference kernels are *not bounded* (as Haar has pointed out), so that in the case of equiconvergence we have found it necessary to put some restriction on the given function to be expanded.

In discussing the *equiconvergence* of the double SL and FCC series (Part III) we assume that the given function is of bounded variation (Tonelli) ([17], p. 443) and obtain *equiconvergence of the SL and FCC series over the square Q* (Theorem III). Since this condition is included in all of the older sufficient conditions for the convergence of the double Fourier series as well as in some of the more modern ones,[3] we get various *sufficient conditions for the convergence of the double SL series* (Theorem IV).

---

* Received July 1, 1942; Revised January 18, 1943.

[1] In this paper integration is Lebesgue integration. Also, a measurable function f is said to be of *class* L^p if $|f|^p$ is integrable.

[2] We may consider instead of convergence some of the various methods of summation of series and define *equisummability* of two series.

[3] Well-known sufficient conditions for the convergence of the double Fourier series are, e. g., that the function be of bounded variation (Hardy) [10] or of bounded variation (Arzela) [7]. Other sufficient conditions are given by L. Tonelli [17] and in generalizations to two variables of the tests of Dini, de la Vallée Poussin, Lebesgue, etc. For a statement of the tests and the connections among them see J. J. Gergen's paper [8].

Part II contains our principal theorem (Theorem I) on the (C, 1, 1) *equisummability of the double SL and FCC series,* namely that the *SL series of an integrable function* $f(x,y)$ *is* $(C,1,1)$ *equisummable with its FCC series at every point* (x,y) *in* Q *for which*

$$\text{(A)}\quad \frac{1}{hk}\int_0^h\int_0^k |f(x+u,y+v)|\,du\,dv \leqq C(x,y) < \infty, \quad (0<|h|\leqq\pi, 0<|k|\leqq\pi)$$

We also prove that the two series are *equisummable almost everywhere* (a. e.) *in* Q (Theorem II).

To get a *summability* theorem for double SL series we consider the corresponding theory for the summability of the double FCC series. In 1918 H. Geiringer ([7], also [17]) following the method of proof introduced by Lebesgue, proved that the Fourier series of $f(x,y)$ is (C, 1, 1) summable to $f(x,y)$ at every point (x,y) where

$$\text{(C)}\qquad \lim_{h\to 0,\, k\to 0}\frac{1}{hk}\int_0^h\int_0^k |f(x+u,y+v)-f(x,y)|\,du\,dv = 0,$$

if, also, the two simple integrals of $|f|$ are bounded. A. Zygmund [19] proved the summability at all points (x,y) for which (C) holds and

$$\text{(D)}\qquad \frac{1}{hk}\int_0^h\int_0^k |f(x+u,y+v)-f(x,y)|\,du\,dv < D(x,y) < \infty,$$
$$(0<|h|\leqq\pi,\ 0<|k|\leqq\pi).$$

Relating these conditions to the problem of "strong derivability" of double integrals, which has been thoroughly investigated in the last few years by S. Saks, A. Zygmund and others [19], he showed that (C) and (D) hold a. e. for a function of class L^p $(p>1)$ and that therefore the Fourier series of such a function is summable a. e. A more recent investigation by B. Jessen, J. Marcinkiewicz and A. Zygmund [16] has extended this result to a class of functions $f(x,y)$ such that $f\log^+|f|$ (cf. footnote [11]) is integrable (denoted by class (B)). By showing that for such functions condition (A) holds a. e.[4] we deduce that the *SL series of a function* $f(x,y)$ *of class* (B) *is* (C, 1, 1) *summable to* $f(x,y)$ a. e.

Finally in part IV we consider a *modification of the Abel-Poisson summation method.* Instead of multiplying the n-th term of the Fourier development, $\sum_{n=0}^{\infty} c_n v_n(\theta)$, of a given function $f(\theta)$, defined on $(-\pi,\pi)$, by r^n, we form the sequence $\{v_n(\theta,r)\}$, $(n=0,1,2,\cdots)$, where $v_n(\theta,r)$ is *har-*

---

[4] For a discussion of the relation between functions of class (B) and conditions (C), (D) and (A) see 4.

monic in the interior of the unit circle and reduces to $v_n(\theta)$ *on the circumference and consider the limit of the series* $\sum_{n=0}^{\infty} c_n v_n(\theta; r)$ *as* r *approaches* $1-0$ (*modified Poisson method of summability*) (**6.** 1). For ordinary Fourier series this reduces to the usual Abel-Poisson summability. This procedure, which can be generalized in various ways to more variables, was introduced by S. Bergman ([4], [5]) in a much more general connection in the theory of functions of two complex variables. Applying this summation to the SL system we show that the *double SL series of an integrable function is equisummable with its FCC series* (Theorem V). Lastly we state a theorem on modified Poisson summability for the development of an integrable function with respect to more general orthonormal systems (Theorem VI).

PART I. The Sturm-Liouville System

1. Definitions and theorems.

1. The SL system ([9], [18]). Consider the *SL orthonormal system* $\{\phi_n(x)\}$, $(n=0,1,2,\cdots)$, defined on $(0 \leqq x \leqq \pi)$. The $\phi_n(x)$ are the characteristic functions of the *differential system*

$$(1)\quad d^2u/dx^2 + (Q(x)+\lambda)u = 0$$

$$du/dx - hu = 0 \text{ for } x=0,\ du/dx + Hu = 0 \text{ for } x=\pi\,(h, H \text{ constants}),$$

where λ is a parameter $(-\infty < \lambda < \infty)$ and $Q(x)$ is continuous. If, also, $Q(x)$ is of bounded variation, Liouville, and Hobson [12], have proved the *asymptotic formula*

$$(2)\quad \phi_n(x) = (2/\pi)^{1/2}\cos nx + \beta(x)\frac{\sin nx}{n} + \frac{p_n(x)}{(n+1)^2}, \quad (n=0,1,2,\cdots),$$

where the function $\beta(x)$ has a continuous derivative, the continuous functions $p_n(x)$ are bounded in absolute value independently of n and x, and for $n=0$ the second term on the right is zero.

Now let $f(x)$ be a function, integrable on $(0,\pi)$. The development of $f(x)$ by the SL system

$$(3)\quad \sum_{n=0}^{\infty} a_n\phi_n(x), \qquad a_n = \int_0^{\pi} f(s)\phi_n(s)\,ds, \qquad (n=0,1,2,\cdots),$$

is called the *SL series* of $f(x)$.

It has been proved that the SL system is *complete with respect to integrable functions* [18].

Using the system $\{\phi_m(x)\}$, we can form the *double SL orthonormal system* $\{\phi_m(x)\phi_n(y)\}$, $(m, n = 0, 1, 2, \cdots)$, where (x, y) is a point in $Q(0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \pi)$.

Now let $f(x, y)$ be an integrable function in Q.[5] Its *double SL series* is

$$(4) \qquad \sum_{m=0,\, n=0}^{\infty} a_{mn}\phi_m(x)\phi_n(y) \text{ [6]}$$

where

$$(5) \qquad a_{mn} = \int_0^{\pi}\int_0^{\pi} f(s, t)\phi_m(s)\phi_n(t)\,ds\,dt, \qquad (m, n = 0, 1, 2, \cdots).$$

In this paper we compare the *double SL series of $f(x, y)$ with its double FCC series*

$$(6) \qquad \sum_{m=0,\, n=0}^{\infty} \lambda_{mn} b_{mn} \cos mx \cos ny$$

$\lambda_{mn} = \frac{1}{4}$ if $m = n = 0$, $= \frac{1}{2}$ if $m = 0$, $n > 0$ or $m > 0$, $n = 0$, $= 1$ if $m > 0$, $n > 0$ where

$$(7) \qquad b_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(s, t)\cos ms \cos nt\, dsdt, \qquad (m, n = 0, 1, 2, \cdots).$$

2. Equiconvergence and (C, 1) equisummability for $f(x)$.

Denote the *partial sums of the SL and cosine kernels,* respectively, by

$$(8) \qquad K_n(x, s) = \sum_{\nu=0}^{n} \phi_\nu(x)\phi_\nu(s),$$

and

$$(9) \qquad C_n(x, s) = (2/\pi)\left(\tfrac{1}{2} + \sum_{\nu=1}^{n} \cos \nu x \cos \nu s\right),$$

and define the "*difference kernel,*" $\Phi_n(x, s)$, by

$$(10) \qquad \Phi_n(x, s) = K_n(x, s) - C_n(x, s).$$

Similarly denote the corresponding *arithmetic means* ((C, 1) means) by

$$(11) \qquad K^*_n(x, s) = \frac{K_0(x, s) + \cdots + K_{n-1}(x, s)}{n}.$$

and

$$(12) \qquad C^*_n(x, s) = \frac{C_0(x, s) + \cdots + C_{n-1}(x, s)}{n}$$

and define the "(C, 1) *difference kernel,*" $\Phi^*_n(x, s)$, as in (10).

---

[5] Whenever we consider $f(x, y)$ beyond the square Q, we extend it as an even-even function of period 2π in each of the variables x and y.

[6] The *completeness of the double SL system* with respect to integrable functions is an immediate consequence of the corresponding property for the simple system ([20] p. 13, Ex. 6).

Haar's proof of the equiconvergence theorem and the analogous proof for the equisummability theorem involve the *boundedness of the difference kernel* [9], that is,

$$(13) \qquad |\Phi_n(x,s)| \leqq \Phi,$$

where Φ is independent of n, x and s, and

$$(14) \qquad |\Phi^*_n(x,s)| = \frac{|\Phi_0(x,s) + \cdots + \Phi_{n-1}(x,s)|}{n} \leqq \Phi.$$

THEOREM 1[7] (Haar). *The SL series of a function $f(x)$, integrable on $(0,\pi)$, is uniformly equiconvergent with its Fourier cosine (FC) series* [9].

THEOREM 2. *The SL series of a function $f(x)$, integrable on $(0,\pi)$, is uniformly equisummable $(C,1)$ with its FC series.*

2. Two properties of the difference kernels $\Phi_n(x,s)$, $\Phi^*_n(x,s)$. The following lemmas enable us to prove general $(C,1,1)$ equisummability theorems for the double SL and FCC series[8] (cf. Part II).

LEMMA I. *The difference kernel $\Phi_n(x,s)$ converges to 0 uniformly with respect to s as $n \to \infty$ if s is in any closed interval not containing the point x.*

Proof. Let x be an arbitrary fixed point in $(0,\pi)$.

From the asymptotic formula **1** (2) for the SL functions we easily see that the sequence $\Phi_n(x,s)$ converges uniformly for s in any closed interval (a,b) in $(0,\pi)$, not containing the point x, as $n \to \infty$. Hence the limit $\Phi(x,s)$ is a continuous function of s in (a,b).

Further $|\Phi_n(x,s)| \leqq \Phi$ so that

$$(1) \qquad \lim_{n\to\infty} \int_a^b \Phi_n(x,s)\,ds = \int_a^b \Phi(x,s)\,ds.$$

But from Haar's theorem

$$(2) \qquad \lim_{n\to\infty} \int_a^b \Phi_n(x,s)\,ds = 0,$$

for $\int_a^b K_n(x,s)\,ds$ is the partial sum of the SL development of the function equal to 1 in (a,b) and 0 elsewhere, and similarly for $\int_a^b C_n(x,s)\,ds$.

Consequently $\Phi(x,s) = 0$ for all s in $(0,\pi)$ different from x.

---

[7] We indicate known theorems and lemmas by arabic numbers and new ones by Roman numerals.

[8] We intend to use Lemmas I and II again in a later discussion on the equiconvergence of the SL and FCC series.

This *same property* is, of course, valid for the kernel $\Phi^*_n(x,s)$ defined in 1.[9]

LEMMA II.

$$\lim_{n\to\infty}\int_0^\pi |\Phi_n(x,s)|\,ds = 0 \text{ for all } x \text{ in } (0,\pi). \tag{3}$$

Proof. This property follows for each x in $(0,\pi)$ from Lemma I by the Lebesgue theorem on the integration of sequences. The same property holds for the kernel $\Phi^*_n(x,s)$.

COROLLARY. *If* $f(x)$ *is integrable over* $(0,\pi)$, *then for all* x in $(0,\pi)$

$$\lim_{n\to\infty}\int_0^\pi |\Phi_n(x,s)|\,f(s)\,ds = 0. \tag{4}$$

PART II. On the (C, 1, 1) Equisummability of the double SL and double FCC Series.

3. Theorem on Equisummability.

1. THEOREM I. *Let* $f(x,y)$ *be integrable in the square* Q $(0\leqq x\leqq\pi,$ $0\leqq y\leqq\pi)$. *Then the double SL series of* $f(x,y)$ *is* (C, 1, 1) *equisummable with its double FCC series at every point* (x,y) *in* Q *for which*

$$\text{(A)}\quad \frac{1}{hk}\int_0^h\int_0^k |f(x+u,y+v)|\,du\,dv \leqq C,\ (0<|h|\leqq\pi,\ 0<|k|\leqq\pi),$$

where $C = C(x,y) < \infty$.[10]

Proof. Denote the SL and FCC (C, 1, 1) partial sums, respectively, by

$$\sigma_{mn}(x,y) = \int_0^\pi\int_0^\pi K^*_m(x,s)K^*_n(y,t)f(s,t)\,ds\,dt \tag{1}$$

$$\sigma'_{mn}(x,y) = \int_0^\pi\int_0^\pi C^*_m(x,s)C^*_n(y,t)f(s,t)\,ds\,dt. \tag{2}$$

Then $\sigma_{mn}(x,y) - \sigma'_{mn}(x,y)$ equals the sum of the three integrals

$$\Psi^{(1)}_{mn}(x,y) = \int_0^\pi\int_0^\pi \Phi^*_m(x,s)\Phi^*_n(y,t)f(s,t)\,ds\,dt, \tag{3}$$

$$\Psi^{(2)}_{mn}(x,y) = \int_0^\pi\int_0^\pi C^*_m(x,s)\Phi^*_n(y,t)f(s,t)\,ds\,dt, \tag{4}$$

$$\Psi^{(3)}_{mn}(x,y) = \int_0^\pi\int_0^\pi \Phi^*_m(x,s)C^*_n(y,t)f(s,t)\,ds\,dt. \tag{5}$$

---

[9] It is well-known that Lemma I is valid for the Fejér kernel $C^*_n(x,s)$. Thence *it also holds for the SL kernel* $K^*_n(x,s)$. However we do not use this fact.

[10] For a further discussion of condition (A) see 4. 2.

It will, obviously, be sufficient to prove that $|\Psi^{(1)}_{mn}|$, $|\Psi^{(2)}_{mn}|$ and $|\Psi^{(3)}_{mn}| \to 0$ as $m, n \to \infty$.

2. The fact that $|\Psi^{(1)}_{mn}(x,y)| \to 0$ follows at once from the uniform boundednesss of Φ^*_m.

Now

$$C^*_m(x,s) = \frac{1}{m\pi}\left\{\frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)} + \frac{\sin^2\frac{1}{2}m(s+x)}{2\sin^2\frac{1}{2}(s+x)}\right\} \tag{6}$$

so that in discussing $\Psi^{(2)}_{mn}(x,y)$ we consider the two integrals

$$\Psi'_{mn}(x,y) = \frac{1}{m\pi}\int_0^\pi\int_0^\pi \frac{\sin^2\frac{1}{2}m(s+x)}{2\sin^2\frac{1}{2}(s+x)}\,\Phi^*_n(y,t)f(s,t)\,dsdt \tag{7}$$

and

$$\Psi''_{mn}(x,y) = \frac{1}{m\pi}\int_0^\pi\int_0^\pi \frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)}\,\Phi^*_n(y,t)f(s,t)\,dsdt. \tag{8}$$

If $x=0$ or π, $\Psi'_{mn}(x,y)$ reduces to the same value as $\Psi''_{mn}(x,y)$. If x is an interior point, we can choose a $\delta>0$ so that $0 \leqq x-\delta < x+\delta \leqq \pi$ and then

$$|\Psi'_{mn}(x,y)| \leqq \frac{1}{m\pi}\,\frac{1}{2\sin^2\frac{1}{2}\delta}\,\Phi\int_0^\pi\int_0^\pi |f(s,t)|\,dsdt. \tag{9}$$

For fixed δ, a sufficiently large m gives

(10) $|\Psi'_{mn}(x,y)| < \eta$, where η is an arbitrary positive number.

To consider the integral $\Psi''_{mn}(x,y)$ we decompose it into

$$\int_0^\pi\int_0^\pi = \left\{\int_0^x + \int_x^\pi\right\}\left\{\int_0^{y-\epsilon} + \int_{y-\epsilon}^y + \int_y^{y+\epsilon} + \int_{y+\epsilon}^\pi\right\}. \tag{11}$$

Let

$$g(h,k) = \int_0^k |f(x+h,y+v)|\,dv,\quad \theta(h,k) = \int_0^h\int_0^k |f(x+u,y+v)|\,dudv. \tag{12}$$

It is readily shown that

$$\frac{1}{m\pi}\,\frac{\sin^2\frac{1}{2}mu}{\sin^2\frac{1}{2}u} \leqq \frac{\pi m}{(1+\frac{1}{2}mu)^2} \quad \text{for } 0 < u \leqq \pi \text{ (e.g. [17], p. 176).} \tag{13}$$

Then

$$\begin{aligned} |I_1| &= \left|\frac{1}{m\pi}\int_x^\pi\int_y^{y+\epsilon}\frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)}\,\Phi^*_n(y,t)f(s,t)\,dsdt\right| \\ &\leqq \frac{1}{m\pi}\,\Phi\int_0^{\pi-x}\int_0^\epsilon \frac{\sin^2\frac{1}{2}mu}{2\sin^2\frac{1}{2}u}\,|f(x+u,y+v)|\,dudv \\ &\leqq \tfrac{1}{2}\pi\Phi\int_0^{\pi-x}\frac{m}{(1+\frac{1}{2}mu)^2}\,g(u,\epsilon)\,du \\ &\leqq \tfrac{1}{2}\pi\Phi\int_0^\pi \frac{m}{(1+\frac{1}{2}mu)^2}\,g(u,\epsilon)\,du. \end{aligned} \tag{14}$$

Integration by parts of the right hand integral with condition (A) gives us

$$(15)\qquad |I_1| \leqq \tfrac{1}{2}\pi\Phi \left\{ \frac{m\theta(\pi,\epsilon)}{(1+\frac{1}{2}m\pi)^2} + \int_0^\pi \frac{m^2\theta(u,\epsilon)}{(1+\frac{1}{2}mu)^3}du \right\}$$

$$\leqq \tfrac{1}{2}\pi\Phi \left\{ \frac{mC\pi\epsilon}{(1+\frac{1}{2}m\pi)^2} + C\epsilon \int_0^\pi \frac{m^2u}{(1+\frac{1}{2}mu)^3}du \right\}$$

$$\leqq \tfrac{1}{2}\pi\Phi \left\{ \frac{mC\pi\epsilon}{(1+\frac{1}{2}m\pi)^2} + 2C\epsilon \right\}.$$

Secondly let

$$(16)\qquad M_n = M_n(\epsilon) = \max_{y+\epsilon \leqq t \leqq \pi} \Phi^*{}_n(y,t).$$

By Lemma I, $M_n \to 0$ as $n \to \infty$ for fixed ϵ. Then

$$(17)\qquad |I_2| = \left| \frac{1}{m\pi} \int_x^\pi \int_{y+\epsilon}^\pi \frac{\sin^2 \frac{1}{2}m(s-x)}{2\sin^2 \frac{1}{2}(s-x)} \Phi^*{}_n(y,t) f(s,t)\, ds dt \right|$$

$$\leqq \tfrac{1}{2}\pi M_n \int_0^\pi \frac{m}{(1+\frac{1}{2}mu)^2} g(u,\pi)\,du.$$

Again integration by parts with condition (A) gives

$$(18)\qquad |I_2| \leqq \tfrac{1}{2}\pi M_n \left\{ \frac{mC\pi^2}{(1+\frac{1}{2}m\pi)^2} + 2C\pi \right\}.$$

Now it is readily seen that Ψ''_{mn} decomposed as in (11), involves four integrals of type I_1 and four of type I_2, satisfying inequalities like (15) and (18) respectively. From inequalities (15) and (18) and Lemma I, it now follows at once that $\lim_{m,n} \Psi''_{mn}(x,y) = 0$. Consequently

$$(19)\qquad \lim_{m\to\infty,\ n\to\infty} |\Psi^{(2)}_{mn}(x,y)| = 0.$$

Similarly for $|\Psi^{(3)}_{mn}(x,y)|$ and the theorem is proved.

4. Summability for the double SL series and further results on Equisummability.

1. It is easily deduced that condition (A) is satisfied at almost every point (x,y) provided that

(B) $f(x,y) \log^+ |f(x,y)|$ belongs to L over Q[11] ([14], p. 221).

On the other hand it is well-known that the Fourier series of functions satisfying condition (B) are summable (C, 1, 1) to $f(x,y)$ almost everywhere [14].[12] It follows that the *double SL series of the function f satisfying* (B) *is summable* (C, 1, 1) *to* $f(x,y)$ *a. e.*

---

[11] $\text{Log}^+ |f(x,y)|$ is defined as equal to $\log |f(x,y)|$ if $|f(x,y)| > 1$, equal to 0 otherwise (cf. e. g. [20], p. 150). $f \log^+ |f|$ of class L implies f of class L ([11], p. 99).

[12] If $f(x,y)$ is extended beyond Q as an even-even function its double Fourier series reduces to the double FCC series.

We shall prove now the following

THEOREM II. *Let $f(x,y)$ be integrable over the square Q. Then the double SL series of $f(x,y)$ is* (C, 1, 1) *equisummable with its double FCC series at almost every point (x,y) in Q.*[13]

Proof. It is sufficient to prove that $\Psi_{mn}^{(2)}(x,y) \to 0$ a. e. We use the fact that $\int_0^\pi \int_0^\pi C^*_m(x,s) f(s,t)\,ds\,dt$ converges to $\int_0^\pi f(x,t)\,dt$ at all points x where the derivative of $\int_0^h \int_0^\pi | f(x+u,t) - f(x,t) |\,du\,dt$ exists and equals 0 (a. e.).[14]

Now

$$|\Psi_{mn}^{(2)}(x,y)| \leqq |J_1| + |J_2| \tag{1}$$

where

$$\begin{aligned} J_1 &= \int_0^\pi \int_0^\pi C^*_m(x,s)\Phi^*_n(y,t) f(x,t)\,ds\,dt \\ &= \int_0^\pi \Phi^*_n(y,t) f(x,t)\,dt \end{aligned} \tag{2}$$

and

$$J_2 = \int_0^\pi \int_0^\pi C^*_m(x,s)\Phi^*_n(y,t)\,(f(s,t) - f(x,t))\,ds\,dt. \tag{3}$$

By Fubini's theorem $f(x,t)$ is integrable with respect to t for a. a. x. Therefore by Theorem 2 the right side of (2) $\to 0$ as $n \to \infty$ for a. a. x and uniformly in y. Hence $J_1 \to 0$ as $n \to \infty$ for a. a. (x,y) in Q.

Again

$$|J_2| \leqq \Phi \int_0^\pi C^*_m(x,s) g(s)\,ds \tag{4}$$

where $g(s) = \int_0^\pi | f(s,t) - f(x,t) |\,dt$. From the theory for simple Fourier series the second factor on the right of (4) $\to 0$, as $m \to \infty$, at all points x for which the derivative of the indefinite integral of $g(s)$ is 0 (a. a. x). Hence $J_2 \to 0$ as $m \to \infty$ for a. a. (x,y) in Q. Consequently $|\Psi^{(2)'}| \to 0$ as $m, n \to \infty$. Similarly for $|\Psi^{(3)}|$ and the theorem is proved.[15]

2. It has been proved that the double FCC series of an integrable function $f(x,y)$ is (C, 1, 1) summable to $f(x,y)$ at all points for which the two

---

[13] This theorem was suggested by the referee.

[14] The latter statement can be proved by a generalization of the proof of the corresponding one-variable theorem given in ([13], vol. 1, p. 582).

[15] It must be noted that this proof of the fact that $|\Psi_{mn}^{(2)}| \to 0$ does not use Lemma I. However the proof seems to us less straight-forward than that of Theorem I.

conditions (C) and (D) (see Introduction) hold and that for functions of class L^p $(p > 1)$ these conditions hold a. e. [19]. We now point out that even for *functions of class* (B) *condition* (C) *holds a. e.* and that moreover (C) *implies* (D) *a. e.*, thus affording a second proof of the summability theorem for such functions; since obviously (D) *implies* (A) *everywhere* we have a proof of the corresponding summability theorem for double SL series which shows the connection of these different conditions.

This fact follows for *condition* (C) from the recently proved theorem that the *integral of a function f of class* (B) *is "strongly differentiable"* [16] *to $f(x,y)$ a. e.*[17] by Lebesgue's argument ([13] vol. 1, p. 582).

Condition (D) can be proved to result from (C) a. e. as follows. If h, k are sufficiently small $(< \delta_1)$ (D) is given by the statement of condition (C). Also (D) follows readily if $h, k \geqq \delta_1$. If $h < \delta_1$ and $k \geqq \delta_1$, then

$$(5)\qquad \frac{1}{hk}\int_0^h\int_0^k |f(x+u, y+v) - f(x,y)|\, dudv$$
$$\leqq \frac{1}{h\delta_1}\int_0^h\Big(\int_0^\pi |f(x+u, t)|\, dt\Big)du + |f(x,y)|.$$

Now the monotone function $\int_0^x\big(\int_0^\pi |f(s,t)|\, dt\big)ds$ is differentiable to $\int_0^\pi |f(x,t)|\, dt$ at a. e. point x in $(0, \pi)$ (even if $f(s,t)$ belongs to L). Consequently for such x and sufficiently small h $(h < \delta_2)$

$$(6)\qquad \frac{1}{h}\int_0^h\int_0^\pi |f(x+u, t)|\, du\, dt \leqq \int_0^\pi |f(x,t)|\, dt + \epsilon.$$

Hence

$$(7)\qquad \frac{1}{hk}\int_0^h\int_0^k |f(x+u, y+v) - f(x,y)|\, du\, dv$$
$$\leqq \frac{1}{\delta_1}\Big(\int_0^\pi |f(x,t)|\, dt + \epsilon\Big) + |f(x,y)|.$$

A similar result holds if $h \geqq \delta_1$ and $k < \delta_1$. Therefore from the results of the four cases condition (D) follows from (C) a. e.

---

[16] *Strong derivability* of a double integral means that

$$(1/hk)\int_0^h\int_0^k f(x+u, y+v)\, dudv \to f(x,y) \quad \text{as } h, k \to 0.$$

[17] Such theorems have been proved by S. Saks, B. Jessen, J. Marcinkiewicz and A. Zygmund, etc. For the particular statement of the theorem used here see ([16], p. 122).

PART III. On the uniform Equiconvergence of the double SL and FCC Series.

5. Equiconvergence theorem and sufficient conditions for the convergence of the double SL series.

1. In this section we prove a theorem on the equiconvergence of the double SL and FCC series of an integrable function $f(x, y)$. Since the difference kernel is not bounded a proof similar to that of Haar's theorem cannot be generalized to two variables. Further, the Fourier kernel is not positive as in the case for Cesàro summability. We prove equiconvergence for integrable functions $f(x, y)$ which are of *bounded variation* as defined by *Tonelli* (b. v. T.). This condition is only slightly stronger than that of bounded variation in each variable and is by no means sufficient for the convergence of the double Fourier series as L. Tonelli has shown by a counterexample ([17], p. 485).

Definition. *A function $f(x, y)$ is said to be of bounded variation (Tonelli) in Q if*

(i) *$f(x, y)$ is of bounded variation in $(0, \pi)$ as a function of y for almost all x,*

(ii) *$f(x, y)$ is of bounded variation in $(0, \pi)$ as a function of x for a. a. y,*

(iii) *the total variation of $f(x, y)$ with respect to y, namely,*

$$(1)\quad V_1(x) = \sup_{\sigma} \sum_{\nu=1}^{k} | f(x, y_\nu) - f(x, y_{\nu-1}) |, \ \sigma;\ 0 = y_0 < y_1 < \cdots < y_k = \pi,$$

is dominated by an integrable function $U_1(x)$, that is,

$$(2)\qquad V_1(x) \leqq U_1(x),$$

(iv) and similarly *the total variation of $f(x, y)$ with respect to x, namely, $V_2(y)$, is dominated by an integrable function $U_2(y)$* ([17], p. 443).[18]

2. Theorem III. *If the integrable function $f(x, y)$ is of b. v. T. in square Q, then its double SL series is equiconvergent in Q with its double FCC series.*

Proof.[19] In analogy to the notation of 3. 1 call the partial sums of the double SL and FCC series

---

[18] L. Tonelli imposed originally integrability on $V_1(x)$ and $V_2(y)$. The extension considered here is due to C. R. Adams and J. A. Clarkson [2] and J. J. Gergen [8].

[19] The proof of this theorem suggested itself by reason of certain similar ideas used by L. Tonelli in *Serie Trigonometriche* ([17], Ch. IX).

$$(3) \qquad S_{mn}(x,y) = \int_0^\pi \int_0^\pi K_m(x,s)K_n(y,t)f(s,t)\,dsdt,$$

$$(4) \qquad s_{mn}(x,y) = \int_0^\pi \int_0^\pi C_m(x,s)C_n(y,t)f(s,t)\,dsdt,$$

respectively. Also, using as before $\Phi_m(x,s) = K_m(x,s) - C_m(x,s)$, we set

$$(5) \qquad \Theta^{(1)}_{mn}(x,y) = \int_0^\pi \int_0^\pi \Phi_m(x,s)\Phi_n(y,t)f(s,t)\,dsdt,$$

$$(6) \qquad \Theta^{(2)}_{mn}(x,y) = \int_0^\pi \int_0^\pi C_m(x,s)\Phi_n(y,t)f(s,t)\,dsdt,$$

$$(7) \qquad \Theta^{(3)}_{mn}(x,y) = \int_0^\pi \int_0^\pi \Phi_m(x,s)C_n(y,t)f(s,t)\,dsdt,$$

and show that the three integrals $|\Theta^{(1)}_{mn}(x,y)|$, $|\Theta^{(2)}_{mn}(x,y)|$ and $|\Theta^{(3)}_{mn}(x,y)|$ $\to 0$ as $m,n \to \infty$.

As for integral (3) 3 the convergence of $\Theta^{(1)}_{mn}(x,y)$ to 0 follows from the uniform boundedness of $\Phi_m(x,s)$.[20]

To consider $\Theta^{(2)}_{mn}(x,y) = \int_0^\pi \int_0^\pi C_m(x,s)\Phi_n(y,t)f(s,t)\,dsdt$ put

$$(8) \qquad F_n(s) = F_n(s,y) = \int_0^\pi \Phi_n(y,t)f(s,t)\,dt.$$

Then $F_n(s)$ *is of bounded variation with respect to* s *in* $(0,\pi)$ *for all* y *in* $(0,\pi)$ *and* $(n = 0,1,2,\cdots)$.

Proof. Let σ be any subdivision of $(0,\pi)$, σ; $0 = s_0 < s_1 < \cdots < s_k = \pi$. Then

$$(9) \qquad \sum_{\nu=1}^{k} |F_n(s_\nu) - F_n(s_{\nu-1})| = \sum_{\iota=1}^{k} \left| \int_0^\pi \Phi_n(y,t)\,(f(s_\nu,t) - f(s_{\nu-1},t))\,dt \right|$$
$$\leqq \int_0^\pi |\Phi_n(y,t)| \sum_{\nu=1}^{k} |f(s_\nu,t) - f(s_{\nu-1},t)|\,dt.$$

By (1) and (2) in the definition of b. v. T

$$\sum_{\nu=1}^{k} |f(s_\nu,t) - f(s_{\nu-1},t)| \leqq V_2(t) \text{ for a. a. } t \text{ and } V_2(t) \leqq U_2(t)$$

(for all t) so that

$$(10) \qquad \sum_{\nu=1}^{k} |F_n(s_\nu) - F_n(s_{\nu-1})| \leqq \int_0^\pi |\Phi_n(y,t)|\,U_2(t)\,dt.$$

---

[20] By applying a well-known convergence theorem due to E. W. Hobson ([13], vol. 2, pp. 422-4) we see that the convergence of $\Theta_{mn}{}^{(1)}(x,y)$ to 0 is uniform. (See 7.2).

Therefore

$$(11)\qquad V_n(\pi) = \sup_{\sigma} \sum_{\nu=1}^{k} |F_n(s_\nu) - F_n(s_{\nu-1})| \leqq \int_0^\pi |\Phi_n(y,t)|\, U_2(t)\,dt.$$

Hence $F_n(s)$ [21] is of bounded variation and can be represented as $F_n(s) = F_n(0) + P_n(s) - N_n(s)$, where the positive, monotonic non-decreasing functions, $P_n(s)$ and $N_n(s)$, are the positive and negative variations, respectively, of $F_n(s)$ in $(0,s)$.

Then

$$(12)\qquad \Theta_{mn}^{(2)}(x,y) = \int_0^\pi C_m(x,s)\,(F_n(0) + P_n(s) - N_n(s))\,ds$$

and we can use the second mean value theorem getting

$$(13)\qquad \Theta_{mn}^{(2)}(x,y) = F_n(0)\int_0^\pi C_m(x,s)\,ds + P_n(\pi - 0)\int_\xi^\pi C_m(x,s)\,ds$$
$$- N_n(\pi - 0)\int_\eta^\pi C_m(x,s)\,ds,\quad (0 \leqq \xi \leqq \pi,\ 0 \leqq \eta \leqq \pi)$$

([13], vol. 1, p. 568).

Now it is well-known that $\left|\int_a^b C_m(x,s)\,ds\right| \leqq M$, where M is independent of m and x for all a, b such that $0 \leqq a \leqq b \leqq \pi$ ([13], vol. 2, p. 510). Hence

$$(14)\qquad |\Theta_{mn}^{(2)}(x,y)| \leqq M(|F_n(0)| + P_n(\pi - 0) + N_n(\pi - 0))$$
$$= M(|F_n(0)| + V_n(\pi - 0)).$$

Now $|F_n(0)| \leqq |F_n(s_0)| + V_n(\pi)$ where s_0 is such that $f(s_0,t)$ is an integrable function of t (a. a. s_0). Since also $V_n(\pi - 0) \leqq V_n(\pi)$, then from inequalities (14) and (11)

$$(15)\qquad |\Theta_{mn}^{(2)}(x,y)| \leqq M(|F_n(s_0)| + 2V_n(\pi))$$
$$\leqq M\left(\left|\int_0^\pi \Phi_n(y,t) f(s_0,t)\,dt\right| + 2\int_0^\pi |\Phi_n(y,t)|\, U_2(t)\,dt\right).$$

By Haar's theorem and the corollary to Lemma II the right side $\rightarrow 0$ as $n \rightarrow \infty$. Hence $|\Theta_{mn}^{(2)}(x,y)| \rightarrow 0$ as $m, n \rightarrow \infty$ for all (x,y) in Q. Similarly for $|\Theta_{mn}^{(3)}|$ and the theorem is proved.

3. Remark on equiconvergence by rows and columns. A double series $\sum_{m=0,\, n=0}^{\infty} A_{mn}$ is said to be *convergent by rows* if all the series $\sum_{n=0}^{\infty} A_{mn}$

---

[21] $F_n(s)$ may be defined by (8) for only a. a. s. For all exceptional s it may be defined as equal to either $F_n(s + 0)$ or $F_n(s - 0)$.

$(m=0,1,2,\cdots)$ and the series $\sum_{n=0}^{\infty}(\sum_{n=0}^{\infty}A_{mn})$ converge. (Similarly for convergence by columns).

By a method similar to that used in the proof of Theorem III the following equiconvergence theorem can be proved.

THEOREM III′. *If the integrable function $f(x,y)$ is of b.v.T in Q, then its double SL series is equisummable by rows and columns with its double FCC series.*

4. Sufficient conditions for convergence of the double SL series. Any one of the following conditions on the function $f(x,y)$ is sufficient for the convergence of its double FCC series and also implies that the function is of b. v. T. [1] so that Theorem III is applicable.

(i) $f(x,y)$ is of bounded variation (Hardy) ([10], [1]),

(ii) $f(x,y)$ is of bounded variation (Arzela) ([3], [7], p. 114),

(iii) $f(x,y)$ satisfies any one of the many conditions sufficient for the convergence of the double Fourier series given by L. Tonelli ([17], pp. 450-73). In all these conditions he demands that $f(x,y)$ be of b. v. T. plus some other restrictions.

(iv) A condition recently introduced by J. J. Gergen [8] is of the same character.

Hence

THEOREM IV. *The double SL series of an integrable function $f(x,y)$ is convergent at all points (x,y) in Q for which any one of conditions* (i) *to* (iv) *is satisfied. If (x,y) is a point of continuity of $f(x,y)$ the sum of the series is $f(x,y)$. If it is a point of ordinary discontinuity the sum is*

$$\tfrac{1}{4}(f(x+0,y+0)+f(x+0,y-0)+f(x-0,y+0)+f(x-0,y-0)).$$

PART IV. Modified Poisson Summability and its Application to the SL Series.

6. Modified Poisson summability method.

1. Let $f(\theta)$ be integrable on $(-\pi,\pi)$. Applying the well-known *Abel-Poisson summation method* to Fourier series, we multiply its n-th term by r^n, getting the series

$$\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}(b_n\cos n\theta+b'_n\sin n\theta)r^n=f(\theta,r), \tag{1}$$

which is absolutely and uniformly convergent for $(-\pi \leqq \theta \leqq \pi)$ if $(0 \leqq r < 1)$. Then if $f(\theta, r)$ has a limit as $r \to 1-0$, the Fourier series is said to be summable by this method. Fatou has proved that for any integrable $f(\theta)$

$$\lim_{r\to 1-0} f(\theta, r) = f(\theta) \text{ a. e. } [6]. \tag{2}$$

The analogous theorem is not valid for two dimensions. If it is assumed that $f(\theta, \phi) \log^+ |f(\theta, \phi)|$ is integrable, summability a. e. does follow [14]. It has been shown that the method works also for more general orthonormal systems $\{v_n(\theta)\}$, $(n = 0, 1, 2, \cdots)$, if $f(\theta)$ belongs to L^2 and if suitable restrictions are imposed on the $v_n(\theta)$ ([15], pp. 186-94). (The SL system satisfies these restrictions.)

To get more general results the Poisson method may be modified as follows. Let $v_n(\theta, r)$, $(n = 0, 1, 2, \cdots)$, be the function harmonic in the interior of the unit circle and reducing to $v_n(\theta)$ on the circumference, that is,

$$v_n(\theta, r) = \int_{-\pi}^{\pi} P(\theta, r, s) v_n(s)\, ds, \qquad (n = 0, 1, 2, \cdots), \tag{3}$$

where

$$P(\theta, r, s) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - s) + r^2} \tag{4}$$

and

$$\lim_{r\to 1-0} v_n(\theta, r) = v_n(\theta) \text{ a. e. (Fatou).} \tag{5}$$

(For $v_n(s) = \cos ns$, $v_n(\theta, r) = \cos n\theta\, r^n$ and similarly for the sine.) We then form the series

$$\sum_{n=0}^{\infty} c_n v_n(\theta, r), \qquad c_n = \int_{-\pi}^{\pi} v_n(s) f(s)\, ds \qquad (n = 0, 1, 2, \cdots) \tag{6}$$

(the *modified Poisson* ((*m. P.*)) *series of* $f(\theta)$ formed with respect to $\{v_n(\theta)\}$) and consider the limit as $r \to 1-0$. *The orthogonal series* $\sum_{n=0}^{\infty} c_n v_n(\theta)$ *of an integrable function* $f(\theta)$ *will be called summable by the* (*m. P.*) *method if*

$$\lim_{r\to 1-0} \sum_{n=0}^{\infty} c_n v_n(\theta, r) \tag{7}$$

exists. The method reduces to the Abel-Poisson for ordinary Fourier series.

Similarly we may define *summability of the double series* $\sum_{m=0,\, n=0}^{\infty} c_{mn} v_m(\theta) v_n(\phi)$. In this case we consider the limit

$$\lim_{\substack{r\to 1-0\\ \rho\to 1-0}} \sum_{m=0,\, n=0}^{\infty} c_{mn} v_m(\theta, r) v_n(\phi, \rho), \tag{8}$$

where $v_m(\theta, r)$ and $v_n(\phi, \rho)$ are defined by (3). However it must be remarked that our method could be generalized in several ways to the case of two variables (see [4]). Carrying through the generalization for the double SL system by introducing double harmonic functions on a bicylinder we can prove *equisummability* (*m. P.*) *of the double SL and FCC series for an integrable function without any further restriction* (7.2). (For the one variable case the proof is exceedingly simple.) In **7.3** we generalize some of our *results* to *other orthonormal systems.*

2. The modified Poisson method for orthogonal series of functions of L^2 [5].

THEOREM 3. $\sum_{n=0}^{\infty} c_n v_n(\theta, r)$ *converges absolutely and uniformly for all* θ *in* $(-\pi, \pi)$ *and* r *in any closed interval in the interior of* $(0, 1)$.[22]

Proof. By the Schwarz inequality

$$\sum_{n=0}^{\infty} | c_n v_n(\theta, r) | \leqq \Big(\sum_{n=0}^{\infty} c_n^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} v_n^2(\theta, r) \Big)^{\frac{1}{2}}. \tag{9}$$

Then by (3) and Bessel's inequality applied to each factor on the right of (9)

$$\begin{aligned} \sum_{n=0}^{\infty} | c_n v_n(\theta, r) | &\leqq \Big(\int_{-\pi}^{\pi} f^2(s)\, ds \Big)^{\frac{1}{2}} \Big(\int_{-\pi}^{\pi} P^2(\theta, r, s)\, ds \Big)^{\frac{1}{2}} \\ &\leqq \frac{1}{2\pi} \, \frac{1+r}{1-r} \, (2\pi)^{\frac{1}{2}} \Big(\int_{-\pi}^{\pi} f^2(s)\, ds \Big)^{\frac{1}{2}}. \end{aligned} \tag{10}$$

THEOREM 4. *Let* $\{v_n(\theta, r)\}$, $(n = 0, 1, 2, \cdots)$ *be an orthonormal system complete with respect to functions of class* L^2. *Then the orthogonal series of* $f(\theta)$ *with respect to this system is equisummable* (*m. P.*) *with its Fourier series.*

The proof of this theorem, namely, that

$$\lim_{r \to 1-0} \Big| \sum_{n=0}^{\infty} c_n v_n(\theta, r) - f(\theta, r) \Big| = 0, \qquad (-\pi \leqq \theta \leqq \pi), \tag{11}$$

is given in [5] by showing that the (m. P.) series $\sum_{n=0}^{\infty} c_n v_n(\theta, r)$ is equiconvergent with the Poisson integral $f(\theta, r)$ for $(0 \leqq r < 1)$.

7. Application to SL series of functions of class L.

1. Equisummability of the simple SL and FC series. The following theorem is easily proved:

---

[22] This proof of Theorem 3 seems to be more direct for real functions than Bergman's proof.

THEOREM V′. *The (m. P.) series* $\sum_{n=0}^{\infty} a_n\phi_n(\theta, r)$ *of any integrable function* $f(\theta)$ *is uniformly equiconvergent on* $(0, \pi)$ *with its Poisson integral* $(0 \leqq r < 1)$. *Uniform equisummability (m. P.) on* $(0, \pi)$ *of the SL series of f with its FC series follows.*

Proof. Since the SL functions, $\{\phi_n(\theta)\}$, $(n = 0, 1, 2, \cdots)$, are defined on $(0, \pi)$ we consider

$$(1) \qquad \phi_n(\theta, r) = \int_0^{\pi} \bar{P}(\theta, r, s)\phi_n(s)\,ds,$$

where

$$(2) \quad \bar{P}(\theta, r, s) = \frac{1}{2\pi}\left\{\frac{1-r^2}{1-2r\cos(\theta-s)+r^2} + \frac{1-r^2}{1-2r\cos(\theta+s)+r^2}\right\}.^{23}$$

If we suppose that $f(\theta)$ is even, then

$$(3) \qquad f(\theta, r) = \int_0^{\pi} \bar{P}(\theta, r, s)f(s)\,ds \qquad (\text{cf. } \mathbf{6}\ (1))$$

and, in particular,

$$(4) \qquad \cos n\theta\, r^n = \int_0^{\pi} \bar{P}(\theta, r, s)\cos ns\,ds, \qquad 1 = \int_0^{\pi} \bar{P}(\theta, r, s)\,ds.$$

Call

$$(5) \qquad \sigma_n(\theta, r) = \sum_{\nu=0}^{n} a_n\phi_\nu(\theta, r),$$

and

$$(6) \qquad \sigma'_n(\theta, r) = \tfrac{1}{2}b_0 + \sum_{\nu=1}^{n} b_\nu \cos \nu\theta\, r^\nu.$$

We prove that

$$(7) \quad \lim_{n\to\infty} |\sigma_n(\theta, r) - \sigma'_n(\theta, r)| = 0 \text{ for all } \theta \text{ in } (0, \pi) \text{ and } (0 \leqq r < 1).$$

Using $\Phi_n(s, u)$ defined in **1** (10), we get by direct computation

$$(8) \quad |\sigma_n(\theta, r) - \sigma'_n(\theta, r)| = \left|\int_0^{\pi}\int_x^{\pi} \Phi_n(s, u)\bar{P}(\theta, r, s)f(u)\,du\,ds\right|$$

$$= \left|\int_0^{\pi} \bar{P}(\theta, r, s)\left(\int_0^{\pi} \Phi_n(s, u)f(u)\,du\right)ds\right| \text{ (by Fubini)}$$

$$\leqq \int_0^{\pi} |\bar{P}(\theta, r, s)|\left|\int_0^{\pi} \Phi_n(s, u)f(u)\,du\right| ds$$

$$\leqq \frac{1}{\pi}\,\frac{1+r}{1-r}\int_0^{\pi}\left|\int_0^{\pi} \Phi_n(s, u)f(u)\,du\right| ds.$$

---

[23] This corresponds exactly to what we have done for the cosine kernel in **1**.

By Haar's theorem this last expression is less than $\epsilon(1+r)/(1-r)$, for $n > N_\epsilon$, ϵ an arbitrary positive number, which proves the theorem.

2. Equisummability of the double SL and FCC series. We now prove the corresponding theorem for the double SL series of an integrable function $f(\theta, \phi)$.

THEOREM V. *The* (*m. P.*) *series* $\sum_{m,n=0}^{\infty} a_{mn}\phi_m(\theta, r)\phi_n(\phi, \rho)$ *of any integrable function* $f(\theta, \phi)$ *is equiconvergent with its double Poisson integral over* Q $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$. *Equisummability* (*m. P.*) *over* Q *of the double SL series of* f *with its FCC series follows.*

Proof. Corresponding to the notation used in (5) and (6), we let

$$\sigma_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} a_{\nu\mu}\phi_\nu(\theta, r)\phi_\mu(\phi, \rho), \tag{9}$$

and

$$\sigma'_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} \lambda_{\nu\mu} b_{\nu\mu} \cos \nu\theta \cos \mu\phi \, r^\nu \rho^\mu, \tag{10}$$

where $a_{\nu\mu}$, $\lambda_{\nu\mu}$ and $b_{\nu\mu}$ are defined as in **1**, and we prove that

$$\lim_{m\to\infty,\ n\to\infty} | \sigma_{mn}(\theta, \phi; r, \rho) - \sigma'_{mn}(\theta, \phi; r, \rho) | = 0, \tag{11}$$

for all (θ, ϕ) in Q and $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$. Now

$$\begin{aligned} \sigma_{mn}(\theta, \phi; r, \rho) &= \sum_{\nu,\mu=0}^{m,n} a_{\nu\mu} \int_0^\pi \int_0^\pi \bar{P}(\theta, r, s) P(\phi, \rho, t)\phi_\nu(s)\phi_\mu(t)\, dsdt \\ &= \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi K_m(s, u) K_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, dudvdsdt \end{aligned} \tag{12}$$

and similarly

$$\begin{aligned} \sigma'_{mn}(\theta, \phi; r, \rho) &= \sum_{\nu,\mu=0}^{m,n} \lambda_{\nu\mu} b_{\nu\mu} \int_0^\pi \int_0^\pi \bar{P}(\theta, r, s)\bar{P}(\phi, \rho, t) \cos \nu s \cos \mu t \, dsdt \\ &= \int_0^\pi \cdots \int_0^\pi C_m(s, u) C_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, dudvdsdt. \end{aligned} \tag{13}$$

Let us set (in analogy to **3**. 1)

$$\Lambda_{mn}^{(1)}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi \Phi_m(s, u) C_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, dudvdsdt, \tag{14}$$

$$\Lambda_{mn}^{(2)}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi C_m(s, u) \Phi_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, dudvdsdt, \tag{15}$$

$$\Lambda_{mn}^{(3)}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi \Phi_m(s, u) \Phi_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, dudvdsdt. \tag{16}$$

We wish to prove that the three integrals $|\Lambda_{mn}^{(1)}|$, $|\Lambda_{mn}^{(2)}|$ and $|\Lambda_{mn}^{(3)}| \to 0$ as $m, n \to \infty$ for all (θ, ϕ) in Q and $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$.

To consider $\Lambda_{mn}^{(1)}(\theta, \phi; r, \rho)$ proceed as in inequality (8) and then

$$(17)\quad |\Lambda_{mn}^{(1)}(\theta, \phi; r, \rho)|$$
$$\leqq \int_0^\pi \int_0^\pi |\bar{P}(\theta, r, s)|\,|P(\phi, \rho, t)|\,\left|\int_0^\pi \int_0^\pi \Phi_m(s, u)\Phi_n(t, v) f(u, v)\,du\,dv\right|$$
$$\leqq \frac{1}{\pi}\,\frac{1+r}{1-r}\,\frac{1+\rho}{1-\rho}\int_0^\pi \int_0^\pi \left|\int_0^\pi \int_0^\pi \Phi_m(s, u)\Phi_n(t, v) f(u, v)\,du\,dv\right| ds\,dt.$$

But we have proved in **5.2** that the inner integral on the right side converges to 0 uniformly with respect to (s, t) in Q as $m, n \to \infty$, so that

$$(18)\qquad |\Lambda_{mn}^{(1)}(\theta, \phi; r, \rho)| < \frac{1+r}{1-r}\,\frac{1+\rho}{1-\rho}\,\epsilon \text{ for } \begin{matrix} m \\ n \end{matrix} > N_\epsilon,$$

ϵ an arbitrary positive number.

Considering $\Lambda_{mn}^{(2)}(\theta, \phi; r, \rho)$ we have that

$$(19)\quad |\Lambda_{mn}^{(2)}(\theta, \phi; r, \rho)|$$
$$= \left|\int_0^\pi \int_0^\pi f(u, v)\left(\int_0^\pi C_m(s, u)\bar{P}(\theta, r, s)\,ds\right)\left(\int_0^\pi \Phi_n(t, v)\bar{P}(\phi, \rho, t)\,dt\right) du\,dv\right.$$
$$\leqq \int_0^\pi \int_0^\pi |f(u, v)|\,\left|\int_0^\pi C_m(s, u)\bar{P}(\theta, r, s)\,ds\right|\,\left|\int_0^\pi \Phi_n(t, v)\bar{P}(\phi, \rho, t)\,dt\right|$$

Now by formula (4)

$$(20)\qquad \left|\int_0^\pi C_m(s, u)\bar{P}(\theta, r, s)\,ds\right| = \left|\tfrac{1}{2} + \sum_{\nu=1}^{n} \cos \nu u \cos \nu\theta\, r^\nu\right|$$
$$\leqq \frac{1}{1-r}.$$

Secondly by Haar's theorem, for arbitrary positive ϵ,

$$(21)\qquad \left|\int_0^\pi \Phi_n(t, v)\bar{P}(\phi, \rho, t)\,dt\right| < \epsilon \text{ for } n > N = N_\epsilon(\phi, \rho),$$

so that

$$(22)\quad |\Lambda_{mn}^{(2)}(\theta, \phi; r, \rho)| \leqq \frac{1}{1-r}\,\epsilon \int_0^\pi \int_0^\pi |f(u, v)|\,du\,dv, \text{ for } n > N \text{ and all } m$$

Hence

$$(23)\quad \lim_{m\to\infty,\ n\to\infty} |\Lambda_{mn}^{(2)}(\theta, \phi; r, \rho)| = 0, \text{ for all } (\theta, \phi) \text{ and } (0 \leqq r < 1,\ 0 \leqq \rho < 1).$$

Similarly for $|\Lambda_{mn}^{(3)}(\theta, \phi; r, \rho)|$ and the theorem is proved.

3. Remark on the modified Poisson method for more general orthonormal systems. The applicability of this method is not limited to the SL

orthonormal system. We have seen in **6.2** that the (m.P) series of any function of class L^2 with respect to a complete orthonormal system coincides with the Poisson integral of f. If we assume that the function $f(\theta)$ is of class L, we can prove the following theorem.

THEOREM VI. *Let* $\{v_n(\theta)\}$, $(n=0,1,2,\cdots)$, *be a complete orthonormal system and let* $L_n(\theta,s)=\sum_{\nu=0}^{n} v_\nu(\theta)v_\nu(s)$. *If the condition*

$$(24) \qquad \left| \int_{-\pi}^{h} L_n(\theta,s)\,ds \right| \leqq M, \qquad (-\pi \leqq h \leqq \pi),[24]$$

(M independent of h, n and θ), is satisfied, then the (m.P) series of an integrable function $f(\theta)$ is equiconvergent with its Poisson integral and the development of $f(\theta)$ is equisummable (m.P.) with its Fourier series.

The proof of this theorem will be given on another occasion. This theorem can be generalized in various ways to more variables.

BRYN MAWR COLLEGE.

REFERENCES.

[1] C. R. Adams and J. A. Clarkson, "On definitions of bounded variation for functions of two variables," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 824-854.

[2] ———, "Properties of functions $f(x,y)$ of bounded variation," *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 711-730.

[3] C. Arzela, "Sulle funzione di due variabili a variazione limitata," *Rendiconti di Bologna* 1904-5 and 1907.

[4] S. Bergman, "A method for summation of series of orthogonal functions of two variables," *Bulletin of the American Mathematical Society*, vol. 47 (1941), p. 555, Abstract 47-7-302-t.

[5] ———, "A remark on the paper 'Sur les Functions analytiques de deux Variables complexes'," *Journal of Mathematics and Physics (MIT)*, vol. 21 (1942), pp. 141-3.

---

[24] S. Kaczmarz and H. Steinhaus ([15], p. 176), introducing the "Lebesgue functions," $\int_{-\pi}^{\pi} |L_n(\theta,s)|\,ds$, $(n=0,1,2,\cdots)$, have proved that a sufficient condition for the convergence a. e. of the orthogonal development of a function of class L^2 is that these functions be bounded independently of n and θ, which is, of course, much stronger than (24). Similarly for the (C, 1) summability of an orthogonal development they assume for the arithmetic means, $L^*_n(\theta,s)$, that $\int_{-\pi}^{\pi} |L^*_n(\theta,s)|\,ds \leqq M$ ([15], p. 194).

[6] P. Fatou, "Séries Trigonométriques et Séries de Taylor," *Acta Mathematica,* vol. 20 (1906), pp. 335-400.

[7] H. Geiringer, "Trigonometrische Doppelreihen," *Monatshefte für Mathematik und Physik,* vol. 29 (1918), pp. 65-144.

[8] J. J. Gergen, "Convergence criteria for double Fourier Series," *Transactions of the American Mathematical Society,* vol. 35 (1933), pp. 29-63.

[9] A. Haar, "Zur Theorie der orthogonalen Funktionensysteme," *Mathematische Annalen,* vol. 69 (1910), pp. 331-371.

[10] G. H. Hardy, "On double Fourier Series and especially those which represent the double ζ-function with real and incommensurable parameters," *Quarterly Journal of Mathematics,* vol. 37 (1905-6), pp. 53-79.

[11] G. H. Hardy and J. E. Littlewood, "A maximal theorem with function-theoretic applications," *Acta Mathematica,* vol. 54 (1930), pp. 81-116.

[12] E. W. Hobson, "On a general convergence theorem and the theory of the representation of a function by series of normal functions," *Proceedings of the London Mathematical Society,* vol. 11 (1908-9), pp. 349-95.

[13] ———, *The Theory of Functions of a Real Variable,* vol. 1 and 2 (1926).

[14] B. Jessen, J. Marcinkiewicz and A. Zygmund, "Note on the differentiability of multiple integrals," *Fundamenta Mathematicae,* vol. 25 (1935), pp. 217-34.

[15] S. Kaczmarz and H. Steinhaus, "Theorie der Orthogonalreihen," *Monografje Matematyczne,* Tom VI, Warsaw (1935).

[16] J. Marcinkiewicz and A. Zygmund, "On the summability of double Fourier Series," *Fundamenta Mathematicae,* vol. 32 (1939), pp. 122-132.

[17] L. Tonelli, *Serie Trigonometriche,* Bologna (1928).

[18] A. Zygmund, "Sur la théorie riemannienne de certains systèmes orthogonaux," I, *Studia Mathematica,* vol. 2 (1930), pp. 97-170; II, *Prace Mat. Fizyczne,* vol. 39 (1932), pp. 73-117.

[19] ———, "On the differentiability of multiple integrals," *Fundamenta Mathematicae,* vol. 23 (1934), pp. 143-9.

[20] ———, "Trigonometrical series," *Monografje Matematyczne,* Tom V, Warsaw (1935).

LINEAR TOPOLOGICAL SPACES.* [1]

By D. G. Bourgin.

---

1. Introduction.[2] A (real) linear topological space, l. t. s., is a linear Hausdorff space in which the operations of addition and multiplication by a real number are continuous. Axioms sufficient to ensure that a space is a l. t. s. have been given by Kolmogoroff [1] and von Neumann [2] and the relation between these axioms has often been discussed [3(2), 4]. For convenience in reference we state an axiom system which, with the exception of axiom 1, [3], is that given by von Neumann.

We understand by $\mathfrak{N}$ a neighborhood basis $\{N\}$ of the origin, θ, in L. For this neighborhood basis and for *all bases* introduced in this work (including those for the conjugate space as well) the basis at a point not the origin is obtained by a translation. Thus the basis at x_0 is $\mathfrak{N} + x_0$.

I. $\frown \{N \mid N \varepsilon \mathfrak{N}\} = \theta$.

II. If $N_1, N_2 \varepsilon \mathfrak{N}$ there is an $N_3 \varepsilon \mathfrak{N}$ such that $N_3 \subset N_1 \frown N_2$.

III. For any $N \varepsilon \mathfrak{N}$ there is an $N_1 \varepsilon \mathfrak{N}$ such that $\alpha N_1 \subset N$ for all $| \alpha | \leq 1$.

IV. For any $N \varepsilon \mathfrak{N}$ there is an $N_1 \varepsilon \mathfrak{N}$ such that $N_1 + N_1 \subset N$.

V. For arbitrary $x \varepsilon L$ and $N \varepsilon \mathfrak{N}$ there is a real number α such that $x \varepsilon \alpha N$.

The symbol L with null element θ will henceforth denote a l. t. s., i. e., a space satisfying these axioms. If L is *locally convex,* then

VI. $N + N \subset 2N$.

A set $B \subset L$ is bounded if for each $N \epsilon \mathfrak{N}$ there is a real number α such that $B \subset \alpha N$ and L is *locally bounded* if it contains a bounded open set. In view of Axiom III, α may be taken non-negative in Axiom V and in the definition of boundedness.

We denote the closed hull of A by $\bar{A}$, the convex hull by $\hat{A}$, the set of inner points in A by $\underline{A}$ and the complement of A by $\sim\!\! A$. The set of all linear functionals on L is designated by L^* and its hull and general elements are denoted by θ^* and f or F respectively. We write $\bar{L}$ and $\underline{L}$ for the lim sup and

---

* Received October 25, 1941; Revised December 5, 1942.

[1] Presented to the American Mathematical Society, October 25, 1941. Some of the results were announced in D. G. Bourgin, "Some properties of real linear topological spaces," *Proceedings of the National Academy of Sciences*, vol. 27 (1941), pp. 539-544.

[2] Numbers in brackets refer to the bibliography.

lim inf respectively and $\emptyset$ for the empty set. The Euclidean n dimensional space is denoted by R_n.

A brief partial survey of some of the contents follows. Some general results for convex sets are developed first and these in part involve regular L convexity and weak closure properties. For locally convex spaces, regular L convexity is no stronger than closure and convexity. Although usually L is no longer an l. t. s. in its weak topology, we can associate a new space of equivalence classes of L which is a locally convex l. t. s. Most of the conclusions involve specialization of L. Thus if L is locally convex there is a valid analogue of Mazur's Theorem [5] on the equivalence of strong and weak closure of convex sets in Banach spaces, and a result of the type of Helly's Theorem obtains if we require besides that L^* be metrizable. If L^* is normable then L is the union of a denumerable number of multiples of some bounded set. A necessary and sufficient condition for the metrizability of L^* is given in terms of the existence of a certain basis for the bounded sets in L. The sufficiency condition is valid for a general l. t. s. If L is locally bounded (locally convex) its completion is locally bounded (locally convex) and the conjugate spaces of L and its completion are equivalent in the Banach sense. The results of Hyers [3a] are extended and sharpened significantly. Thus it is shown that the quasi norm satisfies conditions similar to those for the spaces $L_p, 0 < p < 1$. The exact conditions depend on the choice of the fundamental bounded open set and in general the quasi norm need not be continuous. It is shown, nevertheless, that there always exists a topologically equivalent continuous quasi norm. The weak topology of the conjugate space of a locally bounded l. t. s. yields many results of the sort familiar for vector normed spaces. For instance regular L^* convexity is equivalent to weak closure and, the unit sphere is bicompact in the weak L^* topology, etc. However, the exact relation between transfinite closure and weak closure is not yet settled in this paper.

2. The general linear topological space.

In the main, the results in this section require no specialization of L and are, besides, of use in later sections. It is convenient to introduce the neighborhood systems

$$\mathfrak{U} = \{U\}, \qquad U = \{\lambda x \mid 0 \leq \lambda \leq 1, x \in N\},$$

and $U^s = U \cup -U$. In referring to neighborhoods in L or L^* we shall omit the basis when the meaning is plain. Thus we write U for $U \in \mathfrak{U}$ and N for $N \in \mathfrak{N}$ etc. We assume throughout the paper that L contains more than one element. It is easy to see that equivalent bases $\mathfrak{N}'$ may be chosen in which the neighborhoods are all open (closed) sets. (In the first case the sets U or U^s are, accordingly, open).

LEMMA 1. *The neighborhood systems* $\mathfrak{U}$, $\mathfrak{U}^s$ *and* $\mathfrak{N}$ *are equivalent neighborhood bases for* L.

It is trivial that an arbitrary U contains an N. On the other hand for arbitrary N and some N_1 we have $\alpha N_1 \subset N$ for $|\alpha| \leq 1$. Hence

$$U_1 = \{\lambda x \mid 0 \leq \lambda \leq 1, x \in N_1\} \subset \{\lambda x \mid -1 \leq \lambda \leq 1, x \in N_1\} \subset N.$$

That $U(U^s)$ satisfies the axioms for an l. t. s. may now be shown directly. If $\mathfrak{N}$ satisfies VI this need not be true of all equivalent bases. A not altogether trivial example is furnished by l_1 which may easily be shown to admit the equivalent metric $(x, \theta) = \log \Pi_1^\infty(1 + |x_i|)$. Here $N' = \{x \mid (x, \theta) < \epsilon\}$ is out of accord with VI. A more natural example is the locally convex space s [8] whose metric spheres are plainly not convex. However the convexifications of the equivalent neighborhoods do satisfy VI of course and are understood in all references to locally convex spaces, below.

LEMMA 2. *The space* L *is completely regular.*

An l. t. s. is evidently a uniform space [6] or struct [7] whence the lemma follows immediately. For the actual applications in this paper merely the weaker, regularity property of L is required, in the main.

Let L_α, $\alpha \in \mathfrak{A}$ be an l. t. s. with neighborhood basis $\mathfrak{N}_\alpha$. The topology of the product space $L^{\mathfrak{A}} = \Pi L_\alpha$ is given by the neighborhoods

$$N = \Pi\{N_i \mid i \in \sigma\} \times \Pi\{L_\beta \mid \beta \in \mathfrak{A} - \sigma\}$$

where $N_i \in \mathfrak{N}_i$ and σ is a finite subset of $\mathfrak{A}$. It is well known that $L^{\mathfrak{A}}$ is an l. t. s. Obviously if each L_α is locally convex, $L^{\mathfrak{A}}$ is locally convex.

LEMMA 3. *If* L_α *is locally bounded then a necessary and sufficient condition for* $L^{\mathfrak{A}}$ *to be locally bounded is that* $\mathfrak{A}$ *is finite.*

Evidently local boundedness implies that for some N, $\{N/m, m = 1, 2 \cdots\}$ constitutes a neighborhood basis. Since $\mathfrak{A} - \sigma$ is non-empty, for non-finite $\mathfrak{A}$, this necessary condition cannot be met. An immediate consequence is that if the spaces L_α are Banach spaces and $\mathfrak{A}$ is non-finite then $L^{\mathfrak{A}}$ is not a Banach space. If $\mathfrak{A}$ is finite then if G_α is open ΠG_α is open in $L^{\mathfrak{A}}$ and if besides G_α contains θ and is bounded then the neighborhood basis in $L^{\mathfrak{A}}$ is given by $\{(1/n)\Pi G_\alpha\}$.

We shall say that U encircles θ if every half ray with θ as an end point has an interval in common with U. That the encirclement properly imposes no restrictions on the l. t. s. is evident from the next lemma.

LEMMA 4. *Every N encircles θ.*

From the definition of U it is patent that if $x_0 \epsilon U$ then the interval from θ to x_0 is contained in U. Since N contains some U, if the lemma were false there would be an $\bar{x}$ such that $\lambda \bar{x} \bar{\epsilon} U$ for all $\lambda > 0$ in contradiction with V.

LEMMA 5. *If A is closed and $x_0 \varepsilon L$ is disjunct from A then, for some U' and U'', $(A + \bar{U}') \cap (x_1 + U'') = \emptyset$.*

Since A is closed we can find U_0^s such that $(x_0 + U_0^s) \cap A = \emptyset$. In view of I and IV, $U_0^s \supset U_1^s + U_2 - U_2$ for some U_1^s and U_2. We assert $(A + U_2) \cap (x_0 + U_1^s) = \emptyset$. Otherwise for some $x_1 \varepsilon A$, $y \varepsilon U_1^s$ we should have $x_0 + y_1 \varepsilon x_1 + U_2$. This implies successively $x_1 - (x_0 + y_1) \varepsilon - U_2$ and $x_1 \varepsilon x_0 + y_1 - U_2 \subset x_0 + U_1^s + U_2 - U_2$ in contradiction with the fact that A is disjunct from $x_0 + U_0^s$. In view of Lemma 2 we can find U_3 such that $\bar{U}_3 \subset U$ and hence $(A + \bar{U}_3) \cap (x_0 + U_1) = \emptyset$.

LEMMA 6. *If K_1 and K_2 are convex then $K_1 \pm K_2$ is convex.*

LEMMA 7. *If K is convex then $\bar{K}$ and $\underline{K}$ are convex.*

By a translation we can arrange to have $\theta \varepsilon K$. Assume the first assertion false. Suppose, then, that $x_1, x_2 \varepsilon \bar{K}$ and yet $z = \lambda x_1 + (1 - \lambda) x_2 \varepsilon \bar{K}$ for some $\lambda, 0 < \lambda < 1$. Then for some U, $(z + U) \cap K = \emptyset$. By III and IV we can find U' to satisfy $\lambda U' + (1 - \lambda) U' \subset U$. Let $y_i \varepsilon (x_i + U') \cap K \neq \emptyset$. $i = 1, 2$. Evidently λy_1 and $(1 - \lambda) y_2 \varepsilon K$. Hence we arrive at the contradiction $\emptyset \neq [\lambda(x_1 + U') + (1 - \lambda)(x_2 + U')] \cap K \subset [\lambda x_1 + (1 - \lambda) x_2 + U] \cap K = \emptyset$.

Let $x_1, x_2 \varepsilon K$. With no loss of generality we may assume $x_1 = \theta$. Let U_1 and $x_2 + U_2$ be contained in K. Suppose that for some $\lambda, 0 < \lambda < 1$, $\lambda x_2 \bar{\varepsilon} K$. Let $U_3 \subset U_1 \cap U_2$. Choose y arbitrarily in $\lambda x_2 + U_3$. Thus $y = \lambda x_2 + z, z \varepsilon U_3$. Clearly z and $z + x_2$ are both in K. Moreover $(1 - \lambda) z + \lambda(z + x_2) = y$ is in K by the convexity of K. Hence $\lambda x_2 + U_3 \subset K$ or $\lambda x_2 \varepsilon K$ in contradiction with an earlier assumption.

LEMMA 8. *If $K \neq L$ is convex and $\theta \varepsilon \underline{K}$ there is a triangular pseudo norm*[3] $\| x \|_K$ *satisfying* (1) $0 = \| \theta \|_K \leq \| x \|_K < \infty$, (2) $\| x \|_K \leq 1$ *for*

---

[3] If $x_0 \epsilon K$ we define $\| x \|_{k, x_0}$ by $\inf \{ h > 0 \mid x \epsilon h (K - x_c) \}$ etc.

$x \varepsilon K$, (3) $\| tx \|_K = t \| x \|_K, t \geq 0$, (4) $\| x_1 + x_2 \|_K \leq \| x_1 \|_K + \| x_2 \|_K$, (5) *The frontier points are those for which*

$$\| x \|_K = 1, \text{ (6) } \bar{K} = \{x \mid \| x \|_K \leq 1\}, \ \underline{K} = \{x \mid \| x \|_K \leq 1\}.$$

We define $\| x \|_K$ as $\inf \{h > 0 \mid x \varepsilon hK\}$. Let $U \subset K$, then for any $x \varepsilon L$ we have $x \varepsilon \alpha U$ for some $\alpha > 0$. Clearly $\infty > \alpha \geq \| x \|_K \geq \| \theta \|_K = 0$. Since $\| x \|_K$ is merely the Minkowski functional determined by K, properties (2), (3) and (4) follow by standard arguments.

We have left (5). In contradistinction to the locally convex case, this is not intuitively so evident when one considers the possibilities inherent in neighborhoods with spines. However the proof is direct. Let $U_1{}^s \subset K$, then, (for arbitrary $\epsilon > 0$,) in view of III and IV, $U_1{}^s \supset (2/\epsilon) U_2{}^s$ for some $U_2{}^s$. Then $\| \pm x \|_K < \epsilon$ for $x \varepsilon U_2{}^s$ and so $\| \pm x \|_K$ is continuous at θ. Hence by (4), $\| \pm x \|_K$ is everywhere continuous.[4] If x_1 is a frontier point, every neighborhood of x_1 contains points of K and points of $\tilde{K}$. Thus every neighborhood of x_1 contains some points for which $\| x \|_K \leq 1$ and some for which $\| x \|_K \geq 1$. Therefore $\| x_0 \|_K = 1$. On the other hand if $\| x_0 \|_K = 1$ and U is arbitrary, then by Lemma 4, there is a $\delta > 0$ such that $(1 \pm \delta) x_0 \varepsilon x_0 + U$. Hence by (3), $(1 + \delta) x_0 \bar{\varepsilon} K$, $(1 - \delta) x_0 \varepsilon K$ or x_0 is a frontier point. The forms of $\bar{K}$ and $\underline{K}$ are now seen to be those given in (6).

Extending a familiar usage we shall say K is *regularly L convex* if for every $x_0 \bar{\varepsilon} K$ there is an $f_0 \varepsilon L^*$ such that $f_0(x_0) > \sup_K f_0(x)$.

LEMMA 9. *If K is regularly L convex then K is convex.*

THEOREM 1. *If $K \neq L$ is convex, closed and $\underline{K} \neq \emptyset$ then K is regularly L convex.*

It is no loss of generality to take θ as an inner point. Let $x_0 \bar{\varepsilon} K$. The linear space L_1 is defined by $\{x \mid x = tx_0, -\infty < t < \infty\}$. Write $f_0(tx_0) = t \| x_0 \|_K$ for all t. Since $\| tx_0 \|_K \geq f(tx_0)$, the function $\| x \|_K$ is available for $p(x)$ in the Banach statement of the Hahn-Banach Theorem [8; p. 27]. Thus there is a distributive functional F_0 on L which agrees with f_0 on L_1 and $F_0(x) = \| x \|_K$. Now $| F_0(x) | \leq \sup (\| x \|_K, \| -x \|_K)$. In connection with (5) of the previous lemma we demonstrated the continuity of $\| x \|_K$ and $\| -x \|_K$. Thus $F_0(x)$ is continuous at $x = \theta$ and so throughout L. Hence choosing F_0 so that $F_0(x_0) = \| x_0 \|_K$ it follows from (5) that

---

[4] The argument used to establish such a result for the case of locally convex spaces [2] makes unnecessary use of VI and so could not be cited directly.

$$F_0(x_0) > 1 \geq \sup_K \| x \|_K \geq \sup_K F_0(x).$$

(it is trivial that every non null linear functional on L may be related to a Minkowski functional. Thus let $K = \{x \mid |f(x)| \leq 1\}$ and plainly $\theta \varepsilon K$ since $\{x \mid |f(x)| < 1\}$ is open. Moreover either $f(x_0) = \| x_0 \|$ or $f(-x_0) = \| -x_0 \|$ for the frontier points $\pm x_0$).

LEMMA 10. *If $K \neq L$ is convex, and $\theta \varepsilon \underline{K}$, then $\{x \mid \| x \|_K \leq 1\}$ is regularly L convex.*

3. The weak topology of *L*. The generic $w(L)$ neighborhood of θ is denoted by $W(\theta; f_1, \cdots, f_n; \epsilon)$ (or merely $W) = \{x \mid |f_i(x)| < \epsilon, i = 1, \cdots, n\}$, where $\{f_i\}$ consists of linearly independent elements of L^* in the sequel. The neighborhoods of x_0 are then given by $x_0 + W$. In general it is not true that the system of weak neighborhoods, $\mathcal{W} = \{W\}$, is consistent with the axioms for a l. t. s. For instance in the space S or $L^{\frac{1}{2}}$ it is well known that L^* consists only of the null element θ^*. Hence I is not satisfied.

Let $\Theta = \{x \mid f(x) = 0 \text{ for all } f \varepsilon L^*\}$ and $X = x + \Theta$. The sets Θ, X considered as elements of a new space $\boldsymbol{L}$ are designated by $\boldsymbol{O}$ and $\boldsymbol{X}$ respectively. Evidently if $x \varepsilon W$ then $X = x + \Theta \varepsilon W$ and accordingly W is made up of distinct X sets. Since Θ is closed we define a natural basis at $\boldsymbol{O}$ by

$$\boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon) = \{\boldsymbol{X} \mid X \varepsilon W(\theta; f_1, \cdots, f_n; \epsilon\}.$$

The topology so defined is tacitly understood in subsequent references to $\boldsymbol{L}$. Manifestly then, $\boldsymbol{L}$ is a space of disjunct equivalence classes defined on L.

THEOREM 2. *The space $\boldsymbol{L}$ is a locally convex* l. t. s.

In view of Theorem 3 b kx_1 and $x_1 + x_2$ correspond to $k\boldsymbol{X}_1$ and $\boldsymbol{X}_1 + \boldsymbol{X}_2$ where $x_i \varepsilon X_i, i = 1, 2$ so that $\boldsymbol{L}$ is a linear space. Write $\boldsymbol{X} = Tx$. Obviously T is continuous since $\boldsymbol{W} = T(W)$. Now I and II are satisfied since $\boldsymbol{L}$ is obviously the Hausdorff topological quotient group L/Θ with addition as the group operation. Now

$$\alpha \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon) = \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; |\alpha| \epsilon) \varepsilon \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon)$$

for $|\alpha| \leq 1$, or III is satisfied. Since $|f(x_1 + x_2)| \leq |f(x_1)| + |f(x_2)|$ it is apparent that IV and VI hold. Finally suppose $x_0 \varepsilon X_0$, and let $c_i = f_i(x_0)$, $i = 1, \cdots, n$. If $a > \sup |c_i|$ then $x_0 \varepsilon (a/\epsilon) W(\theta; f_1, \cdots, f_n; \epsilon)$ and so $\boldsymbol{X}_0 \epsilon (a/\epsilon) \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon)$ which is the requirement of V.

COROLLARY. *If $\Theta = \theta$, L is locally convex in its $w(L)$ topology.*

This includes the case of a locally convex L.

It is easy to give examples of linear topological spaces for which Θ is neither θ nor L. Since $L^*_p = \theta^*$, $0 < p < 1$ [9] take

$$L = L^p \times R_1 = \{(x, y) \mid x \varepsilon L^p, y \varepsilon R_1\}.$$

Then $\Theta = \{(x, \theta)\}$.

THEOREM 3. *The set Θ has the following properties:* (a) *Θ is convex and closed.* (b) *$\alpha\Theta = \Theta$ for $\alpha \neq 0$.* (c) *If P is a linear operator on L_1 to L_2 then $P(\Theta_1) \subset \Theta_2$.* (d) *If $L^* \neq \theta^*$ then $\Theta = \emptyset$.* (e) *For regularly L convex K, either K or $\bar{K}$ contains X.* (f) *If L is convex $\Theta = \theta$.*

The first three assertions are obvious. Suppose (d) false. Let $x_1 + U_1 \subset \Theta$. Then x_1 and so U_1 also, are both in Θ. Hence by (b) and V, $L \subset \Theta$ or $L^* = \theta^*$. If (e) is false then x_1 and x_2 may be found in X with $f(x_1) > f(x_2)$ in contradiction with the definition of X. Assertion (f) is a trivial consequence of Lemma 2, and Theorem 1. The converse of (f) is false. For instance in a sequence space the functionals $f^n = (\delta_1^n, \cdots, \delta_i^n, \cdots)$, with δ_i^n the Kronecker δ, cannot all be zero for any x. Hence, for instance, the non-locally convex l_p, $0 < p < 1$ has $\Theta = \theta$. (We remark parenthetically that while the metrizability of l_p, $0 < p < 1$, has long been recognized, the trivial observation that a valid metric for this purpose is $\Sigma \mid x_i \mid^p$ seems to have been overlooked. With this metric, it is at once evident that l_p is locally bounded as is well known. Moreover a familiar type of argument establishes that l_p is complete).

THEOREM 4. *The following conditions on $K \subset L$ are equivalent.* (a) *K is regularly L convex.* (b) *K is convex and $w(L)$ closed.*

That (a) implies (b) is trivial. The demonstration of the converse is of conventional type. Thus if $x_0 \bar{\varepsilon} K$ let $W(x_0; f_1, \cdots, f_n; \epsilon)$ be disjunct from K. We map L on R_n where the correspondence is given by

$$z = \psi(x) = \{f_1(x), \cdots, f_n(x)\}.$$

Then K_n, the map of K, is convex. Now $z_0 = \psi(x_0)$ is not in $\bar{K}_n$, the closure of K_n in R_n. Indeed if $\| z - z_0 \| < \epsilon$ contains points of K_n then $\Sigma \mid f_i(x) - f_i(x_0) \mid^2 < \epsilon^2$ for some $x \varepsilon K$ and accordingly $\mid f_i(x) - f_i(x_0) \mid < \epsilon$ but this is impossible since W and K are disjunct. Some functional $\mathfrak{F}_0 \varepsilon R^*_n$

satisfies $\mathfrak{F}_0(z_0) > \sup\limits_{Kn} \mathfrak{F}_0(z)$. Now $\mathfrak{F}_0(z)$ is of the form $\Sigma \lambda_i f_i(x)$ and hence with $F_0 = \Sigma_1{}^n \lambda_i f_i$ we have

$$\mathfrak{F}_0(z_0) = F_0(x_0) > \sup_{Kn} \mathfrak{F}_0(z) = \sup_{K} F_0(x).$$

Evidently (a) implies that $T(K)$ is closed and convex.

THEOREM 5. *If* $\Theta = \theta$ *and* K *is a closed finite dimensional set then* K *is regularly* L *convex.*

Let $\{x_i \mid i = 1, \cdots, N\} \subset L$ be linearly independent. Since $\Theta = \theta$ the following steps are valid. Choose $f_1 \varepsilon L^*$ so that $f_1(x_1) = a_1{}^1 \neq 0$. Write $f_1(x_2) = a_2{}^1$. Then $f_1(a_2{}^1 x_1 - a_1{}^1 x_2) = 0$. For some $f_2, f_2(a_2{}^1 x_1 - a_1{}^1 x_2) \neq 0$. Write $f_2(x_i) = a_i{}^2, i = 1, 2$. Hence $\Delta = a_2{}^1 a_1{}^2 - a_1{}^1 a_2{}^2 \neq 0$ and so two linear combinations of f_1 and f_2, designated by F^1 and F^2, satisfy $F^i(x_j) = \delta_j{}^i$, $i, j = 1, 2$. An induction shows that we can find $\{f^i\}$ to satisfy $f^i(x_j) = \delta_j{}^i$, $i, j = 1, \cdots, N$, whence it follows incidentally that we can always satisfy $f(x_i) = c_i, i = 1, \cdots, N$, for arbitrary choice of $\{c_i\}$. Let the N dimensional manifold E_N be the closed linear extension of K, with basis $\{x_i \mid i = 1, \cdots, N\}$. If $x_0 \bar{\varepsilon} E_N$ then we can find f_0 to satisfy $f_0(x_i) = \delta_0{}^i$, $i = 0, 1, \cdots, N$. If $x_0 \varepsilon E_N \frown \tilde{K}$ then, for some $U, x_0 + U_N \frown K = \emptyset$, $U_N = U \frown E_N$. Now

$$Q = \{x \mid x = \Sigma_1{}^N \mu_i x_i, \Sigma_1{}^N \mu_i \mid \leq 1\}$$

is a symmetric convex set in E_N. It is well known that E_N is homeomorphic to R_N with Q the homeomorph of a convex bounded set containing an open set in R_N. Hence $\epsilon Q \subset U_N$ if $\epsilon_0 \geqq \epsilon > 0$. Write $x_0 = \Sigma_1{}^N \lambda'_i x_i$. If $\mid \lambda_i - \lambda'_i \mid < \epsilon_0/N$, $i = 1, \cdots, N$, obviously $x - x_0 = \Sigma(\lambda_i - \lambda'_i) x_i \varepsilon \epsilon Q \subset U_N$. Thus $W(x_0; f^1, \cdots, F^n, \epsilon_0/N\} \frown K = \emptyset$ since $f^j(x - x_0) = \lambda_j - \lambda'_j$, and hence in view of Theorem 4 our proof is complete.

The following example shows that $\Theta = \theta$ does not assure that all closed convex sets are regularly L convex. Define K in $l_{1/2}$ by $\{x \mid \| x \| \geq 1, x_i \geq 0\}$. Then $K = \bar{K}$. To see that K is convex consider any segment H with end points x, y in K. We may require $\| x \| = \| y \| = 1$ for if $z \varepsilon H \frown \bar{K}$ then, by continuity, H would contain x', y' of norm 1 with the order x, x', z, y', y. It is easy to verify that $\| \lambda x + 1 - \lambda y \| - 1 = \Sigma' [(A_{ij}{}^2 + \lambda(\lambda - 1) \times (x_i y_j (1 - x_j y_i) + x_j y_i (1 - x_i y_j))^{1/2} - A_{ij}]$, $A_{ij} = \lambda(x_i x_j)^{1/2} + (1 - \lambda) \times (y_i y_j)^{1/2}$. Terms with $i = j$ are omitted and the typical term is clearly non-negative. Hence $H \subset K$. Now $\theta \bar{\varepsilon} K$ yet $f(\theta) = 0 \leq \sup\limits_{K} f(x)$ for all f. Indeed write $f = (a_1, a_2, \cdots)$ with $a_i \leq 0$ for this is the only case in doubt.

Write $x^N = \Sigma_1^N y^i/N^2$, $y^i = (\delta_1^i, \delta_2^i, \cdots)$. Then $\| x^N \| = 1$ and $x^N \varepsilon K$. Now $l_{1/2}^*$ is the space in [8, 9]. Thus $-\infty < a \leq \inf a_i \leq 0$ and $0 \geq f(x^N) = \Sigma_i a_i x_i^N \geq a/N \rightarrow 0$. Hence K is not regularly L convex. Incidentally $\underline{K}^{-} = \phi$.

THEOREM 6. *If $A \subset L$ is $w(L)$ closed then A is closed.*

Suppose $x_0 \bar{\varepsilon} A$ and let $W(x_0; f_1, \cdots, f_n; \epsilon) \cap A = \emptyset$. Since f_i is continuous, $| f_i(x - x_0) | < \epsilon$ for all $x \varepsilon x_0 + U_i$. Thus $x_0 + \bigcap_1^n U_i \subset W$.

4. Locally convex spaces.

The hypothesis of local convexity is *understood* in the statements of the results in this section.

LEMMA 11. *The completion of L is a locally convex* l. t. s.

We assume the material in Tukey's Chapter VI [7] but translate coverings into neighborhoods by his Lemma 3.3. Thus the equivalence classes of Cauchy phalanxes in L are put in $1-1$ correspondence with the elements $\{X\}$ of L'. If X, Y have representatives $\{x(a) \mid \mathfrak{A}\}$, $\{y(b) \mid \mathfrak{B}\}$ we inflate them to phalanxes on the common stack $\mathfrak{C}$ and define $X + Y$ by the representative $\{x_c + y_c \mid \mathfrak{C}\}$ and kX by $\{kx(a) \mid \mathfrak{A}\}$. It is easily verified that these definitions are consistent with the correspondence of elements of L and the convergent Cauchy phalanxes and that L' is the (essentially unique) complete l. t. s. desired[5] when the neighborhood basis is defined by

$$V = \{X \mid \{x(a) + U_1 \mid \mathfrak{A}\} \subset U\}.$$

If $X, Y \epsilon V$ then, for some representative $\{x(a) \mid \mathfrak{A}\}$ and some $U_1 (U, U_1 \varepsilon \mathfrak{U}^s)$, some representatives, $\{x(a) + U_1 \mid \mathfrak{A}\}$, $\{y(b) + U_2 \mid \mathfrak{B}\}$ are in U. The convexity of $\mathfrak{U}^s$ implies $\lambda(x_c + U_3) + (1 - \lambda)(y_c + U_3) = \lambda x_c + 1 - \lambda y_c + U_3 \subset U$ where $U_3 \subset U_1 \cap U_2$ and hence $\lambda X + 1 - \lambda Y$, since it has the representative $\{\lambda x_c + 1 - \lambda y_c \mid \mathfrak{C}\}$, is contained in V.

An alternative argument of independent interest uses the quasi norms associated with a neighborhood basis. Let $||| X |||_{U^s} = \bar{L} \| \{x(a) \mid \mathfrak{A}\} \|_{U^s}$. It is easy to show that $||| X |||_{U^s}$ is independent of the choice of representative for X. Since $\| \ \|_{U^s}$ is triangular,

$$L \| \{x_c + y_c \mid \mathfrak{C}\} \|_{U^s} \leq \bar{L} \| \{x_c \mid \mathfrak{C}\} \|_{U^s} + \bar{L} \| \{y_c \mid \mathfrak{C}\} \|_{U^s}$$

---

[5] Axioms I, II, IV reflect known properties of the completion of an Abelian topological group. Axiom III is easily verified. For V let $X \sim \{x(a) \mid \mathfrak{a}\}$. Suppose that V comes from U. Choose U_1 to satisfy $U_1 + U_1 + U_1 \subset U$. Then $x(a) \epsilon x(a_0) + U_1$, $a \vdash a_0$. For some $k \geq 1$, $x(a_0) \epsilon kU_1$. Then $x(a) + U_1 \subset kU_1 + U_1 + U_1 \subset kU$, $a \vdash a_0$ or $X \epsilon kV$, since $\mathfrak{a}' = \{a \mid a \vdash a_0\}$ can replace $\mathfrak{a}$.

whence it follows readily that $||| \; |||_{U^s}$ is triangular. Order U^s by $U^s \prec U^{s'}$ if and only if $U^s \subset U^{s'}$. It may be verified that $\{||| \; |||_{U^s} \mid U^s \varepsilon \mathfrak{U}^s\}$ is a collection of triangular quasi norms satisfying conditions [3b] guaranteeing that L' is a locally convex l. t. s. which is obviously complete and identical with that obtained in the preceding proof. Incidentally it is plain that the completion of a convex set is convex.

It would be of interest to investigate further the relations between convexity and regular L convexity. If every closed convex set is regularly L convex (this obviously implies $\Theta = \theta$), is L convex or is B bounded if $\hat{B}$ is bounded? It seems probable that both these implications are false.

LEMMA 12. *If K is closed and convex and $x_0 \bar{\varepsilon} K$ then there is a closed convex set K' with $K' \neq \emptyset$ such that $x_0 \varepsilon \bar{K}' \subset \bar{K}$.*

According to Lemma 5, $(K + \bar{N}_1) \frown (x_0 + \underline{N}_2) = \emptyset$. Since N_1 satisfies VI, $\bar{N}_1$ is convex [2] and therefore, in view of Lemmas 6 and 7, $\overline{K + \bar{N}_1}$ is convex.[6] Inasmuch as $x_0 + \underline{N}_2$ is open it follows that $(\overline{K + \bar{N}_1}) \frown + \underline{N}_2) = \emptyset$. Hence the conclusion of the lemma is valid for $K' = \overline{K + \bar{N}_1}$.

THEOREM 7. *If K is convex then K is $w(L)$ closed.*

This is an extension of a fundamental result for vector normed spaces due to Mazur [5]. If $K = L$ the theorem is trivially true. Suppose $x_0 \bar{\varepsilon} K$. The set K' of Lemma 12 is regularly L convex according to Theorem 1. Hence $F_0(x_0) > \sup_{K'} F_0(x) \geq \sup_{K} F_0(x)$ for some $F \varepsilon L^*$. That is to say K is regularly L convex, whence, by Theorem 4, K is $w(L)$ closed.

5. Metrizability of L^*.

Let $B \subset L$ be a bounded set containing θ and let $\mathfrak{B}$ stand for the collection of all such sets. The neighborhoods B^*, of L^*, are defined by $B^* = \{f \mid |f(x)| < 1, x \varepsilon B\}$ and are plainly convex. The set B is said to *determine* B^*. The neighborhoods of f_0 are of the form $f_0 + B^*$. With this neighborhood topology L^* is an l. t. s. Plainly if $B' \supset B''$ then $B'^* \subset B''^*$. In general the converse implication is not valid, but cf. Lemma 14. We shall call the sequence $\{B_n\} \varepsilon \mathfrak{B}$ an *outer countable basis* if for every $B \in \mathfrak{B}$, we can satisfy $B \subset B_n$ for some n. (The notion of an outer countable basis and Theorem 8 were suggested to me in conversation by Professor Kakutani).

---

[6] Even if K_1 and K_2 contain inner points and are closed and convex, $K_1 + K_2$ need not be closed unless a compactness condition is imposed on K_1. For instance in the Euclidean plane let K_1 consist of the branch of the hyperbola $xy = 1$ in the first quadrant and all points lying above this branch. Let K_2 be the reflection of K_1 in the y axis. Then $K_1 + K_2$ consists of the upper half plane exclusive of $y = 0$ and accordingly is not closed.

THEOREM 8. *If L has an outer countable basis $\{B_n\}$ then L^* is metrizable as an F space.*

It is well known that an l. t. s. space may be metrized as a possibly incomplete F space if there is a countable neighborhood basis at any point [3(b), 4]. Accordingly we need merely show there is a countable basis $\{B^*_n\}$ in L^*. Let the neighborhood B^* be arbitrary and let B be any determining set of B^*. By hypothesis $B_n \supset B$ for some n, and hence, $B^* \supset B^*_n$.

Simple examples show that the sufficiency condition stated in this theorem is not, generally, a necessary condition. If B is any determining set of B^* then $\{\lambda x \mid |\lambda| \leq 1, x \varepsilon B\}$ is obviously bounded and also a determining set of B^*. In the sequel *all determining sets are tacitly assumed to satisfy* $B = \{\lambda x \mid |\lambda| \leq 1, x \varepsilon B\}$.

LEMMA 13. *If L is locally convex and $B = -B$ then*

$$\bar{\hat{B}} = \{x \mid |f(x)| \leq 1, f \varepsilon B^*\}.$$

Denote $\{x \mid |f(x)| \leq 1, f \varepsilon B^*\}$ by B'. Now $\hat{B}$ and $\bar{\hat{B}}$ are bounded. Moreover according to Lemma 7, $\bar{\hat{B}}$ is convex. Write $D(=-D)$ for $\bar{\hat{B}}$ (this notation is used in the following theorem also). Now B' is clearly convex and closed and so $B' \supset D$. Also, D is regularly convex in view of Lemma 7 and Theorems 4 and 7. Hence for arbitrary $x_0 \bar{\varepsilon} D$ and some f_0,

$$|f_0(x_0)| = d > a = \sup_D |f_0(x)|.$$

Write f_1 for $2f_0/a + d$. Then

$$|f_1(x_0)| > 1 > \sup_D |f_1(x)|.$$

Accordingly $f_1 \varepsilon D^* \subset B^*$ and therefore $x_0 \bar{\varepsilon} B'$. Thus $D \supset B'$, and so $D = B'$. Evidently we have as a special case

$$\bar{\hat{B}} = \{x \mid |f(x)| \leq 1, f \varepsilon D^*\} = \{x \mid |f(x)| \leq 1, f \varepsilon (\hat{B})^*\}.$$

THEOREM 9. *If L is locally convex then a necessary condition for L^* to be metrizable is that L admit an outer countable basis.*

If L^* is metrizable there is a countable neighborhood basis $\{B^*_n\}$ at θ^*. Denote by $B_n \varepsilon B$ any determining set of B^*_n. Define D_n as $\bar{\hat{B}}_n$. Cf. Lemma 13. We wish to show that $\{D_n\}$ is an outer countable basis. If this were not true, then, for some $B' \varepsilon B$, we should have $B' \cap \bar{D}_n \neq \emptyset$ for all n. Since $D^*_n \subset B^*_n$ it is clear that $D^*_m \subset B'^*$, where m is some fixed value of n. Since D_m is regularly L convex, if $x_0 \varepsilon B' \cap \bar{D}_m$ we have

$$\sup_{D_m} F_0(x) = -\inf F_0(x) = a \geq 0 \quad \text{and} \quad F_0(x_0) - a = \delta > 0.$$

Then, with $F_1 = 2F_0/2a + \delta$, we have $F_1(x_0) > 1 > \sup_{D_m} | F_1(x) |$. That is to say $F_1 \varepsilon D^*_m$ and $F_1 \bar{\varepsilon} B'^*$ or $D^*_m \not\subset B^{*\prime}$ in contradiction with the definition of D^*_m. Accordingly $B' \cap \bar{D}_m = \emptyset$ and hence $\{D_n\}$ is an outer countable set.

Lemma 14. *If L is locally convex and $B^*_1 \supset B^*_2$ then, if B_i is a determining set of B^*_i, $\bar{\hat{B}}_1 \subset \bar{\hat{B}}_2$.*

The boundedness and convexity of $\bar{\hat{B}}_i$ are covered in the proof of Theorem 9. By Lemma 13,

$$\bar{\hat{B}}_1 = \{x \mid | f(x) | \leq 1, f \varepsilon B^*_1\} \subset \{x \mid | f(x) | \leq 1, f \varepsilon B^*_2\} = \bar{\hat{B}}_2.$$

The following theorem extends an important result due to Helly [10].

Theorem 10. *Let L be locally convex and L^* metrizable. Then there is a fixed closed convex bounded set B in L with the property that, whenever $\{f^i\} \subset L^*$ and $\{c_i\}$, $i = 1, \cdots, N$ are so chosen that*

$$| \Sigma_1^N \lambda_i c_i | \leq (M(\Sigma_1^N \lambda_i f^i), \theta^*)$$

for arbitrary $\{\lambda_i\}$, the system $f^i(x) = c_i$ is satisfied by some $x_\epsilon \varepsilon (M + \epsilon)B$ for each $\epsilon > 0$.

By a well known result of Eidelheit and Mazur [11] on equivalent metrics in an F space, we may require that $(\mu f, \theta^*)$ be strictly monotone in $\mu \geq 0$ for each fixed f (the demonstration given by the authors remains valid when the completeness requirement on F is dropped). Moreover $(f, \theta^*) = (-f, \theta^*)$ [8]. For some set in the outer countable basis of L, say

$$B = \bar{\hat{B}}, \{f \mid (f, \theta^*) < 1\} \supset B^*.$$

Denote the general point $\xi_1, \cdots, \xi_N$ in the N dimensional Euclidean space by z. Let

$$K = -K = \{z \mid \xi_i = f^i(x), x \varepsilon (M + \epsilon)B\}.$$

Evidently K is a convex, bounded, continuous map of $(M + \epsilon)B$. If the theorem were untrue $z_0 = (c_1, \cdots, c_N) \bar{\varepsilon} K$. Then some supporting hyperplane, $\Sigma_i^N \lambda_i \xi^i = 1$. of K contains z. Thus

$$|\Sigma_1^N\lambda_i f^i(x)| \leq 1 \quad \text{for} \quad x \in (M+\epsilon)B$$

while

$$\Sigma_1^N\lambda_i c_i = 1 \quad \text{or} \quad |\Sigma\lambda_i f^i(x)| < 1 \quad \text{for} \quad x \in (M+\epsilon/2)B,$$

i. e., $\Sigma\lambda_i f^i \in (M+\epsilon/2)B^*$. Accordingly,

$$((M+\epsilon/2)\Sigma_1^N\lambda_i f^i, \theta^*) < 1 = |\Sigma_1^N\lambda_i c_i|.$$

Then

$$(M\Sigma\lambda_i f^i, \theta^*) < 1 = |\Sigma_1^N\lambda_i c_i|.$$

This is in contradiction with the hypothesis of the theorem.

If the metric satisfied $(\mu f, \theta^*) \leq \mu(f, \theta^*)$, for all f and $\mu \geq 0$, we could of course replace $(M(\Sigma_1^N\lambda_i f^i), \theta^*)$ by $M(\Sigma_1^N\lambda_i f^i, \theta^*)$ but such a metric must obvously be a norm. (Take $\mu' = \mu^{-1} > 0$ and $f' = \mu f$, then it follows that $\mu'(f', \theta^*) \leq (\mu' f', \theta^*) \leq \mu'(f', \theta^*)$). If B is $w(L)$ bicompact or bicompact of order $\aleph_a$, Theorem 10 may be extended to the case that a in $\{c_a \mid a\}$ has power at most $\aleph_a$, by straightforward modification of the writer's proof for the Banach space case [Theorem 15; 12].

THEOREM 11. *If L is locally convex with L^* metrizable and separable then L is separable.*

The hypotheses of the theorem imply the existence of an outer countable basis $\{B_n\}$. Evidently the neighborhood systems

$$\{f \mid |f(x)| < \epsilon, x \in B_n\} \quad \text{and} \quad \{f \mid |f(x)| < 1, x \in B_n\}$$

are equivalent. Since $\cup B_n = L$, if the theorem were false some B_n, denoted henceforth by B, would not be separable. Let Γ be the set $\{f \mid \|f\|_B = \frac{1}{2}\}$ where $\|f\|_B = \sup_B |f(x)|$. In view of the separability of L^* a sequence $\{f^\sigma\} \subset \Gamma$ may be found which satisfies the condition

$$\{f^\sigma\} \cap \{f \mid \|f_0 - f\|_B < \epsilon\} \neq \emptyset;$$

for any $\epsilon > 0$ and $f_0 \in \Gamma$. Since $f^\sigma \in \Gamma$, $|f^\sigma(x^\sigma)| \geq \delta$ for some $x^\sigma \in B$ and $0 < \delta < \frac{1}{2}$. Let L_B and $L_{\{\sigma\}}$ be the closed linear manifolds determined by B and $\{x^\sigma\}$ respectively. Since $L_{\{\sigma\}}$ is separable some x, say x_0, is in L_B but not in $L_{\{\sigma\}}$. Now $L_{\{\sigma\}}$ is convex and closed and by Theorem 7 is a regularly L convex linear manifold. Hence for some

$$f_1 \in L^*, f_1(x_0) > \sup_{L_{\{\sigma\}}} f_1(x) = 0.$$

Moreover, since f_1 is linear, Theorem 2 of Wehausen guarantees that $\| f_1 \|_B < \infty$. Hence $f_0 = f_1/2 \| f_1 \|_B \varepsilon \Gamma$. Now

$$\| f_0 - f^\sigma \|_B \geq | f_0(x^\sigma) - f^\sigma(x^\sigma) | \geq \delta$$

for all σ which is in contradiction with the definition of $\{f^\sigma\}$.

The theorem is still true if the condition of metrizability of L^* is replaced by the requirement that there be a denumerable collection $\{C_n\}$ of bounded sets in L such that[7]

$$\text{(a)} \quad \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} mC_n = L \quad \text{and} \quad \text{(b)} \quad B \subset \bigcup_{m=1}^{\infty} mC_n$$

for each bounded set B and some n.

THEOREM 12. *If L is locally convex and L^* is normable then* $\bigcup_{n=1}^{\infty} nB = L$ *for some bounded total set B* in L.[7]

Define B as in Theorem 10. Let x' be arbitrary and write B' for $\{tx' \mid |t| \leq 1\}$. Then $B^* \supset \{\beta f | \| f \| < 1\} \supset \beta B^*$ for some $\beta > 0$. In view of Lemma 14, $\beta B \supset B'$; that is to say $x' \varepsilon mB$ for any integer $m > \beta$ and hence $\bigcup_n B = L$. Evidently $f(x) = 0$ for all $x \varepsilon B$ implies $f = \theta^*$. The theorem may also be established by considering the topology induced on L by L^{**} and clearly B may be taken as $\{x \mid |f(x)| \leq 1, \| f \| \leq 1\}$.

COROLLARY. *If L is locally convex and of the second category then a necessary and sufficient condition for L to be normable is that L^* be normable.*

The condition is obviously necessary. On the other hand if L contained no bounded neighborhood then B would be nowhere dense and hence L would be of the first category. However the presence of a bounded neighborhood means that L is both locally convex and locally bounded and hence normable.

The following example is of interest here and in connection with Theorem 19 below. Let E be a non-separable, reflexive Banach space. We denote by L the elements of E under the $w(E)$ topology. Then L is a sequentially complete, non-metrizable, locally convex space of the first category. Nevertheless L^* is normed. Incidentally it is apparent that if $\bar{L} \| f^n(x) \| < \infty$ for all $x \varepsilon L$, then $\| f^n \| < M$ (in particular $w^*(L^*)$ convergence of $\{f^n\}$ to f_0 implies $\| f^n \| < M$) yet L is of the first category.

---

[7] It is not true however that if $L = U_{m=1}^{\infty} mB$, then $\{mB\}$ is an outer countable basis, even if L is normed. The Hilbert parallelotope, taken as the set B, in L^2, furnishes an immediate counter example.

6. Local boundedness. Let A be a bounded open set in L and let

$$A' = \{\lambda x \mid |\lambda| \leqq 1,\ x \varepsilon A\}.$$

Then A' is easily verified to be bounded and open in L. Define the quasi norm $\| x \|$ by the usual scheme

$$\| x \| = \inf\{h \mid x \varepsilon hA'\}.$$

Hyers [3a] has shown that this quasi norm satisfies (1) $\| x \| \geqq 1$ and $\| x \| = 0$ implies $x = \theta$. (2) $\| tx \| = | t | \, \| x \|$. (3) For every $\eta > 0$ there is a $\delta > 0$ such that $\| x + y \| < \eta$ whenever $\| x \| < \delta$ and $\| y \| < \delta$. (4) The sets $\{x \mid \| x \| < \epsilon, \epsilon > 0\}$ form a system equivalent to $\mathfrak{U}$. (5) For every $x \varepsilon L$ and $\eta > 0$ there is a $\delta > 0$ such that $\| y \| - \| x \| < \eta$ for $\| y - x \| < \delta$.

We shall extend and sharpen these results significantly.

THEOREM 13. *If L is locally bounded the quasi norm satisfies*

$$\| x_1 + x_2 \| \leqq b(\| x_1 \| + \| x_2 \|)$$

for some $b \geqq 1$ depending on A and L.

We shall refer to b as the *multiplier* of the quasi norm. The neighborhood system at θ may be taken as $\mathfrak{U} = \{U \mid U = A'/n,\ n = 1, \cdots\}$. For some values of p we have $A' + A'/2 \subset pA'$, $p \geqq 1$, by IV. Hence p' exists such that $p'A' \supset tA' + (1 - t)A'$ for all $0 \leqq t \leqq 1$. Let b be the inf of all such p'. If $\| x_i \| = \lambda_i$, $i = 1, 2$, then $x_i \varepsilon (\lambda_i + \epsilon)A'$ for all $\epsilon > 0$. Therefore

$$x_1 + x_2 \varepsilon (\lambda_1 + \lambda_2 + 2\epsilon) \left[\frac{\lambda_1 + \epsilon}{\lambda_1 + \lambda_2 + 2\epsilon} A' + \frac{\lambda_2 + \epsilon}{\lambda_1 + \lambda_2 + 2\epsilon} A'\right] \subset (b + \delta)(\lambda_1 + \lambda_2 + 2\epsilon)A$$

Since this is true for each $\delta, \epsilon > 0$ it follows that

$$\| x_1 + x_2 \| \leqq b(\lambda_1 + \lambda_2) = b(\| x_1 \| + \| x_2 \|).$$

It thus appears that the quasi norm for a locally bounded L is formally similar to that for the spaces L_p, $0 < p < 1$. (The example of the spaces l_p, $0 < p < 1$ shows that functionals exist even when the norm is not triangular). It is important to remark that b should be written more felicitously b_A for, as is easily seen, it varies in general with the choice of the bounded open set, A. The quantity $\beta_L = \inf\{b_A \mid A \text{ bounded and open in } L\}$ is a characteristic of L alone. However β_L may not be a multiplier. Cf. Section **7**. All references to b below are really to a b_A.

Let A_i be a bounded open set in L_i, $i = 1, 2$, and associate A'_i as before. Let P be a linear operator with domain L_1 and range in L_2. The subscript $i = 1, 2$ affixed to a quasi norm or to b will indicate the space under consideration. Define $\| P \|$ by $\| P \| = \sup_{A'_1} \| P(x) \|_2$. The space of such operators is denoted by $\underline{\boldsymbol{P}}$ and it is known that $\underline{\boldsymbol{P}}$ is a locally bounded l. t. s. satisfying (1), (2) and (5), [3(a)].

THEOREM 14. *The quasi norm in* $\boldsymbol{P}$ *satisfies*

$$\| P_1 + P_2 \| \leq b_2(\| P_1 \| + \| P_2 \|).$$

Evidently

$$\begin{aligned} \| P_1 + P_2 \| &= \sup_{A'_1} \| P_1(x) + P_2(x) \|_2 \\ &\leq \sup_{A'_1} b_2(\| P_1(x) \|_2 + \| P_2(x) \|_2) \\ &\leq b_2(\| P_1 \| + \| P_2 \|). \end{aligned}$$

LEMMA 15. *If* L_2 *is normed,* (a) $\underline{\boldsymbol{P}}$ *is a normed vector space and* (b) *If* P *is additive and* $\| P(x) \| \leq M \| x \|$ *for all* $x \varepsilon L_1$, *then* P *is linear* [3a].

Condition (a) is satisfied for instance when $\underline{\boldsymbol{P}}$ is L^*.

THEOREM 15. *The locally bounded space* L *may be completed with preservation of local boundedness.*

Let the equivalent metric neighborhoods of L (cf. Section **5**) be written $S_d = \{x \mid (x, \theta) < d\}$. Consider the typical element of L' to be the class of equivalent Cauchy sequences in L, where two Cauchy sequences are equivalent if they 'mesh' to a Cauchy sequence. The points of L' are denoted by capitals and the distance is defined by $(X, Y) = \lim\,(x^n, y^n)$ where $X \sim \{x^n\}$, $Y \sim \{y^n\}$ i. e., $\{x^n\}$ and $\{y^n\}$ are representative Cauchy sequences of the classes defining X and Y respectively. As is well known this distance is a metric and if $\{x^n\}$ converges to x_0 and $\{y^n\}$ to y_0, then $(X, Y) = (x_0, y_0)$. Moreover in view of the triangle property of the metric, $S_0 = \{X \mid (X, \theta) < \delta\}$ is an open set. Plainly L' is an l. t. s. and is complete in the metric sense. Now A contains s_D, where D is some positive d, by the equivalence of the original and metric topologies. Then s_D is bounded in L and so S_D is bounded in L'. Indeed this is tantamount to the assertion that for arbitrary $e < D$ and some h we have $hS_D \subset S_e$. Let h' satisfy $h's_D \subset s_{e'}$, $e' < e$. Then if $X \varepsilon S_D$ and $X \sim \{x^n\}$ we must have $x^n \varepsilon s_D$ for $n > N$. Hence $(h'x^n, \theta) < e'$

for $n > N$. Thus $(h'X, \theta) = \lim (h'x^n, \theta) < e$. Since S_D is bounded and open, L' is locally bounded. Incidentally $s_D = \{\lambda x \mid x \varepsilon s_D, |\lambda| \leqq 1\}$ and $S_D = \{\lambda X \mid |\lambda| \leqq 1, X \varepsilon S_D\}$.

THEOREM 16. *If L is locally bounded and x^n converges to x_0 then*

$$\| x_0 \| \geqq \bar{L} \| x^n \| \geqq \underline{L} \| x^n \| \geqq \| x_0 \| / b.$$

We have $\| x^n - x_0 \| \to 0$. From Theorem 13 we have

$$b \| x^n \| \geqq \| x_0 \| - b \| x^n - x_0 \| \quad \text{or} \quad \underline{L} \| x_n \| \geqq \| x_0 \| / b.$$

On the other hand, in consequence of (5), $\| x_0 \| > \| x^n \| - \eta$ for $n > N_\eta$. Hence $\bar{L} \| x^n \| \leqq \| x_0 |$.

A consequence is that the quasi norm relative to A' is always upper semi-continuous but not necessarily continuous except at θ and, in fact, Hyers has given an elementary example of a discontinuous quasi norm. These remarks lend interest to the next result.

THEOREM 17. *If L is locally bounded there is a continuous quasi norm.*[8]

Let s_d, defined above, be the basic open bounded set defining the quasi norm. Let $x^n \to x_0$ and denote $\| x^n \|$ by λ_n. We need only the case $\lambda_0 \neq 0$. We assume without loss of generality that $(\mu x, \theta)$ is monotonic increasing in $\mu, \mu \geqq 0$ [11]. Now, for $0 < \delta < \lambda_0$,

$$\begin{aligned} (x_0/(\lambda_0 - \delta), \theta) > d > {} & (x^n/(\lambda^n + \delta), \theta) \\ & \geqq (x_0/(\lambda^n + \delta), \theta) - (x^n - x_0)/(\lambda^n + \delta), \theta). \end{aligned}$$

Thus

$$(x_0/(\lambda_0 - \delta), \theta) \geqq \bar{L}(x_0/(\lambda^n + \delta), \theta).$$

Hence

$$(\lambda_0 - \delta)^{-1} \geqq \bar{L}(\lambda^n + \delta)^{-1}.$$

Thus $\underline{L}\lambda_n \geqq \lambda_0 - 2\delta$ or, since $\delta > 0$ is arbitrary, $\underline{L} \| x_n \| \geqq \| x_0 \|$. In view of Theorem 16,

$$\| x_0 \| \geqq \bar{L} \| x^n \| \geqq \underline{L} \| x^n \| \geqq \| x_0 \|,$$

or $\| x \|$ is continuous. (We have shown incidentally that a certain sphere in the equivalent metric is the A required).

---

[8] This gives an affirmative answer to the "general question" raised by Hyers [3(a), p. 566].

7. An example. This section concerns itself with an example of a locally bounded l. t. s. for which β_L is not a possible multiplier. Consider the linear sequence space $\mathcal{L}$ with the distance given by

$$(x, y) = \sum_{i=1}^{\infty} | x_i - y_i |^{p_i}$$

where

$$p_i = (1 + (\log (i + 1))^{-1/2})^{-1}.$$

Write $q_i = p_i^{-1}$. Evidently p_i increases monotonely to 1. It is easy to verify that (x, y) is a metric and that $\mathcal{L}$ is an l. t. s. Since $(\lambda x, \theta) \leq \lambda^{p_1}(x, \theta)$ for $0 \leq \lambda \leq 1$, it follows that $\mathcal{L}$ is locally bounded. Although it is irrelevant to our purpose, it may be mentioned that the standard procedure establishes that $\mathcal{L}$ is complete. It is plain that an equivalent metric, denoted by $(x, y)_N$, is defined by

$$(x, y)_N = \sum_{i=1}^{N} | x_i - y_i |^{p_N} + \sum_{N+1}^{\infty} | x_j - y_j |^{p_j}.$$

It is an easy matter to show that, with the choice $A' = \{x \mid (x, \theta)_N < 1\}$, the associated multiplier b is $2^{q_N - 1}$. Hence $\beta_{\mathcal{L}} = 1$. We assert that 1 *is not a possible multiplier*. Suppose the assertion false. Then $\mathcal{L}$ is normable and hence locally convex. Then the neighborhood $s = \{x \mid (x, \theta) < 1\}$ contains a convex open set K which includes $s_\epsilon = \{x \mid (x, \theta) < \epsilon\}$ for some $\epsilon > 0$. Hence $s_\epsilon \subset K \subset s$. Now s_ϵ contains

$$x^n = (\epsilon^{q_1}\delta_1^n, \cdots, \epsilon^{q_i}\delta_i^n, \cdots)$$

where δ_i^n is the Kronecker delta. Then

$$z^M = \Sigma_1^M \alpha_n x^n / \Sigma_1^M \alpha_n \varepsilon s_\epsilon$$

where $\alpha_n = (n + 1)^{-q_n}$. Manifestly $\alpha_n \downarrow 0$ and hence the convergence of $\Sigma_1^\infty \alpha_n$ is guaranteed by the fact that

$$\int_1^\infty \frac{dx}{x^{1+(\log x)^{-1/2}}} = \int_0^\infty e^{-w^{1/2}} dw \text{ converges.}$$

Write $\sigma = \Sigma_1^\infty \alpha_n$ where manifestly $\sigma > 1$. Then

$$\begin{aligned}(z^M, \theta) &> (\Sigma_1^M \alpha_n x^n / \sigma, \theta)\\ &= \epsilon \Sigma_1^M (1/n + 1) \sigma^{-p_n} > (\epsilon/\sigma) \Sigma_1^M (n + 1)^{-1} \to \infty,\end{aligned}$$

which is incompatible with $s_\epsilon \subset s$. Actually all that is needed for this example is $p_i \uparrow 1$. It may be remarked parenthetically that if $p_i \downarrow 0$, the resulting space is neither locally bounded nor locally convex.

8. The conjugate space L^*. The $w^*(L^*)$ topology is induced by the neighborhood system $\mathfrak{U}^* = \{U^*\}$ where

$$U^* = U^*(\theta^*; x_1, \cdots, x_n; \epsilon) = \{f \mid |f(x_i)| < \epsilon;\ i = 1, \cdots, n\}.$$

Obviously X_i of Section **3** may replace x_i and L^* is an l. t. s. in its $w^*(L^*)$ topology. We define *regular L^* convexity* for $K \subset L^*$ by the property that if $f_0 \bar{\varepsilon} K$ then there is an $x_0 \,\varepsilon\, L$ for which $f_0(x_0) > \sup_K f(x_0)$. This implies that K is convex.

THEOREM 18. *The statements* (a) $K \subset L^*$ *is regularly* L^* *convex and* (b) K *is convex and* $w^*(L^*)$ *closed, are equivalent.*

The proof follows the pattern of the demonstration of Theorem 4 with the roles of L and L^* interchanged.

LEMMA 16. *If L is locally bounded the system*

$$\mathfrak{U}' = \{U'\}, \qquad U' = \{f \mid |f(x_i)| < \epsilon, x_i \,\varepsilon\, A'\}$$

is equivalent to $\mathfrak{U}^*$.

The lemma is a trivial consequence of the fact that A' is total [8, p. 58] for L^*, i. e., $f(x) = 0$ for all $x \,\varepsilon\, A'$ requires $f = \theta^*$.

THEOREM 19. *If L' is the completion of L then L^* and L'^* are quasi equivalent.*

Quasi equivalence is defined below. Sets and functions in L' are primed and the theorem stems from the fact that with our method of completion L is isomorphic to a dense set in L'. In this sense $L \subset L'$. Since f is uniformly continuous on L it has a unique extension f'. Moreover every f' determines a unique f. Thus $(A) L^*$ and L'^* are in $1 - 1$ correspondence. Using Zorn's lemma we can define a maximal family $\mathfrak{D}' \subset \mathfrak{B}'$ (cf. Section **5**) subject to (a) $B' = \{\lambda X \mid |\lambda| \leq 1, X \,\varepsilon\, B'\}$, $B' \,\varepsilon\, D'$. (b) No set of $\mathfrak{D}'$ is dense in another. Write $B = B' \cap L$. The sets B comprise a maximal family D with properties analogous to (a) and (b). Plainly $(B) \mathfrak{D}$ and $\mathfrak{D}'$ are in $1 - 1$ correspondence. If the sets of $\mathfrak{D}'(\mathfrak{D})$ are ordered by inclusion the quasi norms $\| f' \|_{B'}$ $(\| f \|_B)$ are triangular and satisfy the conditions of Hyers [3b]. They determine the ordinary topology of L^* and L'^*. If B corresponds to B', B is dense in B' so (C) $\| f \|_B = \| f' \|_{B'}$. The satisfaction of (A), (B), (C) constitutes quasi equivalence. For a locally bounded L, L^* and L'^* are easily shown to be complete and the quasi equivalence reduces to Banach equivalence [8].

Many results concerned with the w^* topology of the conjugate space of a vector normed space have valid analogues when L is a locally bounded l. t. s. The following results are of this type where, to avoid triviality, we assume throughout that L^* *contains more than one point.*

THEOREM 20. *If L is locally bounded then necessary and sufficient conditions for the $w^*(L^*)$ sequential convergence of $\{f^n\}$ to f_0 are* (a) $\| f^n \| \leq M$ and (b) $\{f^n(x)\}$ *converges pointwise on a dense set in L.*

THEOREM 21. *If L is separable then S^* is $w^*(L^*)$ sequentially separable.*

THEOREM 22. *If L is locally bounded and S^* is the unit sphere $\{f \mid \| f \| \leq 1\}$ then S^* is bicompact in the $w^*(L^*)$ topology.*

We vary the Banach space type of proof [13] by not making use of the Tychonoff theorem. It is easy to show that S^* is $w^*(L^*)$ complete. Moreover it is immediate that S^* is totally bounded or what is equivalent, largely compact [7; p. 64], in the sense that for every $W(\theta^*; x^1, \cdots, x^r; \epsilon)$ and some finite collection $f^1, \cdots, f^m$ in S^*, $S^* \subset \bigcup_{j=1}^{m} W(f^j; x^1, \cdots, x^r; \epsilon)$. Accordingly S^* is $w^*(L^*)$ bicompact [6, 7, Theorem 8.14]. Similarly if $\theta = \Theta$, $w(L)$ completeness of a bounded set implies $w(L)$ bicompactness.

The definition of transfinite limits given by Banach [8] may be used without change for L^* when L is locally bounded.

LEMMA 17. *For L locally bounded, if*

$$\| f_\xi \| \leq M, \qquad 1 \leq \xi < \tau, \tau$$

a limit ordinal, and $f_\xi \in L^*$, then $f_0 \in L^*$ exists, such that

$$\underset{\xi<\tau}{\underline{L}}\, f_\xi(x) \leq f_0(x) \leq \underset{\xi<\tau}{\overline{L}}\, f_\xi(x)$$

for all $x \in L$ and $\| f_0 \| \leq M$.

Let $p(x) = \overline{L} f_\xi(x)$ in the notation of Banach, p. 28. Then $p(x) \leq M \| x \|$ and $p(x)$ satisfies the conditions of the corollary to the Hahn-Banach Theorem (though $M \| x \|$ cannot be used for the same purpose since the quasi norm is not triangular). Hence an additive functional f_0 exists satisfying $|f_0(x)| \leq M \| x \|$ and therefore, by Lemma 15(b) f_0 is linear. The function f_0 is a *transfinite limit* of $\{f_\xi\}$. A set $K \subset L^*$ is transitively closed if it contains all its transfinite limits and plainly it is then convex.

LEMMA 18. *For L locally bounded and $\| f_\xi \| \leq M$, $1 \leq \xi < \tau$, if f_1 satisfies*

$$\underset{\xi < \tau}{\underline{L}} f_\xi(x) \leq f_1(x) \leqq \underset{\xi < \tau}{\bar{L}} f_\xi(x)$$

for $x \varepsilon Q \subset L$ then there is a transfinite limit f_0, $\| f_0 \| \leq M$, of $\{f_\xi\}$ which agrees with $f_1(x)$ on the closed linear extension of Q.

LEMMA 19. *If L is locally bounded, and if $K \subset L^*$ is either $w^*(L^*)$ closed, regularly L^* convex or transfinitely closed, then K is closed.*

THEOREM 23. *If L is locally bounded and $K \subset L^*$ is bounded then a necessary and sufficient condition that K be $w^*(L^*)$ closed and convex is that K is transfinitely closed.*

The necessity argument may be patterned with obvious changes on Banach, p. 122, paragraph 2, and Theorem 20.

Let $\bar{K}$ be the $w^*(L^*)$ closure of K. Suppose $\bar{K} \neq K$ and let $f_0 \varepsilon \bar{K} \cap \bar{K}$. Well order all the sets, X, of L, i. e., $\{X_\xi \mid 1 \leq \xi < \gamma\}$ where γ is a limit number (since $L^* \neq \theta^*$). We develop a familiar type of transfinite induction argument. The fact that $\| f \| \leqq M$ for all K is tacitly assumed in all statements made below about the existence of various transfinite limits. Since $f_0 \varepsilon \bar{K}$, for any X_η and integer m there is a linear functional in K, say f_η^m which satisfies

$$| f_\eta^{(m)}(X_\eta) - f_0(X_\eta) | < 1/m.$$

Hence

$$\underset{m}{\underline{L}} f_\eta^m(X_\eta) - 1/m = \underset{m}{\underline{L}} f_\eta^m(X_\eta) \leq f_0(X_\eta)$$

$$\leq \underset{m}{\bar{L}} f_\eta^m(X_\eta) + 1/m = \underset{m}{\bar{L}} f_\eta^m(X_\eta).$$

In view of the transfinite closure of K and Lemma 18, some transfinite limit,

---

[9] If, for instance, K is a closed vector subspace of L such an analysis as that given by Banach for the case L is a Banach space meets difficulties in the last step due to the fact that $b > 1$, [8, p. 119].

$f_\eta \varepsilon K$, of $\{f_\eta{}^m\}$ exists satisfying $f_\eta(X_\eta) = f_0(X_\eta)$. Suppose η is not a limit ordinal and that $f^{\eta-1} \varepsilon K$ satisfies $f^{\eta-1}(X_\rho) = f_0(X_\rho)$ for $\{X_\rho \mid 1 \leq \rho < \eta - 1\}$. We have shown the existence of an element $f_{\eta-1}$ in K such that $f_{\eta-1}(X_{\eta-1}) = f_0(X_{\eta-1})$. Define $\{F^\nu \mid 1 \leq \nu < \eta - 1\}$ by $F^\nu = f^{\eta-1}$, $\nu < \eta - 1$ and $F^{\eta-1} = f_{\eta-1}$. The transfinite limit, denoted by f^η, agrees with f_0 on $\{X_\rho \mid 1 \leq \rho < \eta\}$.

Suppose that for some $\eta \leq \gamma$ it is impossible to find an element of K which agrees with f_0 on $\{X_\rho \mid 1 \leq \rho < \eta\}$. Let Q be the set of such η's. This set has a first element η_0 and in view of considerations given above we need only consider the case that η_0 is a limit number. Then for every $\zeta < \eta_0$ there is an f^ζ such that $f^\zeta(X_\rho) = f_0(X_\rho)$ for $1 \leq \rho < \zeta$. Hence

$$\underset{\xi < \eta_0}{\underline{L}} f^\zeta(X_\rho) \leq f_0(X_\rho) \leq \bar{L} f^\zeta(X_\rho) \quad \text{for} \quad 1 \leq \rho < \eta_0.$$

Accordingly an $f' \varepsilon K$ agrees with f_0 on $\{X_\rho \mid 1 \leq \rho < \eta_0\}$ or $\eta_0 \bar{\varepsilon} Q$. Hence $Q = \emptyset$ and $f_0 \varepsilon K$ or $\bar{K} \cap \bar{K} = \emptyset$. The writer has not been able as yet to settle the question of whether the theorem remains valid if the boundedness restriction on K is dropped.

THEOREM 24. *If L is locally bounded and $\Gamma \subset L^*$ is compact then Γ is $w^*(L^*)$ bicompact.*

By reason of earlier theorems in this section the proof follows the pattern of the Banach space case [12].

In view of the existence of a Hamel base every l. t. s. admits non-null distributive functionals [8; p. 231]. Let D^* be the space of distributive functionals on L under the $w^*(D^*)$ topology defined in the obvious way. It is trivial that if L is finite dimensional then $D^* = L^*$.

THEOREM 25. *If $\Theta = \theta$ then L^* is $w^*(D^*)$ dense in D^*.*

This is an obvious consequence of the argument in the first part of the corollary to Theorem 4.

UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.

BIBLIOGRAPHY

1. A. Kolmogoroff, "Zur Normierbarkeit eines topologischen linearen Raumes," *Studia Mathematica*, vol. 5 (1934), pp. 29-33.
2. J. von Neumann, "On Complete Topological Spaces," *Transactions of the American Mathematical Society*, vol. 37 (1935), pp. 1-20.

3. D. H. Hyers, (a) "Locally Bounded linear topological spaces," *Revista de Ciencas*, vol. 41 (1939), pp. 558-574.

(b) "Pseudo-normed linear spaces and Abelian groups," *Duke Mathematical Journal*, vol. 5 (1939), pp. 628-634.

4. J. V. Wehausen, "Transformations in Linear Topological Spaces," *Duke Mathematical Journal*, vol. 4 (1938), pp. 157-169.
5. S. Mazur, "Uber konvex Mengen in linearen normierten Raume," *Studia Mathematica*, vol. 4 (1933), pp. 70-84.
6. A. Weil, *Sur les Espaces à Structure Uniforme et sur le Topologie Générale*; *Actualités Scientifiques et Industrielles*, 551, Paris, Hermann et Cie.
7. J. W. Tukey, "Convergence and Uniformity in Topology," *Annals of Mathematics Studies* (1940).
8. S. Banach, *Théorie des Operations Linéaires.*
9. M. Day, "The spaces L^p with $0 < p < 1$," *Bulletin of the American Mathematical Society*, vol. 46 (1940), pp. 86-823.
10. E. Helly, "Uber Systeme linearer Gleichungen mit unendlich vielen Unbekannten," *Monatshefte für Math. und Physik*, vol. 31 (1921), pp. 60-91.
11. M. Eidelheit and S. Mazur, "Eine Bemerkung über die Raume vom Typus (F)," *Studia*, vol. 7 (1938), pp. 159-161.
12. D. G. Bourgin, "Some Properties of Banach Spaces," *American Journal of Mathematics*, vol. 64 (1942), pp. 597-612.
13. S. Kakutani, "Weak Topology, Bicompact Set and the Principle of Duality," *Proceedings of the Imperial Academy of Tokio*, vol. 16 (1940), pp. 63-67.

TWO OBSERVATIONS CONCERNING ALGEBRAIC EQUATIONS FOR MATRICES.*

By Ivan Niven.

---

1. Let W be a matrix of order n in a field k, and $\psi(z) = 0$ its minimum equation in k of degree $m = m(W) \leqq n$. This integer m is called the index of W. Any polynomial $F(z)$ in k defines a matrix $A = F(W)$, and all matrices of this form constitute a ring $R[W]$ of order m over k; indeed $F(W) = 0$ if and only if $F(z)$ is divisible by $\psi(z)$. Hence $m(A) \leqq m(W)$, and in case $m(A) = m(W)$, the ring $R[A]$ coincides with the ring $R[W]$, so that W is a polynomial in A with coefficients from k. A matrix A is said to be non-derogatory if its characteristic equation is its minimum equation, $m(A) = n$. Now suppose that the matrix A is given, and that W is to be found. Then we have the following theorem:

If a polynomial $F(z)$ and a matrix A in a field k are given, then a solution W of $F(W) = A$ is expressible as a polynomial in A with coefficients from k, $W = f(A)$, if and only if the index of W equals the index of A. In case A is non-derogatory, every solution W of $F(W) = A$ is a polynomial in A, and is likewise non-derogatory.

From now on we assume k to be algebraically closed, so that the minimum function $\psi(z)$ of the given matrix A in k splits into powers of r distinct linear factors $z - \alpha_i$,

$$\psi(z) = \prod_{i=1}^{r} (z - \alpha_i)^{m_i}.$$

2. Let $P(w, z)$ be a polynomial of two variables w, z in k which actually contains w; we wish to solve the equation $P(W, A) = 0$ by a matrix $W = f(A)$ in $R[A]$. Considering the equation $P(w, z) = 0$ as defining an algebraic function $w(z)$, we speak of the finite Taylor expansion

$$(1) \qquad \phi(z) = \beta + \beta_1(z - \alpha) + \cdots + \beta_{h-1}(z - \alpha)^{h-1}$$

---

* Received May 4, 1942.

as a branch at $z = \alpha$ which is regular up to the h-th order if

$$P(\phi(z), z) \equiv 0 \ (\mathrm{mod}\ (z - \alpha)^h).$$

The condition $P(f(A), A) = 0$ amounts to

$$P(f(z), z) \equiv 0 \ (\mathrm{mod}\ \psi(z)),$$

or to the set of r simultaneous congruences

$$P(f(z), z) \equiv 0 \ (\mathrm{mod}\ (z - \alpha_i)^{m_i}) \qquad (i = 1, \cdots, r),$$

whence follows this almost trivial

Principle of Construction.[1] *If for each $i = 1, \cdots, r$ we can obtain a branch $\phi_i(z)$ at $z = \alpha_i$ which is regular up to the order m_i then the simultaneous congruences*

$$f(z) \equiv \phi_i(z) \ (\mathrm{mod}(z - \alpha_i)^{m_i}) \qquad (i = 1, \cdots, r)$$

uniquely determine a solution $W = f(A)$ of our equation $P = 0$. Every solution $W \in R[A]$ is determined in this fashion.

Special cases:

a) For $\alpha = \alpha_i$ choose a root $w = \beta$ of the equation $P(w, \alpha) = 0$. If

$$\frac{\partial P(\beta, \alpha)}{\partial w} \neq 0 \tag{2}$$

we get a uniquely determined regular branch $\phi(z)$ at $z = \alpha$ which begins with the constant term β and may be continued up to an arbitrarily high order.

---

[1] For the general formulation of this principle of which I had proved a special case, I am indebted to Professors MacDuffee and Weyl. The principle is quite comprehensive; our special case (b), for example, embraces the main results in W. E. Roth's paper, "A solution of the matrix equation $P(X) = A$," *Transactions of the American Mathematical Society*, vol. 30 (1928), pp. 579-596. A different approach to the problem is to be found in the first half of M. H. Ingraham's paper, "Rational methods in matrix equations," *Bulletin of the American Mathematical Society*, vol. 47 (1941), pp. 61-70.

b) An equation of the type $F(w) = z$. The condition (2), $F'(\beta) \neq 0$, is not only sufficient, but also necessary for the regular expansion $\phi(z)$ at $z = \alpha$ to permit continuation beyond the initial term β. Combining this with the second conclusion in **1**, we obtain the following result: Under the hypothesis that A is non-derogatory, the equation $F(W) = A$ is solvable if and only if for every *multiple* root α of the minimum equation $\psi(z) = 0$ of A the equation $F(w) - \alpha = 0$ has at least one *simple* root $w = \beta$. If for every $i = 1, \cdots, r$ we choose β_i as a root of $F(w) - \alpha_i = 0$, which is simple in case α_i is a multiple root of $\psi(z) = 0$, then there is exactly one solution $W = f(A)$ satisfying the r conditions $f(\alpha_i) = \beta_i$, and all solutions W are obtained in this fashion.

c) The equation $w^q = z$, q a positive integer. By case (b) the matrix equation $W^q = A$ has a solution which is a polynomial in A if and only if 0 is not a multiple root of the minimum equation of A. The polynomial (1) is seen to have coefficients given by the formulas

$$\beta = \alpha^{1/q},\ \beta_1 = \beta\alpha^{-1}(1/q),\ (i+1)\beta_{i+1} = \beta_i\alpha^{-1}(1/q - i)\quad (i = 1, \cdots, h-2),$$

and thus we have found the branches needed in the construction. Since $\alpha^{1/q}$ has q values unless $\alpha = 0$, we have found the q^r (or q^{r-1} if some α_i is 0) solutions W which are polynomials in A.

PURDUE UNIVERSITY.

ON SOME CUBIC DIOPHANTINE EQUATIONS.*

By E. Rosenthall.

1. In this paper we shall adhere to the following notation: large capital letters $A, B, \cdots$ (with or without subscripts) will represent integers of the quadratic number field $Ra(\rho)$ where $\rho = \frac{1}{2}(-1 + i\sqrt{3})$; the letter ϵ will be reserved for the units of this field. Small latin letters $a, b, \cdots$ represent rational integers, and the conjugate of a number X is denoted by $\bar{X}$. The symbol $(A, B, C, \cdots)$ signifies the g. c. d. of $A, B, C, \cdots$; $A \mid B$ means A divides B, and $(A, B) = 1$ denotes that A and B are coprime.

The integers of the field $Ra(\rho)$ are of the form $c + d\rho$, or of the form $\frac{1}{2}(a + \sqrt{-3b})$ with a and b of the same parity; also $\epsilon\bar{\epsilon} = 1$.

2. In this paper we solve completely in integers of $Ra(\rho)$ a multiplicative equation of the form

$$wW\bar{W} = nN\bar{N} \tag{2.1}$$

and deduce from this the complete rational integer solution of certain interesting cubic diophantine equations. The method of proof will indicate how some types of diophantine equations reducible in a quadratic field may be solved completely.

Since the integers of $Ra(\rho)$ obey the fundamental law of arithmetic, namely unique decomposition into prime factors, multiplicative equations in this field can be solved completely in parametric form by the method of reciprocal arrays.[1]

To solve (2.1) we require the following fundamental lemma.

Lemma 1.[1] *All integral solutions of*

$$XY = ZW$$

are given by $X = US$, $Y = VT$, $Z = UT$, $W = VS$; *and it suffices to take* $(S, T) = 1$.

This lemma (as stated in the paper referred to) is for non-zero integers X, Y, Z, W. However the solution given includes all the possible zero solu-

---

* Received August 19, 1942; Revised February 19, 1943.

[1] E. T. Bell, "Polynomial diophantine systems," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 909-910. Also E. T. Bell, "Reciprocal arrays and diophantine analysis," *American Journal of Mathematics*, vol. 35 (1933), pp. 50-66.

tions without violating the conditions on S and T. For example, the solution $X = Z = W = 0$ is obtained by taking $U = S = 0$, $T = 1$, and S, T are coprime.

3. We now solve equation (2.1).

THEOREM 1. *The complete solution in integers w, W, n, N satisfying equation* (2.1) *is given by*

$$W = SUL, \quad w = tV\bar{V}, \quad N = S\bar{U}\bar{V}, \quad n = tL\bar{L} \tag{3.1}$$

and it suffices to take $(UL, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$.

Proof. By Lemma 1 all required values of w, W, n, N are of the form

$$\begin{aligned} W &= R\,A\,F & N &= R\,B\,H \\ \bar{W} &= S\,B\,G & \bar{N} &= S\,C\,F \\ w &= T\,C\,H & n &= T\,A\,G \end{aligned} \tag{3.2}$$

with the g. c. d. conditions

$$(W, N) = R, \quad (\bar{W}, \bar{N}) = S, \quad (w, n) = T. \tag{3.3}$$

From (3, 3) it follows that $R = \epsilon\bar{S}$; substituting this in (3.2) we see that the parameters must be restricted to satisfy

$$\epsilon\,A\,F = \bar{B}\,\bar{G} \quad \text{and} \quad \epsilon\,B\,H = \bar{C}\,\bar{F}. \tag{3.4}$$

Hence (3.2) becomes

$$\begin{aligned} W &= \bar{S}\,\bar{B}\,\bar{G} & N &= \bar{S}\,\bar{C}\,\bar{F} \\ w &= T\,C\,H & n &= T\,A\,G \end{aligned} \tag{3.5}$$

with conditions (3.4). By Lemma 1 all A, F, B, G satisfying $(3.4)_1$ are given by

$$\bar{B} = A_1B_1, \quad \bar{G} = C_1D_1, \quad \epsilon A = A_1D_1, \quad F = C_1B_1.$$

Then $(3.4)_2$ becomes $\epsilon\bar{A}_1H = \bar{C}\bar{C}_1$ and so

$$\epsilon\bar{A}_1 = A_2B_2, \quad H = C_2D_2, \quad \bar{C} = A_2D_2, \quad \bar{C}_1 = C_2B_2.$$

Substituting these values in (3.5) we have

$$\begin{aligned} W &= \epsilon\bar{S}\bar{A}_2\bar{C}_2(\bar{B}_2B_1)(\bar{B}_2D_1), & N &= \bar{S}A_2C_2D_2(B_2\bar{B}_1) \\ w &= T\bar{A}_2C_2(D_2\bar{D}_2), & n &= T\bar{A}_2C_2(\bar{B}_2D_1)(B_2\bar{D}_1). \end{aligned}$$

Since the integers B_2, B_1, D_1 always occur as factors of the products $\bar{B}_2B_1, \bar{B}_2D_1$ we can put $\bar{B}_2B_1 = K$ and $\bar{B}_2D_1 = L$; we also make the reversible substitution $D_2 = \epsilon\bar{V}$ and replace $\epsilon\bar{S}$ by the parameter S. Then we have

$$(3.6)\quad W = S(\bar{A}_2\bar{C}_2K)L,\ w = (T\bar{A}_2C_2)V\bar{V},\ N = S(A_2C_2\bar{K})\bar{V},\ n = (T\bar{A}_2C_2)L\bar{L},$$

where $T\bar{A}_2C_2$ must be restricted to be a rational integer. Thus (3.6) is a complete parametric representation for the indeterminates satisfying (2.1).

However if we put $\bar{A}_2\bar{C}_2K = U$, $T\bar{A}_2C_2 = t$ then all numbers in (3.6) are included in (3.1). But by direct substitution we see that all the numbers (3.1) are solutions of (2.1). Hence (3.1) is the complete solution of (2.1) where S, U, L, V are arbitrary integers in $Ra(\rho)$, and t is any rational integer.

4. From Theorem 1 we can deduce the complete rational integer solution of $x^3 + y^3 = z^3 + w^3$. This equation can be put in the form (2.1), viz.,

$$(4.1)\qquad (X + \bar{X})X\bar{X} = (Y + \bar{Y})Y\bar{Y}$$

where $X = \frac{1}{2}[(x + y) + \sqrt{-3}(x - y)]$, $Y = \frac{1}{2}[(z + w) + \sqrt{-3}(z - w)]$ are integers of $Ra(\rho)$. If $x = y$, we have the trivial solutions with $x = z$.

In **6** we prove the following theorem.

THEOREM 2. *The set of all integers X, Y (with $X \neq Y$) satisfying (4.1) is given by*

$$(4.2)\qquad X = (\sqrt{-3}p/\theta q)(e + f\rho)LU,\quad Y = (\sqrt{-3}p/\theta q)(e + f\rho)\bar{U}\bar{V}$$

where $e + f\rho = UV \cdot V\bar{V} - \bar{U}\bar{L} \cdot \bar{L}L$, and θ is 1 unless $\sqrt{-3} \mid (e + f\rho)$ and then $\theta = 3$; it suffices to take $q = (e, f)$ and $(LU, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$.

5. To obtain the solution of (4.1) in the form (4.2) we require the following three lemmas.

LEMMA 2. *If the product XY is pure imaginary and if $X = aA$ where A is not divisible by a rational integer then $Y = \sqrt{-3}b\bar{A}/\theta$ where θ is 1, or 3 if $\sqrt{-3} \mid A$.*

Proof. Put $X = aA = a(m + n\rho)$ and $Y = b(s + t\rho)$ where $(m, n) = (s, t) = 1$. Then since XY is pure imaginary we must have

$$ns + mt - nt = 2(ms - nt)$$

from which it follows that $m(2s - t) = n(s + t)$; and since $(m, n) = 1$ we have $2s - t = kn$, $s + t = km$. Hence $s = k(n + m)/3$, $t = k(2m - n)/3$ and since $(s, t) = 1$ we can write

$$s = (n + m)/\theta, \qquad t = (2m - n)/\theta,$$

which gives $Y = b\{(m + n) + (2m - n)\rho\}/\theta = b\sqrt{-3}\bar{A}/\theta$.

LEMMA 3. *If* $D = (l + m\rho)$ *is a g.c.d. of* $a + b\rho$ *and* $c + d\rho$, *then* $(a, b, c, d) = (l, m)$.

Proof. We know that if a rational integer t divides $a + b\rho$ then t divides both a and b.

Therefore if $q = (l, m)$, then $q | D$ and q is a divisor of a and b, c and d; and if $s = (a, b, c, d)$ then $q | s$. Also s divides $a + b\rho$, $c + d\rho$, and therefore $s | D$. Then $s | l$ and $s | m$, and therefore $s | q$. Hence $s | q$, $q | s$ and so $s = q$, $(a, b, c, d) = (l, m)$.

LEMMA 4. *If the matrix of the coefficients is of rank 2, then all integral solutions of the simultaneous equations*

$$AX + BY + CZ = 0, \qquad A_1X + B_1Y + C_1Z = 0$$

are given by

$$X = (E/D)(BC_1 - B_1C), Y = (E/D)(A_1C - AC_1), Z = (E/D)(AB_1 - A_1B)$$

where E *is an arbitrary integer and it suffices to take* $D = (BC_1 - B_1C, A_1C - AC_1, AB_1 - A_1B)$.

For, solving algebraically, X, Y, Z are certainly given by

$$X = \alpha(BC_1 - B_1C), \quad Y = \alpha(A_1C - AC_1), \quad Z = \alpha(AB_1 - A_1B)$$

where α is in the field; writing α in lowest terms E/D, D must be a divisor of $BC_1 - B_1C$, etc. Hence, multiplying up by a suitable factor, it suffices to take D as stated, and E is arbitrary.

The case where the matrix is of rank 0 or 1 will be considered in the application to which the lemma is put.

6. We now prove Theorem 2. Considering each element of (4.1) as independent, then by Theorem 1 all X, Y satisfying (4.1) are of the form

$$(6.1) \qquad X = SUL, \quad X + \bar{X} = tV\bar{V}, \quad Y = S\bar{U}\bar{V}, \quad Y + \bar{Y} = tL\bar{L}$$

with $(UL, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$ and the parameters must be restricted to satisfy

$$SUL + \bar{S}\bar{U}\bar{L} - tV\bar{V} = 0, \qquad S\bar{U}\bar{V} + \bar{S}UV - tL\bar{L} = 0.$$

Considering this system in the indeterminates S, $\bar{S}$, t and taking the matrix of the coefficients to be of rank 2, then by Lemma 4 all S, $\bar{S}$ are given by

$$S = (E/D)(e + f\rho), \qquad \bar{S} = -(E/D)(e + f\rho^2)$$

where E is arbitrary and it suffices to take $D = (e + f\rho, e + f\rho^2)$. From the expressions for S and $\bar{S}$ we get $E\bar{D} + \bar{E}D = 0$, and hence $E\bar{D}$ is pure imaginary; thus by Lemma 2 we can put

$$D = (k + l\rho) = qK, \qquad E = \sqrt{-3}\, pK/\theta \text{ with } q = (k, l),$$

and $\theta = 1$ unless $\sqrt{-3} \mid K$, that is $\sqrt{-3} \mid (e + f\rho)$ and then $\theta = 3$; from Lemma 3 it follows that $q = (e, f)$. Hence

$$E/D = \sqrt{-3}\, p/\theta q$$

where $q = (e, f)$, and Theorem 2 is proved for the case considered.

If the matrix of the coefficients is of rank 1 then $\bar{\epsilon}V = \epsilon L = \bar{U}$ or $U = 0$ giving in (6.1) the trivial solution $X = Y$; and if the matrix is of rank 0 we get $X = Y = 0$, a solution which is included in (4.2).

7. From Theorem 2 we deduce the following result.

THEOREM 3. *All sets of rational integral values x, y, z, w (except $x = z$) satisfying the equation*

$$x^3 + y^3 = z^3 + w^3 \tag{7.1}$$

are given by

$$x = p(a - 2b)/\theta q, \; y = -p(a + b)/\theta q, \; z = p(c - 2d)/\theta q, \; w = -p(c + d)/\theta q$$

where $a + b\rho = UL(e + f\rho)$, $c + d\rho = \bar{U}\bar{V}(e + f\rho)$, and q, θ, $e + f\rho$, are as defined in Theorem 2.

For from Theorem 2 we have

$$\tfrac{1}{2}[(x + y) + \sqrt{-3}(x - y)] = (\sqrt{-3}p/\theta q)(a + b\rho),$$
$$\tfrac{1}{2}[(z + w) + \sqrt{-3}(z - w)] = (\sqrt{-3}p/\theta q)(c + d\rho).$$

Equating real and imaginary parts and solving for x, y, z, w yields the required results. Further it suffices to take $(UL, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$.

8. The complete solution of the equation $mM\bar{M} = nN\bar{N}$ also yields the complete rational integral solution[2] of

$$x^3 + y^3 + z^3 - 3xyz = u^3 + v^3 + w^3 - 3uvw, \tag{8.1}$$

---

[2] R. D. Carmichael, *Diophantine Analysis*, New York, 1915, pp. 63-65, discusses this equation and gives a four parameter rational solution.

where

$$m = x + y + z,\quad M = (x - z) + \rho(y - z),\quad n = u + v + w,$$
$$N = (u - w) + \rho(v - w).$$

Then from (3.1) all rational integers satisfying (8.1) are obtainable from

$$\begin{aligned} &x + y + z = tV\bar{V} \equiv a, && u + v + w = tL\bar{L} \equiv a_1, \\ &(x - z) + \rho(y - z) = SUL \equiv b + c\rho, && (u - w) + \rho(v - w) = S\bar{U}\bar{V} \equiv b_1 \end{aligned} \tag{8.2}$$

and it suffices to take $(L\bar{L}, VV) = (UL, \bar{U}\bar{V}) = 1$. Solving (8.2) for x, y, z, u, v, w gives

$$\begin{aligned} &x = \tfrac{1}{3}(a + 2b - c), && y = \tfrac{1}{3}(a - b + 2c), && z = \tfrac{1}{3}(a - b - c) \\ &u = \tfrac{1}{3}(a_1 + 2b_1 - c_1), && v = \tfrac{1}{3}(a_1 - b_1 + 2c_1), && w = \tfrac{1}{3}(a_1 - b_1 - c_1 \end{aligned} \tag{8.3}$$

for the complete rational integral solution of (8.1).

It is desirable to know how the parameters t, U, L, V are to be selected so that (8.3) always gives integers. These expressions are integers if $a + 2b - c \equiv 0 \pmod 3$ and $a_1 + 2b_1 - c_1 \equiv 0 \pmod 3$. This implies

$$\begin{aligned} tV\bar{V} + SUL + \bar{S}\bar{U}\bar{L} &\equiv 0 \pmod 3 \\ tL\bar{L} + S\bar{U}\bar{V} + \bar{S}UV &\equiv 0 \pmod 3. \end{aligned} \tag{8.4}$$

If both UL and $\bar{U}\bar{V}$ are prime to 3, then it suffices to take S satisfying the linear congruence $AS \equiv B \pmod 3$, where $UL \cdot UV - \bar{U}\bar{L} \cdot \bar{U}\bar{V} = A$, and $-t(V\bar{V}\,\bar{U}\bar{V} - L\bar{L} \cdot UL) = B$. This congruence is solvable if $(3, A) \mid B$. Since A is divisible by $\sqrt{-3}$, B will have to contain the factor $\sqrt{-3}$ or $(\sqrt{-3})^2$ If it happens that U, L, V are selected so that $(V\bar{V} \cdot UV - UL \cdot L\bar{L})$ does not contain this factor then it suffices to select $t \equiv 0 \pmod 3$; otherwise t is arbitrary.

If one of $UL, \bar{U}\bar{V}$ (say UL) is not prime to 3, then it suffices to take $\bar{U}\bar{V}$ prime to 3; whence U must be prime to 3. Thus L must be divisible by $\sqrt{-3}$, and it follows therefore that $L\bar{L} \equiv 0 \pmod 3$ and $SUL + \bar{S}\bar{U}\bar{L} \equiv 0 \pmod 3$. Hence (8.4) becomes

$$tV\bar{V} \equiv 0 \pmod 3,\qquad S\bar{U}\bar{V} + \bar{S}UV \equiv 0 \pmod 3. \tag{8.5}$$

From $(8.5)_1$ we have $t \equiv 0 \pmod 3$ since $V\bar{V}, L\bar{L}$ are coprime; and $(8.5)_2$ holds if and only if $\sqrt{-3}$ divides $S\bar{U}\bar{V}$, whence $\sqrt{-3}$ divides S since $\bar{U}\bar{V}$ is prime to 3. Summing up, we have, in this case, that (8.3) yields integers if and only if S, L (or V) is divisible by $\sqrt{-3}$, and $t \equiv 0 \pmod 3$.

9. As a further illustration of the method used here for solving diophantine equations, we now solve completely in rational integers the equation[3]

$$x^3 + y^3 = z^2. \tag{9.1}$$

For this purpose we require the following lemmas.

Lemma 5. *If the product XYZ is a rational integer then all values of X, Y, Z are given by*

$$X = aAB, \quad Y = b\bar{A}C, \quad Z = c\bar{B}\bar{C}.$$

Proof. Precisely as in Lemma 2 it follows that if the product RS is a rational integer then all R, S are given by $R = aK$, $S = p\bar{K}$. Hence since $X(YZ)$ is rational we can put $X = aK$, $YZ = p\bar{K}$; whence by Lemma 1, we have

$$Y = FG, \quad Z = HJ, \quad p = FJ, \quad \bar{K} = HG.$$

Again, since FJ is rational it suffices to take $F = bC$, $J = c\bar{C}$; make the change in parameters $H \rightarrow \bar{B}$, $G \rightarrow \bar{A}$, and the lemma is proved.

We note that it suffices to take A, cC coprime (Lemma 1); further any factor common to a and c can be absorbed into the B and so we can take $(a, c) = 1$. These facts will be required in **10.**

Lemma 6. *The complete set of integer values X, Y satisfying $X\bar{X} = Y\bar{Y}$ is given by $X = \epsilon ST$, $Y = S\bar{T}$, and it suffices to take T divisible by no rational integer.*

Proof. This equation $X\bar{X} = Y\bar{Y}$ can be considered as a particular case of Theorem 1 by placing $w = n = 1$, from which it follows in (3.1) that $t = 1$, $V = \epsilon_1$, $L = \epsilon_2$. On making the reversible substitution $U = \bar{\epsilon}_1 T$, $\bar{\epsilon}_1 \epsilon_2 = \epsilon$ we obtain the required result.

Lemma 7. *All integers X, z satisfying $X\bar{X} = z^2$ are given by $X = \epsilon aT^2$, $z = aT\bar{T}$.*

For, from Lemma 6 all required X, z are given by $X = \epsilon ST$, $z = S\bar{T}$, where $S\bar{T}$ must be a rational integer; then we can put $S = aT$, since T is divisible by no rational integer.

Lemmas 6 and 7 furnish the complete rational integral solutions of each of the equations[4] $x^2 + my^2 = z^2 + mw^2$ and $x^2 + my^2 = u^2$ for those m in which the integers of the quadratic field $Ra(\sqrt{-m})$ are uniquely decomposable into prime factors. The complete solutions of these equations for any value of m are readily obtained by other methods,[5] but for our purpose the form of solution given here is desirable since it occurs in multiplicative form.

---

[3] L. E. Dickson, *History of the Theory of Numbers*, vol. 2, pp. 578-581 gives an account of the investigations on (9.1).

[4] If $m \equiv 1\,(4)$ replace form $u^2 + mv^2$ by $u^2 + uv + \frac{1}{4}(1 - m)v^2$.

10. We now return to equation (9.1). Defining the integer X as in **4**, this equation takes the form

$$(X + \bar{X})X\bar{X} = z \cdot z \cdot 1. \tag{10.1}$$

Then from (3.1) it follows that all X, $X + \bar{X}$, z of (10.1) are given by

$$X = SUL, \quad X + \bar{X} = tV\bar{V}, \quad z = S\bar{U}\bar{V}, \tag{10.2}$$

where t and L must satisfy $tL\bar{L} = 1$. Thus $t = 1$ (since $L\bar{L} > 0$), $L = \epsilon$; and with the reversible substitution $U = \bar{\epsilon}K$, $V = \epsilon P$, (10.2) becomes

$$X = SK, \quad X + \bar{X} = P\bar{P}; \quad z = S\bar{K}\bar{P},$$

and it suffices to take K divisible by no rational integer (it can be absorbed into S).

Since $S\bar{K}\bar{P}$ must be a rational integer, then

$$S = aAB, \quad \bar{K} = \bar{A}C, \quad \bar{P} = c\bar{B}\bar{C}$$

and it suffices here to have c prime to a and A (see remarks at the end of Lemma 5), and we have

$$X = aA^2R, \quad X + \bar{X} = c^2R\bar{R}, \quad z = acA\bar{A}R\bar{R} \tag{10.3}$$

in which, since B and C always occur as the product $B\bar{C}$, we have put $B\bar{C} = R$.

Further, the parameters of (10.3) must satisfy $c^2R\bar{R} - aA^2R - a\bar{A}^2\bar{R} = 0$ which is equivalent to

$$(c^2R - a\bar{A}^2)(c^2\bar{R} - aA^2) = (aA\bar{A})^2.$$

Put $c^2R - a\bar{A}^2 = H$, then (10.3) becomes

$$X = aA^2\left(\frac{H + a\bar{A}^2}{c^2}\right), \qquad z = acA\bar{A}\left(\frac{H + a\bar{A}^2}{c^2}\right)\left(\frac{\bar{H} + aA^2}{c^2}\right) \tag{10.4}$$

where $(aA\bar{A})^2 = H\bar{H}$. Whence, by Lemma 1,

$$H = \epsilon bT^2, \quad aA\bar{A} = bT\bar{T}. \tag{10.5}$$

Solving $(10.5)_2$ by Theorem 1 we finally get

---

[5] L. E. Dickson, *Introduction to the Theory of Numbers*, pp. 40-41; also see Ex. 5, p. 43.

$$H = \epsilon m G\bar{G}(M\bar{F}\bar{E})^2, \quad a = mE\bar{E}, \quad A = MFG. \tag{10.6}$$

Again, since R is an integer we must have c^2 dividing $H + a\bar{A}^2$, that is it must divide $m\bar{G}\bar{F}^2\bar{E}(\epsilon GM^2\bar{E} + E\bar{M}^2\bar{G})$. Since c is prime to a and A, then we have c^2 prime to each of m, E, F, G, M. Hence c^2 must divide $(E\bar{M}^2\bar{G} + \epsilon\bar{E}M^2G)$.

Substituting (10.6) in (10.4) we have

$$X = (mF\bar{F})^2(E\bar{G})(\bar{E}G)^2M^2\left(\frac{E\bar{M}^2\bar{G} + \epsilon\bar{E}M^2G}{c^2}\right),$$

$$z = c(mF\bar{F})^3(\bar{E}G)^2(E\bar{G})^2M\bar{M}\left(\frac{E\bar{M}^2\bar{G} + \epsilon\bar{E}M^2G}{c^2}\right)\left(\frac{\bar{E}M^2G + \bar{\epsilon}E\bar{M}^2\bar{G}}{c^2}\right).$$

Put $mF\bar{F} = k$ and $E\bar{G} = \bar{K}$, and we get

$$\begin{aligned} X &= k^2K^2\bar{K}M^2\left(\frac{\bar{K}\bar{M}^2 + \epsilon KM^2}{c^2}\right) \\ z &= \bar{\epsilon}ck^3(K\bar{K})^2M\bar{M}\left(\frac{\bar{K}\bar{M}^2 + \epsilon KM^2}{c^2}\right), \end{aligned} \tag{10.7}$$

and it suffices to have c prime to each of K, M and k.

All the integer solutions of (10.1), and hence of (9.1), are given by (10.7), and in order to obtain all the integers in (10.7) we can take c, k arbitrary and M any integer prime to c and then select the coördinates of K to make $\left(\dfrac{\bar{K}\bar{M}^2 + \epsilon KM^2}{c^2}\right)$ integral. This is done in the following way.

Put $K = u + v\rho$, and let $M^2 = m + n\rho$, (whence m, n, c^2 are coprime). Then if we take $\epsilon = 1$, ρ, ρ^2 in succession the congruence $\bar{K}\bar{M}^2 + \epsilon KM^2 \equiv 0 \pmod{c^2}$ becomes

$$u(2m - n) \equiv (m + n)v \pmod{c^2}, \quad u(m - 2n) \equiv (2m - n)v \pmod{c^2},$$
$$u(m + n) \equiv (2n - m)v \pmod{c^2}$$

respectively; and for $\epsilon = -1$, $-\rho$, $-\rho^2$ respectively we get the congruences

$$un \equiv v(n - m) \pmod{c^2}, \quad um \equiv vn \pmod{c^2}, \quad u(n - m) \equiv vm \pmod{c^2}.$$

Since c^2 is prime to at least one of the coördinates m or n it follows that in the last three congruences either the coefficient of u or of v (at least) must be prime to c^2. If this coefficient is the coefficient of u, then select v arbitrary and solve the congruence for the unique u, and similarly solve for the unique v with arbitrary u if the coefficient of v is prime to c^2.

In the first three congruences, also one of the coefficients (that is, of u or v) must be prime to c^2, and so we can always find a unique u or v. For example, (considering the first congruence), suppose that $2m - n$, $m + n$, c^2 have the factor $p \neq 3$ in common. Then $p | 3m$ and hence divides m, n, c^2, which is impossible since c^2, M^2 are coprime.

Also if 3 is a factor of c^2 then $2m - n$ (and hence $m + n$) is prime to 3, for otherwise $M^2 = \frac{1}{2}(2m - n + \sqrt{-3}n)$ would be divisible by $\sqrt{-3}$, and so M^2 would have a factor in common with c^2.

Thus in all cases either the coefficient of u or v is prime to c^2, and K can be found.

11. The equations which we have solved completely in rational integers, viz. (7.1), (8.1), and (9.1) are merely particular cases of the equation (2.1). By suitably selecting the indeterminates w, W, n, N in (2.1) the complete solution to many other interesting equations can be obtained; for example any of the equations $x^3 + y^3 + z^3 - 3xyz = u^2$, $u^2 + 3v^2$, $u^3 + v^3$; or $(x^3 + y^3)(u^3 + v^3) = (w^3 + z^3)$. Moreover Theorem 1 holds in any normal algebraic field whose integers can be decomposed uniquely into prime factors.

McGill University,
Montreal, Canada.

A NOTE ON SOUSLIN'S PROBLEM.*

By Edwin W. Miller.

As is well known, the following properties characterize the linear continuum L as a type of linear order.

(1) L has no first and no last element.

(2) L is continuous; i.e., every cut in L is a Dedekind cut.

(3) There exists a denumerable subset D of L such that between any two elements of L there is an element of D.

As an immediate consequence of (3) we have

(4) Any set of non-overlapping intervals on L is countable.

Souslin raised the following question.[1] Do properties (1), (2) and (4) characterize the linear continuum? In other words, do properties (1), (2) and (4) imply property (3)?

The purpose of the present article is to prove the following

THEOREM. *In order that there exist a linear order which possesses properties* (1), (2) *and* (4) *without possessing property* (3) *it is necessary and sufficient that there exist a partial order* P *of power* $\aleph_1$ *such that*

(a) *if* $Q \subset P$ *and* $\bar{\bar{Q}} = \aleph_1$, *then* Q *contains two comparable elements and two non-comparable elements;*

(b) *if* x *and* y *are non-comparable elements of* P, *then there exists no* z *in* P *such that* $x < z$ *and* $y < z$.

Proof of the necessity. Suppose there exists a linear order M possessing properties (1), (2) and (4) but not property (3). Let I_1 be any interval on M, and let α be any ordinal $< \Omega$. Suppose an interval I_β has been defined on M for every $\beta < \alpha$. There must exist an interval I on M which contains no endpoint of any interval I_β $(\beta < \alpha)$, for otherwise the denumerable set consisting of the end points of the intervals I_β $(\beta < \alpha)$ would be dense in M, and (3) would be satisfied. Let I_α be any proper subinterval of such an interval I. Clearly, if $\beta < \alpha$, then either I_α is a proper subinterval of I_β or else $I_\alpha \cdot I_\beta = 0$.

---

* Received July 10, 1942. Author died July 23, 1942.

[1] *Fundamenta Mathematicae*, vol. 1 (1920), p. 223.

We now define a partial order P whose elements are the intervals I_α $(\alpha < \Omega)$ by setting $I_\beta < I_\alpha$ if and only if $I_\beta \supset I_\alpha$. Clearly, $\bar{\bar{P}} = \aleph_1$. Furthermore, if I_α and I_β are non-comparable then $I_\alpha \cdot I_\beta = 0$, and (b) is at once seen to be satisfied. To see that (a) is satisfied, let Q be any nondenumerable subset of P. Now Q must contain two comparable elements, for otherwise there is determined a nondenumerable infinity of non-overlapping intervals on M. Suppose that every two elements of Q were comparable. It follows that there would exist a decreasing Ω-sequence of intervals on M: $I^{(1)}, I^{(2)}, \cdots I^{(\alpha)}, \cdots$ $(\alpha < \Omega)$. Let a_α be the left-hand endpoint of $I^{(\alpha)}$. The intervals $(a_\alpha, a_{\alpha+1})$ do not overlap, and a contradiction of (4) is obtained.

In proving the conditions sufficient we shall make use of the following theorems.[2]

THEOREM A. *If P is a partial order of power $\aleph_1$, and every subset of P of power $\aleph_1$ contains two comparable elements, then almost every*[3] *element of P is comparable with $\aleph_1$ elements of P.*

THEOREM B. *If P is a partial order of power $\aleph_1$, and every subset of P of power $\aleph_0$ contains two comparable elements, then P contains a linear order of power $\aleph_1$.*

Let B be a partial order, and A a subset of B such that the elements of A are mutually incomparable. We shall say that A is *maximal* (relative to B) provided $x \epsilon B - A$ implies that x is comparable with some element of A.

The notation $x \phi y$ will be used to mean that x and y are incomparable.

Proof of the sufficiency. Let P be a partial order of power $\aleph_1$ possessing properties (a) and (b). In virtue of Theorem A, almost every element of P is comparable with $\aleph_1$ elements of P. Furthermore, it follows from (a) and (b) that for every element x of P, at most $\aleph_0$ elements of P precede x. Accordingly we can remove a countable set from P so as to obtain a partial order P' (of power $\aleph_1$) which possesses the following properties.

(a′) If $Q \subset P'$ and $\bar{\bar{Q}} = \aleph_1$, then Q contains two comparable elements and two non-comparable elements.

---

[2] Theorem A is an immediate consequence of Lemma 5.21 in a paper by Ben Dushnik and the author, entitled "Partially ordered sets," *American Journal of Mathematics*, vol. 63 (1941), pp. 600-610. Theorem B is a special case of Theorem 5.23 of the same paper.

[3] By "almost every" element of P is meant every element with the exception of a countable set.

(b′) $x, y \in P'$, $x \phi y$ implies that there exists no z in P' such that $x < z$ and $y < z$.

(c′) $x \in P'$ implies that there exist $\aleph_1$ elements of P' which follow x.

(d′) $x \in P'$ implies that at most $\aleph_0$ elements of P' precede x.

Now let A_1 be a maximal set of mutually incomparable elements of P' such that $\bar{\bar{A}}_1 = \aleph_0$. The existence of A_1 follows from (a′) and Theorem *B*. Let Q_1 denote the set of all x in P' such that x precedes an element of A_1. In virtue of (d′) we have $\bar{\bar{Q}}_1 \leqq \aleph_0$. Let α be any ordinal $< \Omega$. Suppose that A_β and Q_β have been defined for all $\beta < \alpha$ so that $\bar{\bar{A}}_\beta = \aleph_0$, $\bar{\bar{Q}}_\beta \leqq \aleph_0$, the elements of A_β are mutually incomparable, $A_\beta \subset P' - \sum\limits_{\mu<\beta} (A_\mu + Q_\mu)$ and is maximal relative to $P' - \sum\limits_{\mu<\beta} (A_\mu + Q_\mu)$, and Q_β is the set of all x in P' such that $x <$ an element of A_β. We shall now define A_α and Q_α.

Case 1. α is a limiting ordinal.

In this case, we merely take A_α so that $\bar{\bar{A}}_\alpha = \aleph_0$, the elements of A_α are mutually incomparable, $A_\alpha \subset P' - \sum\limits_{\beta<\alpha} (A_\beta + Q_\beta)$ and is maximal relative to $P' - \sum\limits_{\beta<\alpha} (A_\beta + Q_\beta)$. We then define Q_α as the set of all x in P' such that $x <$ an element of A_α. We have $\bar{\bar{Q}}_\alpha \leqq \aleph_0$.

Case 2. α is not a limiting ordinal.

Denote the immediate predecessor of α by γ, and for any x in A_γ denote by B_x the set of all elements of $P' - \sum\limits_{\beta<\alpha} (A_\beta + Q_\beta)$ which follow x. We have $\bar{\bar{B}}_x = \aleph_1$. Let C_x be such that $\bar{\bar{C}}_x = \aleph_0$, the elements of C_x are mutually incomparable, $C_x \subset B_x$ and is maximal relative to B_x. We now denote $\sum\limits_{x \in A_\gamma} C_x$ by A_α. We have $\bar{\bar{A}}_\alpha = \aleph_0$. Let Q_α denote the set of all x in P' such that $x <$ an element of A_α. We have $\bar{\bar{Q}}_\alpha \leqq \aleph_0$. Now $A_\alpha \subset P' - \sum\limits_{\beta<\alpha} (A_\beta + Q_\beta)$. We shall show that the elements of A_α are mutually incomparable and that A_α is maximal relative to $P' - \sum\limits_{\beta<\alpha} (A_\beta + Q_\beta)$.

Let a and b be any two elements of A_α. If there exists an x in A_γ such that $a + b \subset C_x$, then certainly $a \phi b$. Suppose $a \in C_x$, $b \in C_y$, $x \neq y$. Assume $a < b$. Then $x < a$, since $a \in C_x$ and $C_x \subset B_x$. Hence $x < b$. But $y < b$ since $b \in C_y$ and $C_y \subset B_y$. However, $x \phi y$ since the elements of A_γ are mutually incomparable. But this contradicts (b′).

Now suppose that A is not maximal relative to $P' - \sum\limits_{\beta<\alpha} (A_\beta + Q_\beta)$. Then there exists an x in $P' - \{\sum\limits_{\beta<\alpha} (A_\beta + Q_\beta) + A_\alpha\}$ such that $x \phi A_\alpha$. Since

$x \epsilon P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$ and A_γ is maximal relative to $P' - \sum_{\beta<\gamma} (A_\beta + Q_\beta)$, there exists an element y of A_γ such that $y < x$. Hence $x \epsilon B_y$ and since $x \phi A_\alpha$, we have $x \phi C_y$. But this contradicts the fact that C_y is maximal relative to B_y.

We now put $A = \sum_{\alpha<\Omega} A_\alpha$.

We shall make use of the following properties of the partial order A.

(I) $A = \sum_{\alpha<\Omega} A_\alpha$; $\bar{\bar{A}} = \aleph_1$, $\bar{\bar{A}}_\alpha = \aleph_0$, $A_\alpha \cdot A_\beta = 0$ if $\alpha \neq \beta$.

(II) The elements of A_α are mutually incomparable.

(III) $x \epsilon A_\alpha$ implies that $x < \aleph_0$ elements of $A_{\alpha+1}$.

(IV) $\beta < \alpha$, $x \epsilon A_\beta$, $y \epsilon A_\alpha$ implies that $x < y$ or $x \phi y$.

(V) $\beta < \alpha$, $x \epsilon A_\alpha$ implies that there exists one and only one element x_β of A_β such that $x_\beta < x$; furthermore we have $x_1 < x_2 \cdots x_\beta \cdots < x$.

(VI) $Q \subset A$, $\bar{\bar{Q}} = \aleph_1$ implies that Q contains two comparable elements and two non-comparable elements.

Properties I, II, III, IV and VI are obvious from the way in which A was obtained. To prove V we proceed as follows. Let $x \epsilon A_\alpha$, and let β be any ordinal $< \alpha$. There exists an element x_β of A_β such that x_β is comparable with x, since A_β is maximal relative to $P' - \sum_{\mu<\beta} (A_\mu + Q_\mu)$. In virtue of IV we have $x_\beta < x$. In virtue of II and (b) there cannot be two such elements x_β. Now let μ and ν be ordinals such that $\mu < \nu < \alpha$. We cannot have $x_\mu \phi x_\nu$ since $x_\mu < x$ and $x_\nu < x$. In virtue of IV we must have $x_\mu < x_\nu$. Hence $x_1 < x_2 < \cdots x_\beta < \cdots < x$.

We now proceed to define a linear extension L of A which has the following properties.

(*) L has no first element and no last element.

(**) No denumerable set is dense in L.

(***) Any set of non-overlapping intervals on L is countable.

First, we write the elements of A_1 in a series of type $\omega^* + \omega$: $\cdots a_{-n}, \cdots a_{-1}, a_0, a_1, \cdots a_n, \cdots$.

Next, we write the elements of A_2 in the form a_{nm}, where m runs through all integers to give the set A_2, where the n indicates that $a_n < a_{nm}$, and where, for a fixed n, the integers m form a series of type $\omega^* + \omega$.

Referring to property V, and using the obvious induction, we may assume that the elements of A_α, for any $\alpha < \Omega$, have been given the form $a_{n_1 n_2 \ldots n_\mu \ldots n_\alpha}$, where n_α runs through all the integers to give the whole set A_α, and $a_{n_1 n_2 \ldots n_\mu} < a_{n_1 n_2 \ldots n_\mu \ldots n_\alpha}$. Furthermore, in case α is of the form

$\gamma + 1$ and $a_{n_1 n_2 \ldots n_\gamma}$ is a (fixed) element of A_γ, we may assume that the integers n_α appearing in the terms $a_{n_1 n_2 \ldots n_\gamma n_\alpha}$ of A_α form a series of type $\omega^* + \omega$.

Now, if x' and x'' are any two elements of A, we shall put $x' < x''$ in L if and only if (1) the series subscript of x' is a lower segment of the series subscript of x'', or (2) $n_\alpha' < n_\alpha''$, where α indicates the first place where the series subscripts of x' and x'' differ.

It is easily verified that L is a linear extension of A.

Let x be any element of L. Let n_1 denote the first term in the series subscript of x. Let n'_1 be an integer $< n_1$. Let $x' = a_{n'_1}$. It is clear that $x' < x$, so that L has no first element. In a similar way we prove that L has no last element. Thus (*) is verified.

To prove (**), let D be any denumerable subset of L. There exists an $\alpha < \Omega$ such that $D \subset \sum_{\beta < \alpha} A_\beta$. Let x be an element of A_α and y an element of $A_{\alpha+1}$ such that $x < y$ in A. Denote the series subscript of x by $n_1 n_2 \cdots n_\alpha$, and the series subscript of y by $n_1 n_2 \cdots n_\alpha n_{\alpha+1}$. Let d be any element of D. Denote its series subscript by $n'_1 n'_2 \cdots n'_\beta$. We have $\beta < \alpha$. Now suppose $x < d$ in L. Since $n_1 n_2 \cdots n_\alpha$ cannot be a lower segment of $n'_1 n'_2 \cdots n'_\beta$, there will exist a $\mu \leqq \beta$ such that $n_\lambda = n'_\lambda$ for $\lambda < \mu$ and $n_\mu < n'_\mu$. Since the series subscript of x is a lower segment of the series subscript of y, it is clear that we shall have $y < d$ in L. In other words, d cannot lie between x and y, and (**) is verified.

We now prove (***). Let p and q be any two elements of L such that $p < q$ in L. We shall show first that there exists a z in L such that $p < z$ in A, $z < q$ in L, but $z \phi q$ in A. Denote the series subscript of p by $m_1 m_2 \cdots m_\mu$ and the series subscript of q by $n_1 n_2 \cdots n_\nu$.

Case 1. $m_1 m_2 \cdots m_\mu$ is a lower segment of $n_1 n_2 \cdots n_\nu$.

In this case we may choose for z an element whose series subscript is of the form $m_1 m_2 \cdots m_\mu m_{\mu+1}$, where $m_{\mu+1} < n_{\mu+1}$.

Case 2. There exists an ordinal λ such that $\lambda \leqq \mu$, $\lambda \leqq \nu$, $m_\beta = n_\beta$ for $\beta < \lambda$, and $m_\lambda < n_\lambda$.

In this case we may choose for z any element whose series subscript is of the form $m_1 m_2 \cdots m_\mu m_{\mu+1}$.

Now let (p', q') and (p'', q'') be non-overlapping intervals of L. There exist elements z' and z'' of L such that $p' < z'$ and $p'' < z''$ in A, $z' < q'$ and $z'' < q''$ in L, but $z' \phi q'$ and $z'' \phi q''$ in A. We must have $z' \phi z''$ in A. For assume the contrary. We may assume $z' < z''$ in A. Then the series subscript of z' is a lower segment of the series subscript of z''. But $z' < q'$ in L, and $z' \phi q'$ in A. It follows easily that $z'' < q'$ in L. We have, then, $p' < z' < z'' < q'$ in L. But this is impossible since (p', q') and (p'', q'') do not overlap. It is now clear that a nondenumerable infinity of non-overlapping intervals would give rise to a nondenumerable infinity of mutually incomparable elements, z, of A. But this contradicts VI.

Now it may be that L is not continuous. However, the argument showing that L has property (***) shows also that no element of L has an immediate successor. Hence L contains no "jumps." Accordingly we can fill in the voids (by adding new elements to L) so as to obtain a continuous linear order $\mathscr{L}$.

Obviously (*) still holds for $\mathscr{L}$.

Furthermore, between any two elements of $\mathscr{L}$ there will be two elements of L. Hence (***) holds for $\mathscr{L}$.

Moreover, it is easily shown that (**) holds for $\mathscr{L}$, since (**) holds for L and (***) holds for $\mathscr{L}$.

THE UNIVERSITY OF MICHIGAN.

THE BEHAVIOR OF THE KERNEL FUNCTION AT BOUNDARY POINTS OF THE SECOND ORDER.*

By Stefan Bergman.

---

1. Introduction. A large chapter of the theory of functions is devoted to the study of the behavior of analytic functions on the boundary of their domains of definition. In the case of functions of one variable and of sufficiently regular domains, say, of domains each bounded by a rectifiable Jordan curve, no necessity arises for considering different classes of boundary points.

We meet quite a different situation in the case of analytic functions of two complex variables. The latter are defined in four dimensional domains (domains of the space of two complex variables). There is no analogue of Riemann's mapping theorem and, as we shall see, there exist essentially different classes of boundary points.

A two-dimensional simply connected domain is completely characterized by the analytic function $z^* = f(z, t)$ which maps it conformally into a circle, taking the point t into the origin, and possessing at this point the derivative 1 (mapping function). A domain, $\mathfrak{D}$, of the space of two complex variables can be characterized to a certain extent by the so-called kernel function, $K_{\mathfrak{D}}(z_1, z_2; \bar{t}_1, \bar{t}_2)$, which can be defined in the following way. Let

$$\{\phi_\nu(z_1, z_2)\}, \quad (\nu = 1, 2, 3, \cdots,) \qquad z_k = x_k + iy_k,$$

be a closed system of orthonormal functions in the domain considered, i. e. a sequence of regular analytic functions with the property that

$$(1.1) \qquad \iint\limits_{\mathfrak{D}} \phi_\nu(z_1, z_2)\overline{\phi_\mu(z_1, z_2)}\, d\omega = 1 \text{ for } \nu = \mu$$

$$= 0 \text{ for } \nu \neq \mu, \quad (\nu, \mu = 1, 2, 3, \cdots),$$

$$d\omega = dx_1 dy_1 dx_2 dy_2,$$

and such that the relation

$$(1.2) \qquad \sum_{\nu=1}^{\infty} \left| \iint\limits_{\mathfrak{D}} f(z_1, z_2)\overline{\phi_\nu(z_1, z_2)}\, d\omega \right|^2 = \iint\limits_{\mathfrak{D}} |f(z_1, z_2)|^2\, d\omega$$

---

* Received May 16, 1942.

holds for every analytic function $f(z_1, z_2)$ defined in $\mathfrak{D}$ for which the last integral is finite.[1] Then [2]

$$(1.3) \qquad K_{\mathfrak{D}}(z_1, z_2; \bar{t}_1, \bar{t}_2) = \sum_{\nu=1}^{\infty} \phi_\nu(z_1, z_2)\overline{\phi_\nu(t_1, t_2)}.$$

([1], § 1).

We shall classify boundary points of a four-dimensional domain according to the behavior of the kernel function at these points. It appears that this classification determines to a certain extent the behavior of any analytic function at these points. In the following we shall use two theorems which characterize the dependence of the kernel function upon the domain.

THEOREM A. *If a PT (pseudo-conformal transformation) i. e. a one-to-one transformation by a pair of analytic functions,*

$$(1.4) \qquad z^*_1 = w_1(z_1, z_2), \qquad z^*_2 = w_2(z_1, z_2),$$

maps the domain $\mathfrak{D}$ *into* $\mathfrak{D}^*$, *then*

$$(1.5) \quad K_{\mathfrak{D}}(z_1, z_2; \bar{z}_1, \bar{z}_2) = K_{\mathfrak{D}^*}(z^*_1, z^*_2; \bar{z}^*_1, \bar{z}^*_2) \,|\, \partial(z^*_1, z^*_2)/\partial(z_1, z_2) \,|^2.$$

THEOREM B. *If the domain* $\mathfrak{D}$ *is contained in the domain* $\mathfrak{A}$ $(\mathfrak{D} \subset \mathfrak{A})$ *then*

$$(1.6) \qquad K_{\mathfrak{D}}(z_1, z_2; \bar{z}_1, \bar{z}_2) \geqq K_{\mathfrak{A}}(z_1, z_2; \bar{z}_1, \bar{z}_2).$$

(See [1], § 1; [3], p. 100 and V; [4], IV).

Remark. The relation (1.5) enables us to define the kernel function for non-schlicht domains which can be transformed pseudo-conformally into schlicht ones.

Theorem B enables us to give bounds for the kernel function of a domain with the aid of *standard domains of comparison*; i. e., by introducing simpler domains which either contain the domain considered or are contained in it, and for which the kernel function can be easily calculated or at least estimated.

---

[1] The existence of systems $\{\phi_\nu(z_1, z_2)\}$ can be proved for a large class of domains, in particular for all bounded domains ([4], p. 28).

The numbers in brackets refer to the bibliography p. 700. The Roman number indicates the chapter of the corresponding paper.

For the convenience of the reader we often indicate several places in which the proof or the explanations of the results used are given, so that our indications partially overlap.

[2] The series (1.3) converges uniformly in every closed sub-domain of $\mathfrak{D}$. The kernel function is therefore a regular function of $(z_1, z_2) \epsilon \mathfrak{D}$, $(t_1, t_2) \epsilon \mathfrak{D}$ ([4] p. 23).

Of course, the kernel function can be defined also for two dimensional domains $\mathfrak{B}^2$. If $\mathfrak{B}^2$ is simply connected, the kernel function is connected with the mapping function by a simple relation. See [4], p. 33.

In order to illustrate the basic idea of the method used, let us apply it to the case of a domain of one complex variable. Suppose $\mathfrak{B}^2$ is a bounded domain of the z-plane which we suppose for the sake of simplicity to be convex. We assume further that the radius of curvature of the boundary $\mathfrak{E}^1$ of $\mathfrak{B}^2$ in the neighborhood of the boundary point Q is positive. Thus we can inscribe a circle $\mathfrak{T}^2$, $\mathfrak{T}^2 \subset \mathfrak{B}^2$, whose boundary curve passes through Q and has a common tangent with $\mathfrak{E}^1$ at Q. We assume that Q is the origin and that the inner normal at Q coincides with the positive x-axis. The half plane $\mathfrak{A}^2 = \mathrm{E}[x > 0]$ will obviously contain $\mathfrak{B}^2$. ($\mathfrak{T}^2$ is an interior and $\mathfrak{A}^2$ an exterior domain of comparison). Since $\mathfrak{T}^2 \subset \mathfrak{B}^2 \subset \mathfrak{A}^2$ we have

$$(z+\bar{z})^2 \mathrm{K}_{\mathfrak{A}^2}(z,\bar{z}) \leqq (z+\bar{z})^2 \mathrm{K}_{\mathfrak{B}^2}(z,\bar{z}) \leqq (z+\bar{z})^2 \mathrm{K}_{\mathfrak{T}^2}(z,\bar{z}).$$

Since

$$\mathrm{K}_{\mathfrak{A}^2}(z,\bar{z}) = \pi^{-1}(z+\bar{z})^{-2}, \qquad \mathrm{K}_{\mathfrak{T}^2}(z,\bar{z}) = \pi^{-1}(z+\bar{z}+|z|^2/r)^{-2}$$

(r being the radius of $\mathfrak{T}^2$) we have

$$\lim_{z\to 0}\,(z+\bar{z})\mathrm{K}_{\mathfrak{A}^2}(z,\bar{z}) = \lim_{z\to 0}\,(z+\bar{z})^2 \mathrm{K}_{\mathfrak{T}^2}(z,\bar{z}) = \pi^{-1},$$

provided that we approach Q in the sector $[2\,|z|/(z+\bar{z})] \leqq 1/\cos\alpha$, $|\alpha| < \pi/2$. It follows that

$$\lim_{z\to 0}\,(z+\bar{z})^2 \mathrm{K}_{\mathfrak{B}^2}(z,\bar{z}) = 1/\pi.$$

Notation. Manifolds will be denoted by German letters, the upper index indicating the dimension of the manifold. However, we omit this upper index in the case of four-dimensional manifolds.

In operating with sets we use the usual symbols: S, $+$ (sumset), $-$ (difference set), $\cdot$ (intersection), $\times$ (topological product of domains) etc.[3] E. g. we denote by $\underset{\gamma\,\epsilon\,\mathfrak{M}^m}{\mathrm{S}}\ \mathfrak{E}^n(\gamma)$ the sumset of a family of sets which depends upon a parameter γ running through the region $\mathfrak{M}^m$. (Note that the sets $\mathfrak{E}^n(\gamma)$ considered in this paper are frequently families of disjoint manifolds lying in the four-dimensional space $[\mathfrak{E}^n(\gamma)\cdot\mathfrak{E}^n(\gamma') = 0$, for $\gamma \neq \gamma']$ so that $\underset{\gamma\,\epsilon\,\mathfrak{M}^m}{\mathrm{S}}\ \mathfrak{E}^n(\gamma)$ has the dimension $n+m$).

We denote by $\mathrm{E}\,(\ldots)$ the set of points whose coordinates satisfy the relations indicated in the brackets.

2. The order of a boundary point. One of the methods of classifying the boundary points of a four-dimensional domain is by establishing connections between the geometrical properties of the immediate neighborhood of the boundary points and the behavior of the kernel function at these points. Evidently, the geometry considered cannot be simply the Euclidean four-dimensional geometry, but must be the geometry of the space of two complex variables, i. e. the geometry in which analytic[4] planes, surfaces and hypersurfaces are distinguished from ordinary planes, surfaces and hypersurfaces. ([3], I).

---

[3] See F. Hausdorff, *Lehrbuch der Mengenlehre*, 2nd ed., 1927, p. 1.

A rough geometrical classification of the boundary points is achieved in the following way. Let $Q = \{Z_1, Z_2\}$ be a boundary point of the four-dimensional domain $\mathfrak{D}$. We suppose at first that the boundary of $\mathfrak{D}$ is a hypersurface which possesses at the point Q an analytic tangential plane, or at least that Q is a point of intersection of several hypersurfaces possessing at Q analytic tangential planes. We consider all *analytic* surfaces which pass through Q. The following cases may occur:

(a) There exists one surface which has no points in common with the boundary of $\mathfrak{D}$, except of course the point Q.

(b) There exists a surface a segment of which lies in the boundary of $\mathfrak{D}$. Q is an interior point of this segment.

(c) There exist two or more surfaces which have segments in common with the boundary. Q is the point of intersection of these surfaces and belongs to the boundary of one of the above segments. (We may call this intersection an "edge").

(d) There exists one surface a segment of which lies entirely inside of the domain $\mathfrak{D}$, except for the point Q. Q is an interior point of this segment.

Note that these properties are invariant with respect to these PT's which are defined in a domain containing $\mathfrak{D}$. At first it might seem that the same boundary point could possess two or more of the properties indicated above. We shall see that under certain conditions this can not happen.

We shall now classify the boundary points with the aid of the kernel function. If the point (z_1, z_2) approaches the boundary point Q, the kernel function $K_{\mathfrak{D}}(z_1, z_2, \bar{z}_1, \bar{z}_2)$ will, in the general case, become infinite. If the expression

$$(2.1) \qquad \left| \sqrt{|Z_1 - z_1|^2 + |Z_2 - z_2|^2} \right|^T K_{\mathfrak{D}}(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

---

[4] In order to avoid any ambiguity we note that the words "analytic plane," "analytic surface," etc. are to be understood in the usual meaning of these terms in the theory of functions of several complex variables.

remains between two positive bounds when $\{z_1, z_2\}$ approaches $Q = (Z_1, Z_2)$ in a manner which may be subjected to certain restrictions, we shall call Q an R_τ-point (R-point of the order τ). If furthermore, the expression (2.1) possesses a definite positive limit provided that $\lim \arg (Z_2 - z_2)$, $\lim \arg (Z_1 - z_1)$ and $\lim |(Z_2 - z_2)/(Z_1 - z_1)|$ ($\neq 0$ or ∞) exist, we shall call Q an L_τ-point (L-point of the order τ). Note that the restrictions concerning the manner in which (z_1, z_2) approaches Q may vary according to τ and that numerous other subdivisions are possible.

If certain additional hypotheses are made, a boundary point possessing one of the properties (a)-(d) is an R-point of the order τ and[4]

$$\tau = 3 \text{ in the case (a)}, \qquad \tau = 4 \text{ in the case (c)},$$

$$\tau = 2 \text{ in the case (b)}, \qquad \tau = 0 \text{ in the case (d)}.$$

This can be relatively easily understood by considering simple examples. (See [1], p. 17). The precise formulations of sufficient conditions for a boundary point to be an R-point or an L-point of the order τ requires, however, a more thorough investigation of the geometrical properties of the boundary. For boundary points of the third order this has been done in [1], § 5.[5] In this paper we shall concern ourselves with the points of the second order. The method used will be that of domains of comparison.[6]

3. Standard domains for points of the second order. Consider a domain $\mathfrak{S}$ possessing the following three properties:

---

[5] Formulating the sufficient conditions to insure that a point Q is a limit point of the third order, we supposed in [1], p. 11, that there exists a one-to-one transformation of the form

$$z_1 = z^*_1 + 2bz^*_1 z^*_2 + cz^{*\,2}_2 + \cdots, \qquad z_2 = z^*_2$$

so that the equation of the boundary hypersurface (in a sufficiently small neighborhood) can be written in the form $2x^*_1 = ay^{*\,2}_1 + \sigma\,|z^*_2|^2 + \cdots$ [See [1], (3.17) and (3.18)]. We note that if the projection $\mathfrak{P}^2$ of the domain on the z_2 plane omits two points say a_1 and a_2, which satisfy the equation $a_1 + a_2 = -2ba_1a_2$, then such a transformation exists. The *PT*

$$z_1 = z^*_1 + 2bz^*_1 z^*_2 + cz^{*\,2}_2 + dz^*_1 z^{*\,2}_2, \quad z_2 = z^*_2, \quad d = 1/a_1a_2,$$

transforms the expression (3.13) of [1] into $2x^*_1 = ay^{*\,2}_1 + \sigma\,|z^*_2|^2 + \cdots$ and is one-to-one in the domain, for $1 - 2bz_2 + dz_2^2 = d(z_2 - a_1)(z_2 - a_2) \neq 0$, $z_2 \in \mathfrak{P}^2$.

[6] The points of the order 0 are "points of non existence": i. e. every analytic function regular in the interior of the domain is, in general, regular also at such boundary points. Points of the fourth order appear in particular in domains possessing a distinguished boundary surface; all points of this surface are, in general, of the fourth order.

(α) The point[7] $Q(0, Z_2)$ is a boundary point of $\mathfrak{S}$;

(β) $\mathfrak{S}$ is invariant under the transformation

$$z'_1 = \mu z_1,\ z'_2 = z_2 \text{ for every } \mu > 0;$$

(γ) there is a sequence of points $\{z_1^{(\nu)}, z_2^{(\nu)}\}$ in $\mathfrak{S}$ approaching Q and such that

$$\phi \equiv \lim_{\nu\to\infty} \arg z_1^{(\nu)} \text{ exists, } \lim_{\nu\to\infty} \left| \frac{z_1^{(\nu)}}{z_2^{(\nu)} - Z_2} \right| \text{ exists, } (e^{i\phi}, Z_2) \in \mathfrak{S}.$$

For such a domain $\mathfrak{S}$ we have by (β) and Theorem A

$$(3.1)\quad | z_1^{(\nu)} |^2 K_{\mathfrak{S}}(z_1^{(\nu)}, z_2^{(\nu)}; \overline{z_1^{(\nu)}}, \overline{z_2^{(\nu)}}) = K_{\mathfrak{S}}(e^{i\phi_\nu}, z_2^{(\nu)}; e^{i\phi_\nu}, \overline{z_2^{(\nu)}}),\ \phi_\nu = \arg z_1^{(\nu)}$$

Next, by (γ), the point $(e^{i\phi}, Z_2)$ lies in $\mathfrak{S}$ and hence by continuity of K the expression (3.1) approaches a limit as $\nu \to \infty$. In view of the existence of the second limit in (γ) this means that $Q(0, Z_2)$ is an L-point of order two for the sequence $\{z_1^{(\nu)}, z_2^{(\nu)}\}$.

This example shows us how to find conditions for points of the second order. We shall construct domains $\mathfrak{S}$ possessing the above properties and use domains obtained from $\mathfrak{S}$ by suitable PT's as domains of comparison.

Let r be a positive number and ϑ_1 and ϑ_2 two real numbers such as $0 < \vartheta_2 - \vartheta_1 < 2\pi$. By $\mathfrak{S}^2(\vartheta_1, \vartheta_2, r)$ we shall denote the angular domain $\mathrm{E}[0 < | z | < r,\ \vartheta_1 < \arg z < \vartheta_2]$, by $\mathfrak{S}^2(\vartheta_1, \vartheta_2)$ the angular domain $\mathrm{E}[0 < | z | < \infty,\ \vartheta_1 < \arg z < \vartheta_2]$.

Suppose that $\mathfrak{H}^2$ is a simply connected domain of the γ-plane bounded by a rectifiable Jordan curve, and that $\vartheta_1(\gamma)$ and $\vartheta_2(\gamma)$ are two real valued functions defined in $\mathfrak{H}^2$ satisfying the following conditions:

1. $\vartheta_1(\gamma)$ and $\vartheta_2(\gamma)$ are uniformly continuous and continuously differentiable.

2. There exists a positive constant ω such that

$$(3.1)\qquad \omega < \vartheta_2(\gamma) - \vartheta_1(\gamma) < \pi.$$

3. There exist four real constants, ϑ_1^0, ϑ_2^0, ϑ_1^1, ϑ_2^1, such that

$$(3.2)\qquad -\pi/2 < \vartheta_1^1 < 0 < \vartheta_2^1 < \pi/2 < \vartheta_1^0 < \pi < \vartheta_2^0 < 3\pi/2,$$

$$(3.3)\qquad \vartheta_2^0 - 2\pi < \vartheta_1(\gamma) < \vartheta_1^1, \qquad \vartheta_2^1 < \vartheta_2(\gamma) < \vartheta_1^0.$$

We may now consider the four dimensional domain $\mathfrak{S}$ given by

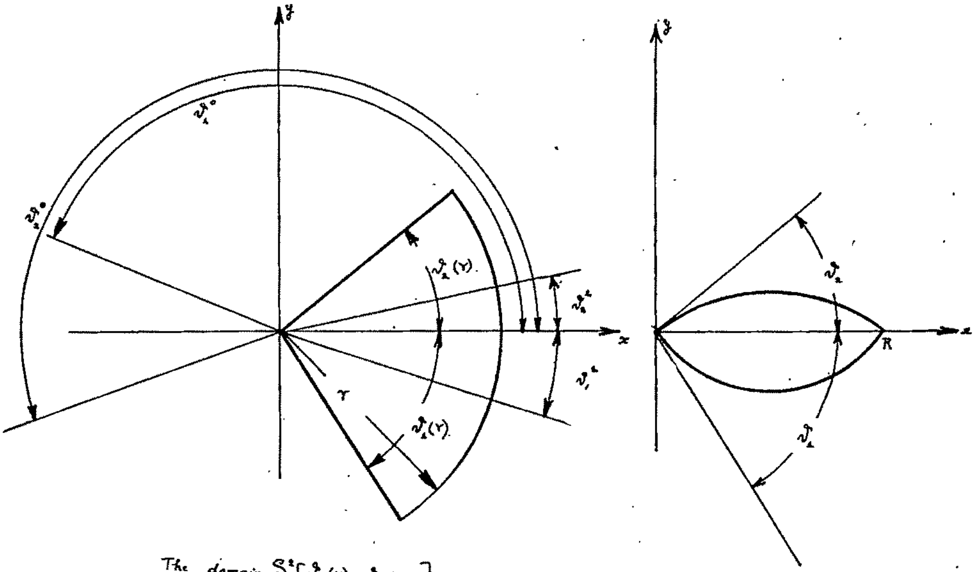
---

[7] For convenience we take the boundary point in question to be the point $Q(0, Z_2)$.

$$\mathfrak{S} = \underset{\gamma \in \mathfrak{N}^m}{S} E[z_2 = \gamma, z_1 \in \mathfrak{S}^2(\vartheta_1(\gamma), \vartheta_2(\gamma))]. \tag{3.4}$$

Such a domain we shall call a standard domain of the $\mathfrak{S}$-type (See [1], § 7).

Our next task consists in investigating the behavior of the kernel function $K_{\mathfrak{S}}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ of our standard domain in the neighborhood of the boundary points $\{0, Z_2\}$, $Z_2 \in \mathfrak{H}^2$. This can be done without the actual calculation of the kernel function.



The domain $S^2[\vartheta_1(\gamma), \vartheta_2(\gamma), \gamma]$.

The domain $J^2(\vartheta_1, \vartheta_2, R)$.

Fig. 1.

We first introduce the concept *A-approach with respect to a standard domain of the $\mathfrak{S}$-type.* Consider a given $\mathfrak{S}$-domain and a sequence of points $\{z_1^{(\nu)}, z_2^{(\nu)}\}$ situated in this domain and such that

$$\{z_1^{(\nu)}, z_2^{(\nu)}\} \to (0, Z_2), \qquad \nu \to \infty. \tag{3.5}$$

If there exists a positive ϵ and a closed subdomain $\mathfrak{H}_0^2 \subset \mathfrak{H}^2$ such that

$$(z_1^{(\nu)}, z_2^{(\nu)}) \in \underset{\gamma \in \mathfrak{H}_0^2}{S} E[z_2 = \gamma,\ z_1 \in \mathfrak{S}^2(\vartheta_1(\gamma) + \epsilon, \vartheta_2(\gamma) - \epsilon)], \qquad (\nu = 1, 2, \cdots), \tag{3.6}$$

we call the approach (3.5) a sectorial one. If furthermore,

$$\lim_{\nu\to\infty} \arg z_1^{(\nu)} \quad \text{and} \quad \lim_{\nu\to\infty} |z_1^{(\nu)}/(z_2^{(\nu)} - Z_2)| \tag{3.7}$$

exist, we call the approach (3.6) an A-approach.[8]

---

[8] In [1], p. 29, we denoted this approach as $AII(\alpha)$-approach, $\alpha = \lim_{\nu\to\infty} \arg z_1^{(\nu)}$.

Consider now an arbitrary four-dimensional domain $\mathfrak{D}$. Let $(0, Z_2)$ be a boundary point of $\mathfrak{D}$ and suppose that the part of the boundary situated in a sufficiently small neighborhood of $(0, Z_2)$ contains a part of the analytic plane $z_1 = 0$, namely $\mathrm{E}[z_1 = 0,\ z_2 \epsilon \mathfrak{B}^2]$, where $\mathfrak{B}^2$ is a simply connected domain in the z_2-plane containing the point $z_2 = Z_2$. Let $\phi(z_1, z_2)$ be a function defined in $\mathfrak{D}$. We shall say that this function possesses *A*-limits at the point $(0, Z_2)$ if

$$\lim_{\nu\to\infty} \phi(z_1^{(\nu)}, z_2^{(\nu)}) \tag{3.8}$$

exists for every sequence of points $(z_1^{(\nu)}, z_2^{(\nu)})$, $(z_1^{(\nu)}, z_2^{(\nu)}) \epsilon \mathfrak{D}$, $(\nu = 1, 2, \cdots)$, satisfying the following conditions:

1\.

$$\{z_1^{(\nu)}, z_2^{(\nu)}\} \to (0, Z_2), \qquad \nu \to \infty. \tag{3.9}$$

2. The approach (3.9) is an *A*-approach with respect to a certain standard domain $\mathfrak{S}$ for which

$$\mathfrak{S} \cdot \mathrm{E}[|z_1| < \delta, \qquad |z_2| < \infty] \subset \mathfrak{D},$$

δ being a sufficiently small positive number.

We shall now show that *for an $\mathfrak{S}$-domain the expression*

$$|z_1|^2 \mathrm{K}_{\mathfrak{S}}(z_1, z_2;\ \bar{z}_1, \bar{z}_2) \tag{3.11}$$

possesses at any boundary point $(0, Z_2)$, $Z_2 \epsilon \mathfrak{H}^2$, *positive A-limits.* To this end we apply to the domain $\mathfrak{S}$ the *PT*

$$\tilde{z}_1 = \mu z_1, \quad \tilde{z}_2 = z_2 \qquad (\mu > 0). \tag{3.12}$$

Since this *PT* maps $\mathfrak{S}$ into itself we have by Theorem A (see **Introduction**)

$$|z_1|^2 \mathrm{K}_{\mathfrak{S}}(z_1, z_2; \bar{z}_1, \bar{z}_2) = |z_1/\mu|^2 \mathrm{K}_{\mathfrak{S}}(z_1/\mu, z_2;\ \bar{z}_1/\mu, \bar{z}_2). \tag{3.13}$$

This formula shows that

$$r^2 \mathrm{K}_{\mathfrak{S}}(re^{i\phi}, z_2;\ re^{-i\phi}, \bar{z}_2) \tag{3.14}$$

is a function of z_2 and ϕ only. This function is continuous, for the kernel function is regular in the interior of the domain. Clearly, it is positive. Therefore the limit

$$\lim_{\substack{r\to 0\\ \phi\to\Phi\\ z_2\to Z_2}} r^2 \mathrm{K}_{\mathfrak{S}}(re^{i\phi}, z_2, re^{-i\phi}, z_2) \tag{3.15}$$

exists, provided that the approach $(re^{i\phi}, z_2) \to (0, Z_2)$ is an *A*-approach. Thus our assertion is proved.

Consider the *PT*

$$(3.20) \qquad z^*_1 = z_1/(1 + R^{-1}z_1), \qquad z^*_2 = z_2, \qquad (R \text{ real}).$$

It transforms $\mathfrak{S}$ into a domain we shall denote by $\mathfrak{J}$. The linear transformation

$$(3.21) \qquad z^*_1 = z_1/(1 + R^{-1}z_1)$$

maps $\mathfrak{S}^2(\vartheta_1, \vartheta_2)$ into the domain $\mathfrak{J}^2(\vartheta_1, \vartheta_2, R)$ which is bounded by two circular arcs passing through the origin and through the point R and forming with the real axis the angles ϑ_1 and ϑ_2, respectively. [Clearly, $\mathfrak{J}^2(\vartheta_1, \vartheta_2, R) \subset \mathfrak{S}^2(\vartheta_1, \vartheta_2, R)$]. Hence, $\mathfrak{J}$ can be represented in the form

$$\mathfrak{J} = \mathop{S}_{\gamma \in \mathfrak{H}^2} E[z_2 = \gamma,\ z_1 \in \mathfrak{J}^2(\vartheta_1(\gamma), \vartheta_2(\gamma), R)].$$

Such a domain we shall call standard domain of the $\mathfrak{J}$-type. Its kernel function possesses at any point $(0, Z_2)$ positive A-limits. This follows immediately from Theorem A, for the Jacobian of the transformation (3.20) is equal to 1 at every point of the analytic plane $z_1 = 0$.

4. *R*-points of the second order.

In this and the following sections we shall consider a domain $\mathfrak{D}$ which possesses the property (b) (See p. 682). That is to say, we suppose that there exists an analytic surface $\mathfrak{F}^2$ which possesses a segment in common with the boundary of D. $Q = (Z_1, Z_2)$ is an interior point of this segment. We shall find sufficient additional conditions to insure that Q be a point of the second order.

Let $\mathfrak{P}_2{}^2(\mathfrak{D})$ be the projection of $\mathfrak{D}$ on the plane $z_1 = 0$; i.e. the set of all points z_2, such that for suitably chosen z_1, $(z_1, z_2) \in \mathfrak{D}$. We suppose the following hypotheses to be fulfilled.

Hypothesis 4.1. The analytic surface $\mathfrak{F}^2$ may be represented in the form

$$(4.1) \qquad z_1 = h(z_2), \qquad\qquad Z_1 = h(Z_2),$$

$h(z_2)$ being an analytic function which is regular and univalent in $\mathfrak{P}_2{}^2(\mathfrak{D})$.

The *PT*

$$(4.2) \qquad z^*_1 = z_1 - h(z_2), \qquad z^*_2 = z_2 - Z_2$$

takes $\mathfrak{D}$ into another domain whose boundary contains a segment, $\mathfrak{H}^2$, of the analytic plane $z_1 = 0$. Q is taken into the point $(0, 0)$. We may consider the coordinates z^*_1, z^*_2 to have been introduced from the beginning and replace hypothesis 4.1 by

Hypothesis 4.1a. The boundary of $\mathfrak{D}$ contains a segment, $\mathfrak{H}^2$, of the analytic plane $z_1 = 0$, $0 \in \mathfrak{H}^2$.

Hypothesis 4. 2. There exist two positive constants ρ and α, $\alpha < \pi/2$, a regular harmonic function $\omega(\gamma)$ defined in a simply connected domain $\mathfrak{E}^2 \supset \mathfrak{P}_2{}^2(\mathfrak{D})$, and a closed subdomain $\overline{\mathfrak{W}}^2$ of $\mathfrak{H}^2$, $0 \in \mathfrak{H}^2$, such that

$$(4.\ 3) \qquad \mathfrak{S}^2[\omega(\gamma)-\alpha, \omega(\gamma)+\alpha, \rho] \subset \mathfrak{D}\cdot \mathrm{E}(z_2=\gamma), \qquad \gamma \in \mathfrak{W}^2,$$

and

$$(4.\ 4) \qquad \mathfrak{S}^2[-\omega(\gamma)-\pi+\alpha, \omega(\gamma)+\pi-\alpha] \supset \mathfrak{D}\cdot \mathrm{E}(z_2=\gamma), \qquad \gamma \in \mathfrak{E}^2.$$

Let $\psi(\gamma)$ be the conjugate harmonic function to $\omega(\gamma)$. We may suppose that $\psi(\gamma)$ does not vanish in $\mathfrak{E}^2$. The *PT*

$$(4.\ 5) \qquad z^*{}_1 = z_1 e^{\psi(\gamma)-i\omega(\gamma)}, \qquad z^*{}_2 = z_2$$

maps $\mathfrak{D}$ into a domain which satisfies hypothesis 4. 1a. The direction $\omega(\gamma)$ is taken into the positive x_1-direction. We call $z^*{}_1, z^*{}_2$ the *normal co-ordinates with respect to the boundary point Q, the analytic surface $\mathfrak{H}^2$ and the harmonic function* $\omega(\gamma)$. We may suppose that the normal co-ordinates have been introduced from the beginning and replace hypothesis 4. 2 by the following

Hypothesis 4. 2a. There exists a simply connected domain $\mathfrak{E}^2 \supset \mathfrak{P}_2{}^2(\mathfrak{D})$ and a closed subdomain $\mathfrak{W}^2$ of $\mathfrak{H}^2$, which contains 0 as an interior point, such that

$$(4.\ 6) \qquad \mathfrak{S}^2(-\alpha, \alpha, \rho) \subset \mathfrak{D}\cdot \mathrm{E}(z_2-\gamma), \qquad \gamma \in \mathfrak{W}^2,$$

and

$$(4.\ 7) \qquad \mathfrak{S}^2(-\pi+\alpha, \pi-\alpha) \supset \mathfrak{D}\cdot \mathrm{E}(z_2=\gamma), \qquad \gamma \in \mathfrak{E}^2.$$

If our hypotheses are satisfied, the domain $\mathfrak{D}$ belongs to the class for which we defined the concept of A-approach. We shall prove

THEOREM I. *If $\mathfrak{D}$ satisfies the hypotheses* 4. 1 *and* 4. 2, $Q = (0, 0)$ *is an R-point of the second order.*

Proof. We must show that the expression

$$(4.\ 8) \qquad |\, z_1{}^{(\nu)}\, |^2\, \mathrm{K}_{\mathfrak{D}}(z_1{}^{(\nu)}, z_2{}^{(\nu)}; \bar{z}_1{}^{(\nu)}, \bar{z}_2{}^{(\nu)})$$

remains between two positive bounds as the point $(z_1{}^{(\nu)}, z_2{}^{(\nu)})$ approaches our boundary point $(0, 0)$ in the way defined in **3** (A-approach). By hypothesis 4. 2a the $\mathfrak{J}$-domain

$$(4.\ 9) \qquad \mathfrak{J} = \underset{\gamma \in \mathfrak{W}^2}{\mathrm{S}} \mathrm{E}[z_2 = \gamma, z_1 \in \mathfrak{J}^2(-\alpha, \alpha, \rho)]$$

is an interior, and the $\mathfrak{S}$-domain

$$\mathfrak{A} = \mathop{\mathrm{S}}_{\gamma\,\epsilon\,\mathfrak{E}^2} E[z_2 = \gamma, z_1 \,\epsilon\, \mathfrak{S}^2(-\pi + \alpha, \pi - \alpha)] \tag{4.10}$$

an exterior domain of comparison for $\mathfrak{D}$:

$$\mathfrak{J} \subset \mathfrak{D} \subset \mathfrak{A}. \tag{4.11}$$

For sufficiently large ν, the points $(z_1^{(\nu)}, z_2^{(\nu)})$ are situated within $\mathfrak{J}$. We have then by Theorem B

$$\begin{aligned} |z_1^{(\nu)}|^2 K_{\mathfrak{A}}(z_1^{(\nu)}, z_2^{(\nu)}; z_1^{(\nu)}, z_2^{(\nu)}) &\leqq |z_1^{(\nu)}|^2 K_{\mathfrak{D}}(z_1^{(\nu)}, z_2^{(\nu)}; z_1^{(\nu)}, z_2^{(\nu)}) \\ &\leqq |z_1^{(\nu)}|^2 K_{\mathfrak{J}}(z_1^{(\nu)}, z_2^{(\nu)}; z_1^{(\nu)}, z_2^{(\nu)}). \end{aligned} \tag{4.12}$$

In **3** we proved that the expressions on the right and left hand side in (4.12) possess non-vanishing limits as $\nu \to \infty$. This proves our theorem.

5. Sufficient conditions for *L*-points of second order. In this section we shall formulate conditions which are sufficient to insure that a boundary point is an L-point of second order (L_2). We suppose that our domain $\mathfrak{D}$ satisfies the hypotheses 4.1a and 4.2a.

Hypotheses 5.1. $\mathfrak{H}^2$ is a convex domain.

We write the equation of its boundary curve in the form

$$z_2 = R(\phi)e^{i\phi}, \qquad 0 \leqq \phi \leqq 2\pi. \tag{5.1}$$

Let k be a positive number. We shall denote by $\mathfrak{H}^2(1+k)$ the domain bounded by the curve

$$z_2 = (1+k)Re^{i\phi}, \qquad 0 \leqq \phi \leqq 2\pi. \tag{5.2}$$

A convex domain possesses the following simple property:

LEMMA 1. *For every positive k there exists a positive ϵ, such that all points, ζ, which may be represented in the form*

$$\zeta = e^{i\xi}z_2, \quad z_2 \,\epsilon\, \mathfrak{H}^2(1+k), \quad -\epsilon \leqq \xi \leqq \epsilon, \tag{5.3}$$

are situated within $\mathfrak{H}^2(1+2k)$.

Hypothesis 5.2. $\mathfrak{D}$ is a bounded domain.

For the sake of simplicity we may suppose that this hypothesis is replaced by

Hypothesis 5.2a. If $(z_1, z_2) \,\epsilon\, \mathfrak{D}$, then $|z_1| < 1/2$, $|z_2| < 1$.

Hypothesis 5. 3. There exists a positive function $\kappa(t)$,

$$\lim_{t\to 0} \kappa(t) = 0, \tag{5. 4}$$

such that for every point $(z_1, z_2) \epsilon \mathfrak{D}$ with $|z_1| \leqq t$

$$z_2 \epsilon \mathfrak{H}^2[1 + \kappa(t)]. \tag{5. 5}$$

Hypothesis 5. 4. There exists a positive constant η_0 such that the intersection

$$\mathfrak{D} \cdot \mathrm{E}[|z_1| < \eta_0] \cdot \mathrm{E}(z_2 = \gamma), \qquad \gamma \epsilon \mathfrak{E}^2, \tag{5. 6}$$

(See Hypothesis 4. 2a) is either empty or a simply connected domain which is bounded by two curves,[9] $\mathfrak{a}_k^1(\gamma)$ starting at the origin and lying in the upper (lower) z_1-half-plane for $k = 2$ $(k = 1)$, and, perhaps, by a part of the circle $|z_1| = \eta_0$ (See Fig. 2).

Hypothesis 5. 5. There exists for every closed sub-domain $\bar{\mathfrak{T}}^2$ of $\mathfrak{H}^2$, $0 \epsilon \mathfrak{T}^2$, a positive constant $\eta_1(\mathfrak{T}^2) < \eta_0$ such that for $\gamma \epsilon \mathfrak{T}^2$ and $|y_1| < \eta_1(\mathfrak{T}^2)$ the curves $\mathfrak{a}_k^1(\gamma)$ (introduced in Hypothesis 5. 4) may be represented in the form

$$x_1 = A_k(y_1, \gamma), \qquad 0 \leqq (-1)^k y_1, \qquad |y_1| \leqq \eta_1(\mathfrak{T}^2), \tag{5. 7}$$

Furthermore, if $\Theta_k(y_1, \gamma) = \cot[\partial A_k(y_1, \gamma)/\partial y_1]$,

$$\theta_k(\gamma) = \Theta_k(0, \gamma). \qquad \gamma \epsilon \mathfrak{H}^2; \tag{5. 8}$$

$\Theta_k(y_1, \gamma)$ is a continuously differentiable function of its real variables, and

$$|\operatorname{grad} \theta_k(\gamma)| < C, \tag{5. 9}$$

$$\alpha - \pi < \theta_1(\gamma) < -\alpha, \quad \alpha < \theta_2(\gamma) < \pi - \alpha. \tag{5. 10}$$

Hypothesis 5. 6. There exist two functions $C_k(y_1, \gamma)$, defined for $|y_1| < \eta_0$, $\gamma \epsilon \bar{\mathfrak{E}}^2$, possessing continuous derivatives with respect to their three real variables, satisfying the conditions

$$C_1(0, \gamma) = C_2(0, \gamma) \tag{5. 11}$$

$$\begin{aligned} C_k(0, \gamma) &= 0 \text{ for } \gamma \epsilon \mathfrak{H}^2, \\ &\geqq 0 \text{ for } \gamma \epsilon \mathfrak{E}^2 - \mathfrak{H}^2, \end{aligned} \tag{5. 12}$$

and such that for every point $(z_1, z_2) \epsilon \mathfrak{D}$ with $|z_1| < \eta_0$, $z_2 \epsilon \mathfrak{H}^2$,

$$\begin{cases} x_1 > C_2(y_1, z_2) \text{ for } y_1 \geqq 0, \\ x_1 > C_1(y_1, z_2) \text{ for } y_1 \leqq 0. \end{cases} \tag{5. 13}$$

---

[9] To avoid repetition, we shall omit the statement " $k = 1, 2$ " hereafter. Whenever the index k, occurs, it is to have the values 1 and 2.

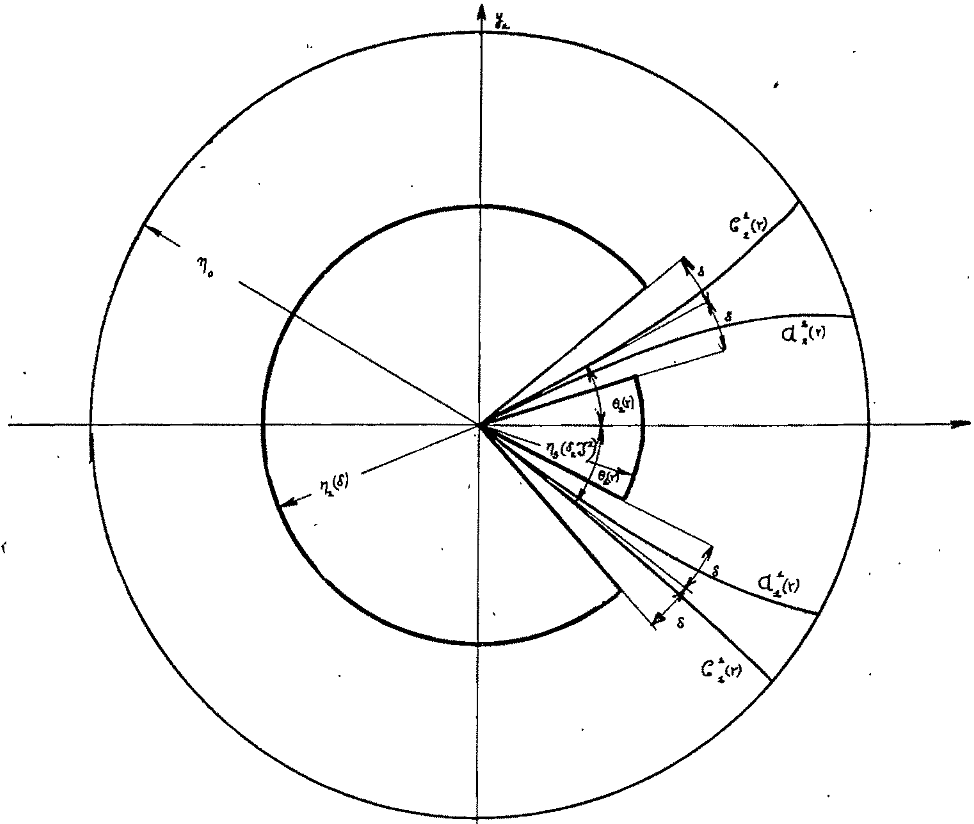
(We denote the curves $x_1 = C_k(y_1, \gamma)$ by $\mathfrak{c}_k^1(\gamma)$.

If we write $\theta_k(y_1, \gamma) = \cot [\partial C_k(y_1, \gamma)/\partial y_1]$,

$$(5.14) \qquad \theta_k(\gamma) = \theta_k(0, \gamma), \qquad \gamma \in \mathfrak{E}^2 - \mathfrak{H}^2,$$

$\theta_k(y_1, \gamma)$ is still a continuously differentiable function of its three real variables, and the inequalities (5.9) and (5.10) are valid for $\gamma \in \mathfrak{E}^2 - \mathfrak{H}^2$.

The following is an immediate consequence of our hypotheses.



The intersection of $\mathfrak{D} \cdot E[|z_2| < \eta_0]$ with a plane $x_2 = r$.

Fig. 2.

LEMMA 2. (A) *For every sufficiently small positive* δ *there exists a positive* $\eta_2(\delta) < \eta_0$,

$$(5.15a) \qquad \lim_{\delta \to 0} \eta_2(\delta) = 0$$

such that

$$(5.15) \qquad \mathfrak{S}^2[\theta_2(\gamma) + \delta, 2\pi + \theta_1(\gamma) - \delta, \eta_2(\delta)] \cdot \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma) = 0, \ \gamma \in \mathfrak{E}^2.$$

(B) *For every sufficiently small positive* δ, *and for every closed subdomain* $\bar{\mathfrak{T}}^2$ *of* $\mathfrak{H}^2$, $0 \in \mathfrak{T}^2$, *there exists a positive* $\eta_3(\delta, \mathfrak{T}^2) < \eta_1(\mathfrak{T}^2)$ *such that*

(5.16) $\mathfrak{S}^2[\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \eta_3(\delta, \mathfrak{T}^2)] \subset \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma), \quad \gamma \in \mathfrak{T}^2.$

Proof. We choose $\eta_2(\delta)$ so small that

(5.17) $|\theta_k(\gamma) - \theta_k(\eta, \gamma)| < \delta$ for $\gamma \in \mathfrak{C}^2$, $0 \leqq (-1)^k \eta \leqq (-1)^k \eta_2(\delta)$.

This is possible since $\theta_k(y, \gamma)$ are uniformly continuous. Let $z_1 = x_1 + iy_1$ be a point of

(5.18) $$\mathfrak{S}^2[\theta_2(\gamma) + \delta, 2\pi + \theta_1(\gamma) - \delta, \eta_2(\delta)].$$

We must show that (z_1, γ) is an exterior point of $\mathfrak{D}$.

It will suffice to show that, say, for $y_1 \geqq 0$,

(5.19) $$x_1 < C_2(y_1, \gamma)$$

(see Hypothesis 5.6). We have, say, for $\theta_2(\gamma) < \pi/2$,

(5.20) $$x_1 < y_1 \cot[\theta_2(\gamma) + \delta];$$

and by the mean value theorem,

(5.21) $$C_2(y_1, \gamma) = C_2(0, \gamma) + y_1 \cot \theta_2(\xi y_1), \quad 0 < \xi < 1.$$

Since $|y_1| < \eta_2(\delta)$ we have by (5.17) and (5.21)

(5.22) $$C_2(y_1, \gamma) > C_2(0, \gamma) + y_1 \cot[\theta_2(\gamma) + \delta].$$

By (5.20) and (5.22), (5.19) is proved, for $C_2(0, \gamma)$ is non-negative, according to (5.12). The cases $y_1 < 0$, $\theta_2(\gamma) > \pi/2$ etc. may be treated in the same way. Our assertion (A) is thus proved. The proof of (B) is similar. We apply the Hypotheses 5.4 and 5.5 and need only show that if $z_2 \in \mathfrak{T}^2$ and $(z_1, z_2) \in \mathfrak{D}$ then

$$x_1 > A_2(y_1, z_2) \text{ for } 0 \leqq y_1 \leqq \eta_1(\mathfrak{T}^2),$$

and

$$x_1 > A_1(y_1, z_2) \text{ for } -\eta_1(\mathfrak{T}^2) \leqq y_1 \leqq 0,$$

But this follows immediately from the fact that each circle

$$\mathrm{E}[z_2 \in \bar{\mathfrak{T}}^2, |z_1| < \eta_1(\mathfrak{T}^2)]$$

is divided by the curves $\mathfrak{a}_k^1(z_2)$ into two parts which vary continuously with z_2, one of which (the one containing positive real points z_1) contains inner points of $\mathfrak{D}$.

We shall prove

Theorem II. *If the domain* $\mathfrak{D}$ *satisfies the hypotheses* 4. 1-4. 2 *and* 5. 1-5. 6, *the boundary point* $Q = (0, 0)$ *is an L-point of the second order.*

The proof will be given in the following sections.

6. Construction of the domains of comparison for an L_2-point. In order to prove Theorem II we shall construct an interior and an exterior domain of comparison for the domain $\mathfrak{D}$, where $\mathfrak{D}$ satisfies Hypotheses 4. 1-4. 2 and 5. 1-5. 6.

(i) *Interior domain of comparison.* Its construction is possible by virtue of Lemma 2, (B). Let δ be a sufficiently small positive number, $\mathfrak{T}^2$ a closed subdomain of $\mathfrak{H}^2$, $0 \in \mathfrak{T}^2$. We write

$$\tau = \eta_3(\delta, \mathfrak{T}^2) \sin \alpha. \tag{6.1}$$

Then, for $\gamma \in \mathfrak{T}^2$,

$$\mathfrak{J}^2[\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \tau] \subset \mathfrak{S}^2[\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \eta_3(\delta, \mathfrak{T}^2)], \tag{6.2}$$

whence it follows by Lemma 2, (B) that

$$\mathfrak{J} = \underset{\gamma \in \mathfrak{T}^2}{\mathrm{S}} \ \mathrm{E}[z_2 = \gamma, z_1 \in \mathfrak{J}^2(\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \tau)] \tag{6.3}$$

is an interior domain of comparison:

$$\mathfrak{J} \subset \mathfrak{D}. \tag{6.4}$$

Remark. We may choose for $\mathfrak{T}^2$ the domain

$$\mathfrak{T}^2 = \mathfrak{H}^2[1 - 2\kappa(t)], \tag{6.5}$$

(see Hypothesis 5. 3) where

$$t = \tfrac{1}{2}\eta_2(\delta). \tag{6.6}$$

Then our interior domain of comparison is determined by one positive parameter, δ.

(ii) *Exterior domain of comparison.* We shall obtain this domain by applying the *PT*

$$z^*_1 = z_1, \ z^*_2 = z_2/(1 - \sigma z_1^{1/\mu}), \quad (0 < \sigma < 1, \ \mu > 0), \tag{6.7}$$

to the domain [11]

---

[10] Concerning the visualization of the domains of two complex variables see [2] and [5]. (Fig. 3 is a "still" of a model of $\mathfrak{J}$.)

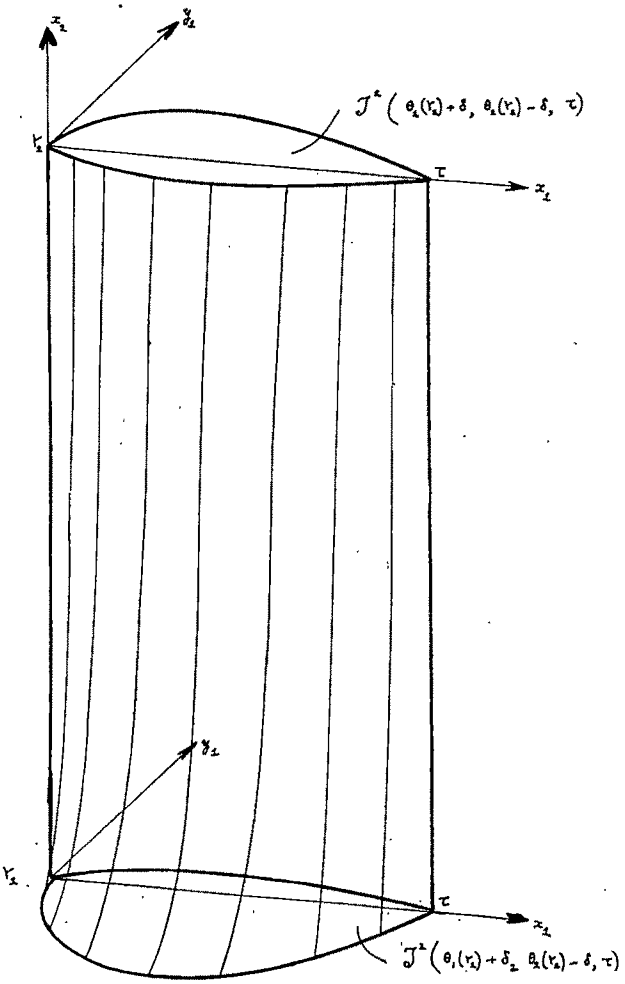
[11] Note that according to our definition the second vertex of

$$\mathfrak{J}^2(\theta_2(\gamma) + 2\delta, 2\pi + \theta_1(\gamma) - 2\delta, \rho)$$

lies on the *negative* real axis.

$$(6.8)\quad \mathfrak{G} = \underset{\gamma\,\epsilon\,\mathfrak{R}^2}{\mathrm{S}}\ \mathrm{E}[z_2 = \gamma, |z_1| < 1, z_1 \notin \mathrm{E}(x_1 < -\rho; y_1 = 0), \\ z_1 \notin \mathfrak{J}^2(\theta_2(\gamma) + 2\delta, 2\pi + \theta_1(\gamma) - 2\delta, \rho)]$$

where δ is a sufficiently small positive number, $\rho > 0$, and $\mathfrak{R}^2$ is a fixed domain for which we have $\mathfrak{E}^2 \supset \mathfrak{R}^2 \supset \bar{\mathfrak{H}}^2$.



x_2
y_2
γ_2
$\mathfrak{J}^2(\theta_2(\gamma_2)+\delta, \theta_2(\gamma_2)-\delta, \tau)$
τ
x_1
y_1
γ_1
τ
x_1
$\mathfrak{J}^1(\theta_1(\gamma_1)+\delta_2\ \theta_2(\gamma_2)-\delta, \tau)$

Fig. 3.

An intersection of $\mathfrak{S}$ with space $y_2 = \text{const.}$

Remark. Note that $\mathfrak{G}$ contains only points (z_1, z_2) with $|\arg z_1| < \pi$. The transformation (6.7) is therefore a one-to-one transformation. We may write $z_1^{1/\mu} = (re^{i\phi})^{1/\mu} = r^{1/\mu}e^{i\phi/\mu}$.

Before choosing the values of the parameters in (6.7) and (6.8), we shall prove some very simple lemmas.

LEMMA 3. *For given N, $N > 1$, and t, $0 < t < 1$, and for*

$$\sigma = N/(N+1), \tag{6.9}$$

there exists a $\mu_0 > 0$ such that

$$|[1 - \sigma z_1^{1/\mu}]^{-1}| > N \tag{6.10}$$

for

$$\mu > \mu_0, \qquad |z_1| > t. \tag{6.11}$$

Proof. We have

$$|1 - \sigma z_1^{1/\mu}|^{-2} > |1 + \sigma^2 - 2\sigma t^{1/\mu} \cos(\pi/\mu)|^{-1} \rightarrow (1-\sigma)^{-2} = (N+1)^2 \tag{6.12}$$

as $\mu \rightarrow \infty$.

LEMMA 4. *For a fixed σ, $0 < \sigma < 1$, and $\epsilon > 0$, there exists $\mu_1 > 0$ such that for $\mu > \mu_1$,*

$$|\arg z_2 - \arg z^*_2| < \epsilon \tag{6.13}$$

[See (6.7)].

Proof. By Hypothesis 5.2a, $|z_1| < 1/2$. Hence

$$|\arg(1 - \sigma z_1^{1/\mu})| \leqq \arctan [2^{1/\mu} \sin(\pi/\mu)/(1 - \sigma 2^{-1/\mu})] \rightarrow 0, \tag{6.14}$$

as $\mu \rightarrow \infty$.

LEMMA 5. *For fixed values of σ $(0 < \sigma < 1)$, δ $(\delta > 0)$, and μ $(\mu > 0)$, there exists a positive χ such that*

$$|\theta_k(z^*_2) - \theta_k(z_2)| < \delta/2 \tag{6.15}$$

for $|z_1| < \chi$.

Proof. By (6.7), (5.9) and hypothesis 5.2a we have

$$|\theta_k[z_2(1 - \sigma z_1^{1/\mu})] - \theta_k(z_2)| \leqq C\chi^{1/\mu}\sigma. \tag{6.16}$$

We shall now determine the parameters in (6.7) and (6.8). We choose a sufficiently small positive δ and write

$$t = (1/2)\eta_2(\delta), \tag{6.17}$$

$$\mathfrak{R}^2 = \mathfrak{H}^2[1 + 2\kappa(t)], \tag{6.18}$$

$$(1/N) = \min_{0 \leqq \phi \leqq 2\pi} R(\phi), \tag{6.19}$$

$$\sigma = N/(N+1). \tag{6.20}$$

See Lemma 2, (5.1) and Hypothesis 5.3.

Next we determine a positive ϵ which satisfies the conditions expressed in Lemma 1 with $k = \kappa(t)$. With these values of t, N, σ and ϵ we find numbers

μ_0 and μ_1 which satisfy the inequalities indicated in Lemmas 3 and 4. Then we choose

$$(6.21) \qquad \mu = \max(\mu_0, \mu_1).$$

With this value of μ we determine χ satisfying the condition of Lemma 5, and such that

$$(6.22) \qquad 0 < \chi < t.$$

Finally we write

$$(6.23) \qquad \rho = \chi \sin \alpha.$$

See Hypothesis 4.2.

With the parameters determined above we construct the domain $\mathfrak{G}$ [see (6.8)] and apply to it the *PT* (6.7). We obtain a domain $\mathfrak{A}$ for which we shall prove that

$$(6.24) \qquad \mathfrak{A} \supset \mathfrak{D}.$$

Let $(z_1{}^0, z_2{}^0)$ be a point of $\mathfrak{D}$. By $(\zeta_1{}^0, \zeta_2{}^0)$ we denote the point which we obtain from $(z_1{}^0, z_2{}^0)$ by the transformation inverse to (6.7), i.e. let

$$(6.25) \qquad \zeta_1{}^0 = z_1{}^0, \quad \zeta_2{}^0 = z_1{}^0[1 - \sigma(z_1{}^0)^{1/\mu}].$$

We must show that $(z_1{}^0, z_2{}^0) \in \mathfrak{A}$. To this end we consider three cases.

1°. $|z_1{}^0| > t$. By (6.22) and (6.23), $\rho < t$. By (6.8) and (6.18), the intersection of $\mathfrak{G}$ with the plane $z_1 = z_1{}^0$ is $\mathrm{E}[z_1 = z_1{}^0, z_1 \in \mathfrak{H}^2(1 + 2\kappa(t))]$. Hence it contains the circle $\mathrm{E}[z_1 = z_1{}^0, |z_2| < \min\limits_{0 \leqq \phi \leqq 2\pi} R(\phi)]$. By the transformation (6.7) this circle has been expanded in a ratio greater then $[\min\limits_{0 \leqq \phi \leqq 2\pi} R(\phi)]^{-1}$ [see Lemma 3 and (6.19)]. Thus the intersection of $\mathfrak{A}$ with the plane $z_1 = z_1{}^0$ contains the whole circle $\mathrm{E}[z_1 = z_1{}^0, |z_2| < 1]$. Since $|z_2{}^0| < 1$, therefore $(z_1{}^0, z_2{}^0)$ is a point of $\mathfrak{A}$.

2°. $\chi \leqq |z_1{}^0| \leqq t$. By Hypothesis 5.3, $z_2{}^0$ is an interior point of $\mathfrak{H}^2[1 + \kappa(t)]$. We obtain the intersection of $\mathfrak{A}$ with $\mathrm{E}[z_1 = z_1{}^0]$ by applying to $\mathfrak{H}^2[1 + 2\kappa(t)]$ the transformation $z^*{}_2 = z_2 e^{-i\phi}/p$, $pe^{i\phi} = 1 - \sigma(z_1{}^0)^{1/\mu}$. By Lemma 1 and Lemma 4, the rotation $\zeta = e^{-i\phi} z_2$ applied to the domain $\mathfrak{H}^2[1 + 2\kappa(t)]$ furnishes a domain which still contains $\mathfrak{H}^2[1 + \kappa(t)]$. Since $p < 1$ the transformation $z^*{}_2 = \zeta/p$ is a positive dilatation. Thus $\mathfrak{A} \cdot \mathrm{E}(z_1 = z_1{}^0)$ contains the domain $\mathfrak{H}^2[1 + \kappa(t)]$. Thus $(z_1{}^0, z_2{}^0) \in \mathfrak{A}$.

3°. $|z_1{}^0| < \chi$. Again we have $z_2{}^0 \in \mathfrak{H}^2[1 + \kappa(t)]$, and we can show in the same way as in case 2° that $\zeta_2{}^0 \in \mathfrak{H}^2[1 + 2\kappa(t)]$. In order to show that $(\zeta_1{}^0, \zeta_2{}^0) \in \mathfrak{G}$, it will suffice to prove that

$$(6.26) \qquad \theta_1(\zeta_2{}^0) - 2\delta < \arg \zeta_1{}^0 < \theta_2(\zeta_2{}^0) + 2\delta.$$

Since $(z_1^0, z_2^0) \in \mathfrak{D}$ and $z_1^0 = \zeta_1^0$, we have by Lemma 2 (A)

$$\theta_1(z_2^0) - \delta < \arg \zeta_1^0 < \theta_2(z_2^0) + \delta, \tag{6.27}$$

since $\chi < \eta_2(\delta)$. (6.26) follows immediately by Lemma 5. But $(\zeta_1^0, \zeta_2^0) \in \mathfrak{G}$, therefore $(z_1^0, z_2^0) \in \mathfrak{A}$. The assertion (6.2) is thus proved.

We have constructed an exterior domain of comparison $\mathfrak{A}$. Note that it is determined by one positive parameter, δ.

7. Limit relation for L_2-points. In this section we shall complete the proof of Theorem II. Let $\{(z_1^{(\nu)}, z_2^{(\nu)})\}$, $(\nu = 1, 2, \cdots)$, be a sequence of points which approach the boundary point $Q = (0, 0)$ in the A-sense (see p. 686) as $\nu \to \infty$. We must prove that the expression

$$| z_1^{(\nu)} |^2 \mathrm{K}_{\mathfrak{D}}(z_1^{(\nu)}, z_2^{(\nu)}; \bar{z}_1^{(\nu)}, \bar{z}_2^{(\nu)}) \tag{7.1}$$

possesses a non-vanishing limit for $\nu \to \infty$. For the sake of simplicity we shall denote this expression by $K_\nu(\mathfrak{D})$. By $K_\nu(\mathfrak{B})$ we shall denote the analogous expression for some other domain $\mathfrak{B}$. We choose a sufficiently small positive δ and construct the corresponding interior and exterior domains of comparison $\mathfrak{J}$ and $\mathfrak{A}$. For sufficiently large ν, the points $(z_1^{(\nu)}, z_2^{(\nu)})$ lie within $\mathfrak{J}$. Since

$$\mathfrak{J} \subset \mathfrak{D} \subset \mathfrak{A} \tag{7.2}$$

we have by Theorem B

$$K_\nu(\mathfrak{A}) \leqq K_\nu(\mathfrak{D}) \leqq K_\nu(\mathfrak{J}). \tag{7.3}$$

$\mathfrak{J}$ is a standard domain, and therefore

$$\lim_{\nu \to \infty} K_\nu(\mathfrak{J}) = L_{\mathfrak{J}}(0, \phi)$$

exists and is a positive continuous function of ϕ, $\phi = \lim\limits_{\nu \to \infty} \arg z_1^{(\nu)}$. Thus

$$\underline{\lim}\, K_\nu(\mathfrak{A}) \leqq \underline{\lim}\, K_\nu(\mathfrak{D}) \leqq \overline{\lim}\, K_\nu(\mathfrak{D}) \leqq \overline{\lim}\, K_\nu(\mathfrak{J}). \tag{7.4}$$

$\mathfrak{A}$ has been obtained from $\mathfrak{G}$ by the transformation (6.7). The Jacobian of this transformation converges uniformly to 1 as $(z_1, z_2) \to (0, 0)$. By Theorem A (see p. 680)

$$\underline{\lim}\, K_\nu(\mathfrak{A}) = \underline{\lim}\, | z^*{}_1^{(\nu)} |^2 \mathrm{K}_{\mathfrak{G}}(z^*{}_1^{(\nu)}, z^*{}_2^{(\nu)}; \bar{z}^*{}_1^{(\nu)}, z^*{}_2^{(\nu)}). \tag{7.5}$$

Consider instead of $\mathfrak{G}$ the domain

$$\mathfrak{M} = \underset{\gamma \in \mathfrak{R}^2}{\mathrm{S}} \mathrm{E}[z_2 = \gamma, z_1 \notin \mathfrak{J}^2(\theta_2(\gamma) + 2\delta, \pi + \theta_1(\gamma) - 2\delta, \rho')], \tag{7.6}$$

where δ and $\mathfrak{R}^2$ have the previous significance (see p. 695) and

$$\rho' = \rho \sin \alpha. \tag{7.7}$$

Since $\mathfrak{M} \supset \mathfrak{G}$

$$(7.8) \quad |z_1^{(\nu)}|^2 K_{\mathfrak{G}}(z_1^{(\nu)}, z_2^{(\nu)}; \bar{z}_1^{(\nu)}, \bar{z}_2^{(\nu)}) \geqq |z_1^{(\nu)}|^2 K_{\mathfrak{M}}(z_1^{(\nu)}, z_2^{(\nu)}; \bar{z}_1^{(\nu)}, \bar{z}_2^{(\nu)})$$

$\mathfrak{M}$ is a standard domain. Therefore the limit of the right hand side of (7.8) exists. Since $\arg z_1$ is unchanged by the transformation (6.7), this limit is equal to $L_{\mathfrak{M}}(0, \phi)$. Thus we have

$$(7.9) \qquad L_{\mathfrak{M}}(0, \phi) \leqq \underline{\lim}\, K_\nu(\mathfrak{D}).$$

The *PT*

$$(7.10) \qquad \zeta_1 = z_1/[1 + z_1/\rho'], \quad \zeta_2 = z_2$$

maps $\mathfrak{M}$ into an $\mathfrak{S}$-domain

$$(7.11) \qquad \mathfrak{S}_a = \underset{\gamma \in \mathfrak{T}^2}{\mathrm{S}} \mathrm{E}[z_2 = \gamma,\ z_1 \in \mathfrak{S}^2(\theta_1(\gamma) - 2\delta,\ \theta_2(\gamma) + 2\delta)].$$

The *PT*

$$(7.12) \qquad \zeta_1 = z_1/[1 - \epsilon^{-1} z_1], \quad \zeta_2 = z_2$$

maps $\mathfrak{J}$ into an $\mathfrak{S}$-domain

$$\mathfrak{S}_i = \underset{\gamma \in \mathfrak{T}^2}{\mathrm{S}} \mathrm{E}[z_2 = \gamma,\ z_1 \in \mathfrak{S}^2(\theta_1(\gamma) + \delta,\ \theta_2(\gamma) - \delta)].$$

The Jacobians of the transformations (7.10) and (7.12) are both equal to 1 at (0, 0), and in both cases we have

$$(7.13) \qquad |\arg z_1 - \arg \zeta_1| \to 0, \quad \zeta_1 \to 0, \ z_1 \to 0.$$

By Theorem A,

$$(7.14) \qquad L_{\mathfrak{M}}(0, \phi) = L_{\mathfrak{S}_a}(0, \phi), \quad L_{\mathfrak{J}}(0, \phi) = L_{\mathfrak{S}_i}(0, \phi).$$

By (7.4), (7.5), (7.9) and (7.14) we finally have

$$(7.15) \qquad L_{\mathfrak{S}_a}(0, \phi) \leqq \underline{\lim}\, K_\nu(\mathfrak{D}) \leqq \overline{\lim}\, K_\nu(\mathfrak{D}) \leqq L_{\mathfrak{S}_i}(0, \phi).$$

Now we apply to $\mathfrak{S}_i$ the *PT*

$$(7.16) \qquad \zeta_1 = z_1^{1+b}, \quad \zeta_2 = (1 + 2\kappa)(1 - 2\kappa)^{-1} z_2,$$

where

$$(7.17) \qquad \kappa = \kappa(t)$$

and

$$(7.18) \qquad b = [3\delta + 4C\kappa^2(t)]/[\alpha - 2\delta].$$

(We have then

$$(7.19) \qquad \arg \zeta_1 = (1 + b) \arg z_1.)$$

We denote by $\mathfrak{S}_j$ the domain obtained in this manner, and state that

$$(7.20) \qquad \mathfrak{S}_j \supset \mathfrak{S}_a.$$

In order to prove this assertion we must only show that

$$(7.21)\quad |(1+b)[\theta_k\left(\frac{1-2\kappa}{1+2\kappa}z_2\right)-(-1)^k\delta]| > |\theta_k(z_2)+2(-1)^k\delta|.$$

(For, the transformation $\zeta = \frac{1+2\kappa}{1-2\kappa}z$ maps $\mathfrak{T}^2$ into $\mathfrak{R}^2$.) We have, say for $k=2$, by the mean value theorem,

$$(7.22)\quad |\theta_2\left(\frac{1-2\kappa}{1+2\kappa}z_2\right)-\theta_2(z_2)| \leqq 4C\kappa^2,$$

(see (5.9)) and therefore

$$(7.23)\quad (1+b)[\theta_2\left(\frac{1-2\kappa}{1+2\kappa}z_2\right)-\delta]-\theta_2(z_2)-2\delta$$
$$\geqq -4C\kappa^2+b\alpha-b\delta-3\delta$$
$$= b(\alpha-\delta)-(3\delta+4C\kappa^2) > b(\alpha-2\delta)-(3\delta+4C\kappa^2)=0.$$

Thus (7.20) is proved. By Theorem B and (7.5),

$$(7.24)\quad L_{\mathfrak{S}_j}(0,\phi) \leqq L_{\mathfrak{S}_a}(0,\phi) \leqq L_{\mathfrak{S}_i}(0,\phi).$$

Since

$$(7.25)\quad \frac{\partial(\zeta_1,\zeta_2)}{\partial(z_1,z_2)} = \frac{1+2\kappa}{1-2\kappa}(1+b)z_1^b,$$

we have by Theorem A,

$$(7.26)\quad \mathrm{K}_{\mathfrak{S}_j}(\zeta_1,\zeta_2;\bar\zeta_1,\bar\zeta_2)$$
$$= (1+2\kappa)^2(1-2\kappa)^{-2}(1+b)^{-2}|\zeta_1|^{-2b}\mathrm{K}_{\mathfrak{S}_i}(z_1,z_2;\bar z_1,\bar z_2),$$

and

$$(7.27)\quad |z_1|^{2(1+b)}\mathrm{K}_{\mathfrak{S}_j}(\zeta_1,\zeta_2;\bar\zeta_1,\bar\zeta_2)$$
$$= (1-2\kappa)^2(1-2\kappa)^{-2}(1+b)^{-2}|z_1|^2\mathrm{K}_{\mathfrak{S}_i}(z_1,z_2;\bar z_1,\bar z_2),$$

i. e.

$$(7.28)\quad |\zeta_1|^2\mathrm{K}_{\mathfrak{S}_j}(\zeta_1,\zeta_2;\bar\zeta_1,\bar\zeta_2) = |z_1|^2\mathrm{K}_{\mathfrak{S}_i}(z_1,z_2;\bar z_1,\bar z_2).$$

Hence it follows by (7.19) that

$$(7.29)\quad L_{\mathfrak{S}_j}(0,\phi) = (1-2\kappa)^2(1+2\kappa)^{-2}(1+b)^{-2}L_{\mathfrak{S}_i}[0,\phi/(1+b)].$$

Since $L_{\mathfrak{S}_i}(0,\phi)$ is a continuous function of ϕ, and since for sufficiently small δ, b and κ will be arbitrarily small [see (7.18), (7.17), (5.15a) and (5.4)], the difference

$$(7.30)\quad |L_{\mathfrak{S}_j}(0,\phi)-L_{\mathfrak{S}_i}(0,\phi)|$$

can be made arbitrarily small. The same is, therefore, true for

$$(7.31)\quad |\underline{\lim}\,\mathrm{K}_\nu(\mathfrak{D})-\overline{\lim}\,\mathrm{K}_\nu(\mathfrak{D})|$$

[see (7.15) and (7.24)]. Thus the limit

$$\lim \mathrm{K}_\nu(\mathfrak{D}) = L_{\mathfrak{D}}(0, \phi) \tag{7.32}$$

exists; and Theorem II is proved.

By using analogous methods we can formulate sufficient conditions in order that all boundary points $(0, Z_2)$ situated in a sufficiently small neighborhood of Q be L-points of the second order. To each of these points we then associate the limit $L_{\mathfrak{D}}(Z_2, \phi)$, which we define for $(0, Z_2)$ in the similar way as before $L_{\mathfrak{D}}(0, \phi)$ for $(0, 0)$. It is then possible to prove that $L_{\mathfrak{D}}(Z_2, \phi)$ is a continuous function of Z_2 and ϕ.

I take the opportunity of thanking Lipman Bers for some suggestions in connection with this paper.

BIBLIOGRAPHY.

Stefan Bergman,

1. "Über die Kernfunktion eines Bereiches and ihr Verhalten am Rande," *Crelle's Journal fur die reine und angewandte Mathematik*, vol. 169 (1933), pp. 1-42, and vol. 172 (1934), pp. 89-128.

2. "Zur Veranschaulichung der Kreiskörper and Bereiche mit ausgezeichneter Randfläche," *Jahresbericht d. Deutschen Mathematiker Vereinigung*, vol. 42 (1933), pp. 238-252.

3. "Theory of pseudo-conformal transformations and its connection with differential geometry," Notes on lectures delivered at the Massachusetts Institute of Technology, 1939-1940 (available at the Brown University library).

4. "Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques," Interscience Publishers, Inc., New York, 1941.

5. "The visualization of domains of the theory of functions of two complex variables," *Journal of Mathematics and Physics*, vol. 20 (1941), pp. 107-117.

Brown University.

PLANE SECTIONS THROUGH A POINT OF A NON-HOLONOMIC SURFACE.*

By Buchin Su.

In a previous paper[1] we showed that if there are given two arbitrary non-asymptotic tangents t_1 and t_2 of a (holonomic) surface S at an ordinary point O and a plane π_1 through t_1, we can always determine a quadric Q through the two asymptotic tangents of S at O such that the osculating conic of the section of S produced by π_1 and that of the section produced by the corresponding plane π_2 through t_2 lie on Q. This quadric has at O a contact of the second order with S and forms a pencil when π_1 turns about t_1. The present paper develops a similar theory for a non-holonomic surface in ordinary space. According to the non-holonomy we obtain some results different from those in the holonomic case. Take, for example, a quadric Q of the type referred to; there exist in general *four* other tangents through each of which a plane can be drawn such that the osculating conic of the corresponding section at O lies on Q. Moreover, we can determine certain lines and quadratic cones at a point of a non-holonomic surface by means of the neighborhoods of the third and the fourth order, which do not exist in the holonomic case. Especially, a covariant line is constructed by means of the neighborhood of the third order and a new extension of the Darboux pencil of quadrics is established.

1. Let a non-holonomic surface V_3^2 in ordinary space be defined by an equation of Pfaff, namely,

$$dz = p(x, y, z)\,dx + q(x, y, z)\,dy, \tag{1}$$

where the functions p and q do not satisfy the condition of integrability. At a generic point O of V_3^2 we can determine a projective system of reference in which the two asymptotic tangents, supposed to be distinct at O, are $x = z = 0$ and $y = z = 0$, so that the tangent plane of V_3^2 at O is $z = 0$, and such that the axis z and "the line at infinity" of the tangent plane are in correspondence of the projectivity of cell due to Bompiani,[2] so that at 0

---

* Received June 1, 1942.

[1] Cf. B. Su, "Osculating conics of the plane sections through a point of a surface," *American Journal of Mathematics*, vol. 65 (1943), pp. 439-449.

[2] Cf. E. Bompiani, "Sulle varietà anolonome, I. Alcuni teoremi generali; II. Le V_3^2 di S_3 proiettivo," *Rendiconti dei Lincei*, (6), vol. 27 (1938), pp. 37-52.

$$(2)\qquad p=q=0,\quad p_x=q_y=0,\quad p_z=q_z=0,$$

where the subscript denotes partial differentiation with respect to the indicated argument.

Consider the section of V_3^2 produced by a plane π through a non-asymptotic tangent t:

$$(3)\qquad z=y-nx=0;$$

if π be expressed by the equation

$$(4)\qquad z=\rho(y-nx),$$

where $\rho \neq 0, \neq \infty$, then the osculating conic of the section at O is given by the equations[3]

$$(5)\quad \begin{cases} \dfrac{n}{2\rho}(p_y+q_x)Y=\dfrac{n^2}{4\rho^2}(p_y+q_x)^2x^2-\frac{1}{6}\left(\dfrac{A_3}{\rho}+\dfrac{A_0n}{\rho^2}\right)xY \\ +\left\{\dfrac{1}{qn^2(p_y+q_x)^2}\left(A_3+\dfrac{A_0n}{\rho}\right)^2-\dfrac{1}{12n(p_y+q_x)}\left(B_4+\dfrac{C_3}{\rho}+\dfrac{D_0n}{\rho^2}\right)\right\}Y^2= \\ z=\rho Y, \end{cases}$$

where we have placed

$$(6)\qquad Y=y-nx$$

and

$$(7)\quad \begin{cases} A_0=(p_y+q_x)(p_y+2q_x), \\ A_3=p_{xx}+(2p_{xy}+q_{xx})n+(2q_{xy}+p_{yy})n^2+q_{yy}n^3, \\ D_0=(p_y+q_x)(p_y+2q_x)(p_y+3q_x), \\ C_3=q_{yy}(7p_y+9q_x)n^3+\{2p_{yy}(2p_y+3q_x)+q_{xy}(11p_y+15q_x)\}n^2 \\ \qquad +\{p_{xy}(5p_y+9q_x)+2q_{xx}(2p_y+3q_x)\}n+p_{xx}(3q_x+p_y), \\ B_4=p_{xxx}+\{3p_{xxy}+3p_{xz}(p_y+q_x)+q_{xxx}\}n \\ \qquad +\{3p_{xyy}+3p_{yz}(p_y+q_x)+3q_{xxy}+3q_{xz}(p_y+q_x)\}n^2 \\ \qquad +\{p_{yyy}+3q_{xyy}+3q_{yz}(p_y+q_x)\}n^3+q_{yyy}n^4. \end{cases}$$

2. Let us now take a tangent quadric Q of V_3^2 at O and determine whether three or more osculating conics of V_3^2 lie on Q. Since Q is given by an equation of the form

$$(8)\qquad z=ax^2+2bxy+cy^2+z(k_2x+k_3y+k_4z),$$

---

[3] Cf. B. Su, "On the projective differential geometry of a non-holonomic surface in ordinary space," *Annali di Matematica*, (4), vol. 19 (1940), pp. 289-313. The author is much indebted to Prof. Enea Bortolotti for his valuable correction and supplement. Cf. E. Bortolotti, "Sulla geometria proiettiva differenziale di una superficie anolonoma nello spazio ordinario," *Ibidem*, vol. 19 (1940), pp. 315-325.

a, b, c, k_2, k_3, k_4 being constants, the section of Q produced by the plane π is the conic

$$(9)\quad \begin{cases} -\dfrac{1}{\rho}Y + \dfrac{1}{\rho^2}(a + 2bn + cn^2)x^2 + \left\{2\,\dfrac{b+cn}{\rho^2} + \dfrac{k_2 + k_3 n}{\rho}\right\}xY \\ \qquad\qquad + \left(k_4 + \dfrac{k_3}{\rho} + \dfrac{c}{\rho^2}\right)Y^2 = 0, \\ \qquad\qquad z = \rho Y. \end{cases}$$

A comparison of these equations with (5) shows that the latter conic becomes an osculating conic of $V_3{}^2$ at O when and only when

$$(10)\quad \begin{cases} a + 2bn + cn^2 = \frac{1}{2}(p_y + q_x)n, \\ \dfrac{n}{2}(p_y + q_x)\{2\,\dfrac{b+cn}{\rho} + k_2 + k_3 n\} = \frac{1}{6}\left(\dfrac{A_0 n}{\rho} + A_3\right), \\ -\dfrac{n}{2}(p_y + q_x)\left(k_4 + \dfrac{k_3}{\rho} + \dfrac{c}{\rho^2}\right) \\ = \dfrac{1}{qn^2(p_y + q_x)^2}\left(A_3 + \dfrac{A_0 n}{\rho}\right)^2 - \dfrac{1}{12n(p_y + q_x)}\left(B_4 + \dfrac{C_3}{\rho} + \dfrac{D_0 n}{\rho^2}\right). \end{cases}$$

For a quadric Q given by (8) there are in general two tangents (3) satisfying the first equation of (10), but no value of ρ can be obtained so as to satisfy the remaining two equations. Therefore we may conclude that no osculating conic of the section of $V_3{}^2$ at O can lie on a general tangent quadric Q of $V_3{}^2$ at O.[4]

There is, however, an exceptional case where the first equation of (10) is satisfied identically in n, so that

$$a = 0, \quad b = \tfrac{1}{4}(p_y + q_x), \quad c = 0.$$

The corresponding quadric (8) becomes

$$(11)\qquad z = \tfrac{1}{2}(p_y + q_x)xy + z(k_2 x + k_3 y + k_4 z),$$

which has at O a contact of the second order with $V_3{}^2$; while the conditions (10) take the form

$$(12)\qquad k_2 + k_3 n = -\tfrac{1}{6}(p_y - q_x)\,\frac{1}{\rho} + \frac{A_3}{3n(p_y + q_x)},$$

$$(13)\qquad k_4 + \frac{k_3}{\rho} = \frac{3n(p_y + q_x)B_4 - 4A_3{}^2}{18n^3(p_y + q_x)^3} + \frac{3C_3 - 8(p_y + 2q_x)A_3}{18n^2(p_y + q_x)^2\rho} - \frac{(p_y + 2q_x)(p_y - q_x)}{18n(p_y + q_x)\rho^2}.$$

---

[4] If we determine ρ from the second equation of (10), where n is a root of the first equation, and substitute it into the last equation, there is found a relation between a, b, c, k_2, k_3 and k_4, which does not hold in the general case.

3. As an analogue of the problem for a surface we propose to determine a quadric Q of the system furnished by equation (11) containing both the osculating conics of sections of V_3^2 produced by planes through two arbitrary non-asymptotic tangents t_1, t_2 at O. Write the equations of t_i $(i=1,2)$ as

$$z = y - n_i x = 0$$

and the equation of the corresponding plane π_i as

$$z = \rho_i (y - n_i x)\,;$$

it follows from (12) and (13) that four conditions

$$(14)\quad k_2 + k_3 n_i = -\tfrac{1}{6}(p_y - q_x)\frac{1}{\rho_i} + \frac{A_3(n_i)}{3n_i(p_y+q_x)},$$

$$(15)\quad k_4 + \frac{k_3}{\rho_i} = \frac{3n_i(p_y+q_x)B_4(n_i) - 4A_3^2(n_i)}{18n_i^3(p_y+q_x)^3} + \frac{3C_3(n_i) - 8(p_y+2q_x)A_3(n_i)}{18n_i^2(p_y+q_x)^2\rho_i} - \frac{(p_y+2q_x)(p_y-q_x)}{18n_i(p_y+q_x)\rho_i^2} \qquad (i=1,2)$$

should be satisfied for arbitrarily assigned n_1, n_2, where $A_3(n_i)$, $C_3(n_i)$ and $B_4(n_i)$ are given by (7), n being replaced by n_i.

Solving (14) for k_2 and k_3 we have

$$(16)\quad \left\{\begin{aligned} k_2 &= -\tfrac{1}{6}(p_y - q_x)\frac{\rho_1 n_1 - \rho_2 n_2}{\rho_1\rho_2(n_1-n_2)} \\ &\quad + \frac{1}{3(p_y+q_x)}\left\{\frac{n_1+n_2}{n_1 n_2}p_{xx} + 2p_{xy} + q_{xx} - q_{yy}n_1 n_2\right\} \\ k_3 &= \tfrac{1}{6}(p_y-q_x)\frac{\rho_1-\rho_2}{\rho_1\rho_2(n_1-n_2)} \\ &\quad + \frac{1}{3(p_y+q_x)}\left\{-\frac{1}{n_1 n_2}p_{xx} + 2q_{xy} + p_{yy} + q_{yy}(n_1+n_2)\right\}. \end{aligned}\right.$$

In order that the two equations (15) should be compatible, it is necessary and sufficient that

$$\begin{aligned} k_3\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) &= \frac{1}{18(p_y+q_x)^3}\Bigg[3(p_y+q_x)\left\{\frac{1}{n_1^2}B_4(n_1) - \frac{1}{n_2^2}B_4(n_2)\right\} \\ &\qquad - 4\left\{\frac{1}{n_1^3}A_3^2(n_1) - \frac{1}{n_2^3}A_3^2(n_2)\right\}\Bigg] \\ &\quad + \frac{1}{18(p_y+q_x)^2}\Bigg[3\left\{\frac{1}{n_1^2\rho_1}C_3(n_1) - \frac{1}{n_2^2\rho_2}C_3(n_2)\right\} \\ &\qquad - 8(p_y+2q_x)\left\{\frac{1}{n_1^2\rho_1}A_3(n_1) - \frac{1}{n_2^2\rho_2}A_3(n_2)\right\}\Bigg] \\ &\quad - \frac{(p_y+2q_x)(p_y-q_x)}{18(p_y+q_x)}\left\{\frac{1}{\rho_1^2 n_1} - \frac{1}{\rho_2^2 n_2}\right\}. \end{aligned}$$

Substitution of k_3 given by (16) shows that the two planes π_1 and π_2 are in the following correspondence:

$$
\begin{aligned}
(17)\qquad & \frac{p_y - q_x}{n_1 - n_2} \Big\{ \frac{(2n_1 + n_2)p_y + (n_1 + 2n_2)q_x}{n_1 \rho_1^2} - 6(p_y + q_x)\, \frac{1}{\rho_1 \rho_2} \\
& \qquad + \frac{(n_1 + 2n_2)p_y + (2n_1 + n_2)q_x}{n_2 \rho_2^2} \Big\} \\
& + \frac{1}{p_y + q_x} \Big[6(p_y + q_x) \Big\{ \frac{p_{xx}}{n_1 n_2} - (2q_{xy} + p_{yy}) - q_{yy}(n_1 + n_2) \Big\} \\
& \qquad + 3\, \frac{1}{n_1^2}\, C_3(n_1) - 8(p_x + 2q_y) \frac{1}{n_1^2}\, A_3(n_1) \Big] \frac{1}{\rho_1} \\
& - \frac{1}{p_y + q_x} \Big[6(p_y + q_x) \Big\{ \frac{p_{xx}}{n_1 n_2} - (2q_{xy} + p_{yy}) - q_{yy}(n_1 + n_2) \Big\}. \\
& \qquad + 3\, \frac{1}{n_2^2}\, C_3(n_2) - 8(p_y + 2q_x) \frac{1}{n_2^2} A_3(n_2) \frac{1}{\rho_2} \\
& + \frac{1}{(p_y + q_x)^2} \Big[3(p_y + q_x) \Big\{ \frac{B_4(n_1)}{n_1^2} - \frac{B_4(n_2)}{n_2^2} \Big\} \\
& \qquad\qquad - 4 \Big\{ \frac{A_3^2(n_1)}{n_1^3} - \frac{A_3^2(n_2)}{n_2^3} \Big\} \Big] = 0.
\end{aligned}
$$

Thus we have the

THEOREM. *If two planes through a generic point O of a non-holonomic surface V_3^2 are drawn such that each of them passes through one of two arbitrarily assigned non-asymptotic tangents and the osculating conics of the sections of V_3^2 produced by these planes lie on a quadric Q tangent to V_3^2 at O, then Q must have at O a contact of the second order with V_3^2 and the correspondence between these two planes is* (2, 2).

In the holonomic case (17) becomes a linear equation in $1/\rho_1$ and $1/\rho_2$, and therefore the correspondence is a perspectivity, as we have shown before.[5]

4. It is natural to inquire whether other osculating conics of V_3^2 at O belong to the same quadric Q thus determined. If there is such a conic, then, expressing the plane of the conic by the equation

$$(18)\qquad z = \rho_3(y - n_3 x),$$

we have

$$(19)\qquad k_2 + k_3 n_3 = -\tfrac{1}{6}(p_y - q_x)\, \frac{1}{\rho_3} + \frac{A_3(n_3)}{3n_3(p_y + q_x)},$$

---

[5] Cf. B. Su, *loc. cit.*[1]

$$(20)\quad k_4+\frac{k_3}{\rho_3}=\frac{3n_3(p_y+q_x)B_4(n_3)-4A_3{}^2(n_3)}{18n_3{}^3(p_y+q_x)^3}+\frac{3C_3(n_3)-8(p_y+2q_x)A_3(n_3)}{18n_3{}^2(p_y+q_x)^2\rho_3}-\frac{(p_y+2q_x)(p_y-q_x)}{18n_3(p_y+q_x)\rho_3{}^2}.$$

Substituting the value of k_3 given by (16) into (19) and reducing, we obtain

$$(21)\quad (p_y-q_x)\left\{(n_2-n_3)\frac{1}{\rho_1}+(n_3-n_1)\frac{1}{\rho_2}+(n_1-n_2)\frac{1}{\rho_3}\right\}+\frac{2}{p_y+q_x}(n_1-n_2)(n_2-n_3)(n_3-n_1)\{p_{xx}(n_1n_2n_3)^{-1}+q_{yy}\}=0.$$

As the coefficient of the term $\rho_3{}^{-1}$ in the last equation is different from zero, we can express $\rho_3{}^{-1}$ as a function of n_3. Further substitution of the latter into (20) suffices to demonstrate that the resulting equation is an algebraic equation of the sixth degree in n_3.[6]

For each root of this equation we can find the corresponding ρ_3 from (20) and therefore obtain the osculating conic on the quadric Q in consideration. Hence we obtain the

THEOREM. *If a quadric Q contains two osculating conics of $V_3{}^2$ at O with distinct tangents, then it contains also four other osculating conics of $V_3{}^2$ at O.*

For a holonomic surface the equation (21) becomes particularly simple and *uniquely* determines the third tangent.[7]

Conversely, *on a quadric which has at O a contact of the second order with $V_3{}^2$ there are in general six osculating conics of $V_3{}^2$ at O.*

In fact, the equation of such a quadric is of the form (11), where k_2, k_3, k_4 are given *a priori*. If ρ be eliminated from (12) and (13), the resulting equation is evidently of the sixth degree in n and consequently gives six tangents of $V_3{}^2$ at O. In virtue of (12) we have the corresponding value of ρ and therefore the osculating conic of $V_3{}^2$ at O.

5. On account of the correspondence (17) between the planes π_1 and π_2 we can further find a line L_{12} associated with the given tangents t_1 and t_2. For this purpose, let us consider the two planes π_2' and π_2'' corresponding to a plane π_1 under (17) and construct the harmonic conjugate $\bar{\pi}_2$ of the tangent plane $z = 0$ with respect to π_2' and π_2''; the equation to $\bar{\pi}_2$ is obviously of the form

---

[6] Two roots of this equation are evidently n_1 and n_2; to these correspond the original conics.

[7] Cf. Su, *loc. cit.*[1]

$$\tag{22} \tfrac{1}{2}(1/\rho_2' + 1/\rho_2'')z = y - n_2 x.$$

where $1/\rho_2'$, $1/\rho_2''$ are roots of the quadratic equation (17) in $1/\rho_2$. A simple calculation shows that these planes π_1 and $\bar{\pi}_2$ are in perspective position, the line of intersection being on the same plane π_{12}:

$$\tag{23} \begin{aligned} &2(p_y^2 - q_x^2)[(p_y + 2q_x)y + (2p_y + q_x)n_2 x] \\ &\quad - z[p_{xx}\{6(p_y + q_x)(1/n_1 - 1/n_2) + (p_y - q_x)(1/n_2)\} - (p_y + 5q_x)p_{xy} \\ &\quad + 2(2p_y + q_x)q_{xx} - 6(p_y + q_x)q_{yy}n_2(n_1 - n_2) + (p_y - q_x)q_{yy}n_2^2 \\ &\quad + \{(5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}\}n_2] = 0. \end{aligned}$$

In a similar way we may define the plane π_{21} by interchanging π_1 and π_2. From the equation (23) and that obtained by interchanging n_1 and n_2 in (23) it follows that the intersection L_{12} of π_{12} and π_{21} is given by the equations

$$\tag{24} \left\{ \begin{aligned} &2(2p_y + q_x)(p_y^2 - q_x^2)x + z[(7p_y + 5q_x)q_{yy}(n_1 + n_2) \\ &\quad + 2(p_y + 2q_x)p_{yy} - (5p_y + q_x)q_{xy} - (11p_y + 13q_x)p_{xx}(n_1 n_2)^{-1}] = 0. \\ &2(2q_x + p_y)(q_x^2 - p_y^2)y + z[(7q_x + 5p_y)p_{xx}(1/n_1 + 1/n_2) \\ &\quad + 2(q_x + 2p_y)q_{xx} - (5q_x + p_y)p_{xy} - (11q_x + 13p_y)q_{yy}n_1 n_2] = 0. \end{aligned} \right.$$

If we consider all the lines L_{12} corresponding to the tangents apolar with respect to the asymptotic tangents at O, then the locus is a quadratic cone, namely,

$$\tag{25} \begin{aligned} &(11p_y + 13q_x)(11q_x + 13p_y)p_{xx}q_{yy}z^2 \\ &\quad = [2(2p_y + q_x)(p_y^2 - q_x^2)x + \{2(p_y + 2q_x)p_{yy} - (5p_y + q_x)q_{xy}\}z] \\ &\quad \cdot [2(2q_x + p_y)(q_x^2 - p_y^2)y + \{2(q_x + 2p_y)q_{xx} - (5q_x + p_y)p_{xy}\}z]. \end{aligned}$$

The polar line $\mathfrak{L}$ of the tangent plane $z = 0$ with respect to the cone (25) is determined by the neighborhood of the third order of V_3^2 at O. From (25) we have the equations of $\mathfrak{L}$:

$$\tag{26} \left\{ \begin{aligned} y &= \frac{2(2p_y + q_x)q_{xx} - (p_y + 5q_x)p_{xy}}{2(p_y + 2q_x)(p_y^2 - q_x^2)}\, z, \\ x &= \frac{2(2q_x + p_y)p_{yy} - (q_x + 5p_y)q_{xy}}{2(q_x + 2p_y)(q_x^2 - p_y^2)}\, z. \end{aligned} \right.$$

6. We are now in a position to investigate the line of intersection of two planes π_1, π_2 in the correspondence (17).

In virtue of (7) and (17) there is no difficulty in determining the locus of the line of intersection. The result of carrying out the computation is as follows:

$$
\begin{aligned}
(27)\qquad & (p_y - q_x)\left\{\frac{p_y + 2q_x}{n_1 n_2}\, y^2 + (2p_y + q_x)x^2\right\} \\
& + \frac{yz}{p_y + q_x}\Big[(5p_y + 7q_x)p_{xx}\,\frac{n_1 + n_2}{n_1^2 n_2^2} + \{(p_y + 5q_x)p_{xy} - 2(2p_y + q_x)q_{xx}\} \\
& \qquad\qquad + (13p_y + 11q_x)q_{yy}\Big] \\
& - \frac{xz}{p_y + q_x}\Big[(11p_y + 13q_x)\,\frac{p_{xx}}{n_1 n_2} + (q_x + 5p_y)q_{xy} - 2(2q_x + p_y)p_{yy} \\
& \qquad\qquad + (7p_y + 5q_x)q_{yy}(n_1 + n_2)\Big] \\
& + \Big[4\left(\frac{p_{xx}^2}{n_1^3 n_2^3} - q_{yy}^2\right)\cdot(n_1^2 + n_1 n_2 + n_2^2) \\
& \qquad + 8\left\{\frac{p_{xx}(2p_{xy} + q_{xx})}{n_1^2 n_2^2} - (2q_{xy} + p_{yy})q_{yy}\right\}(n_1 + n_2) \\
& + 4\{(2p_{xy} + q_{xx})^2 + 2p_{xx}(2q_{xy} + p_{yy})\}\frac{1}{n_1 n_2} \\
& - 4\{(2q_{xy} + p_{yy})^2 + 2q_{yy}(2p_{xy} + q_{xx})\} \\
& + 3(p_y + q_x)\Big\{ - p_{xxx}\,\frac{n_1 + n_2}{n_1^2 n_2^2} + p_{yyy} + 3q_{xyy} \\
& + 3q_{yz}(p_y + q_x) + q_{yyy}(n_1 + n_2) \\
& - (3p_{xxy} + 3p_{xz}(p_y + q_x) + q_{xxx})\,\frac{1}{n_1 n_2}\Big\}\Big](p_y + q_x)^{-2}z^2 = 0.
\end{aligned}
$$

This quadratic cone K_{12} associated with two non-asymptotic tangents t_1 and t_2 is evidently determined by the neighborhood of the fourth order of V_3^2 at O. It is easily seen that the polar of the tangent plane of V_3^2 at O with respect to K_{12} just coincides with the line L_{12} associated with the same pair of tangents.

In particular, we consider the three lines L_{12} associated with two of *the Segre tangents*

$$(28)\qquad p_{xx}x^3 - q_{yy}y^3 = 0.$$

Put

$$(29)\qquad \lambda = (p_{xx}/q_{yy})^{1/3}, \quad \epsilon^3 = 1 \quad (\epsilon \neq 1);$$

then the lines in question are given by

$$
\begin{aligned}
(30)\qquad & x : y : z \\
& = \frac{p_y + 2q_x}{p_y^2 - q_x^2}\{4(p_y + 2q_x)q_{yy}\lambda\epsilon^r +)5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}\} \\
& : \frac{q_x + 2p_y}{q_x^2 - p_y^2}\{4(q_x + 2p_y)p_{xx}\frac{1}{\lambda}\,\epsilon^{2r} + (5q_x + p_y)p_{xy} - 2(q_x + 2p_y)q_{xx}\} \\
& : 2(p_y + 2q_x)(q_x + 2p_y) \qquad\qquad (r = 1, 2, 3).
\end{aligned}
$$

Consequently, *the harmonic conjugate of the tangent plane $z = 0$ with respect to the trihedron formed by these lines is the line $\mathfrak{L}$,* as may easily be derived from (30) and (26).

The same result holds for three tangents in the cubic involution

$$p_{xx}x^3 = \mu q_{yy}y^3 = 0, \tag{31}$$

and especially for *three Darboux tangents.*

7. It is worth noticing that the quadratic cone K_{12} associated with two Darboux tangents or two Segre tangents gives certain covariant lines. Take, for example, two Segre tangents:

$$\begin{cases} z = y - n_i x = 0 & (i = 1, 2), \\ n_1 = \lambda\epsilon^r, \quad n_2 = \lambda\epsilon^s \end{cases} \tag{32}$$
$$(r \neq s,\ s \neq t,\ t \neq r; \quad r, s, t = 1, 2, 3)$$

and substitute them into (27); we easily arrive at the cone:

$$\begin{aligned} (33) \qquad & (p_y - q_x) \left\{ \frac{p_y + 2q_x}{\lambda^2}\, \epsilon^t y^2 + (2p_y + q_x)x^2 \right\} \\ & + \frac{yz}{p_y + q_x}\, [4(2p_y + q_x)q_{yy} + \frac{1}{\lambda^2}\, \{(p_y + 5q_x)p_{xy} - 2(2p_y + q_x)q_{xx}\}\epsilon^t] \\ & - \frac{xz}{p_y + q_x}\, [4(2q_x + p_y)p_{xx}\frac{\epsilon^t}{\lambda^2} + (5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}] \\ & + \frac{z^2}{(p_y + q_x)^2}\, [3(p_y + q_x) \left\{ \frac{p_{xxx}}{p_{xx}} q_{yy} + 3q_{xyy} + 3q_{yz}(p_y + q_x) + p_{yyy} \right. \\ & \left. - (q_{xxx} + 3p_{xxy} + 3p_{xz}(p_y + q_x) + \frac{q_{yyy}}{q_{yy}}\, p_{xx})\lambda^{-2}\epsilon^t \right\} \\ & - 4\{(2q_{xy} + p_{yy})^2 + 4(2p_{xy} + q_{xx})q_{yy}\} \\ & + 4\{(2p_{xy} + q_{xx})^2 + 4(2q_{xy} + p_{yy})p_{xx}\}\lambda^{-2}\epsilon^t] = 0. \end{aligned}$$

Consequently *the three quadratic cones K_{12} associated with two of three Segre tangents belong to a pencil and therefore determine four lines of intersection,* namely, the lines given by the equations

$$\begin{aligned} (34) \quad & (p_y - q_x)(p_y + 2q_x)y^2 + \frac{yz}{p_y + q_x}\{(p_y + 5q_x)p_{xy} - 2(2p_y + q_x)q_{xx}\} \\ & - 4\,\frac{p_{xx}(p_y + 2q_x)}{p_y + q_x}\, xz + \frac{z^2}{(p_y + q_x)^2}\, [4(2p_{xy} + q_{xx})^2 + 16(2q_{xy} + p_{yy})p_{xx} \\ & - 3(p_y + q_x)\{q_{xxz} + 3p_{xxy} + 3p_{xz}(p_y + q_x) + \frac{q_{yyy}}{q_{yy}}\, p_{xx}\}] = 0, \end{aligned}$$

$$(35)\quad (q_x - p_y)(q_x + 2p_y)x^2 + \frac{xz}{p_y + q_x}\{(q_x + 5p_y)q_{xy} - 2(2q_x + p_y)p_{yy}\}$$
$$- 4\frac{q_{yy}(q_x + 2p_y)}{p_y + q_x}\, yz + \frac{z^2}{(p_y + q_x)^2}[4(2q_{xy} + p_{yy})^2 + 16(2p_{xy} + q_{xx})q_{yy}$$
$$- 3(p_y + q_x)\{p_{yyy} + 3q_{xyy} + 3q_{yz}(p_y + q_x) + \frac{p_{xxx}}{p_{xx}}\, q_{yy}\}] = 0.$$

The equations (34) and (35) represent the cones in the pencil touching the tangent plane $z = 0$ along the asymptotic tangents $y = z = 0$ and $x = z = 0$ respectively. The polar plane of the tangent $x = z = 0$ with respect to (34) and that of $y = z = 0$ with respect to (35) intersect in the line $\mathcal{L}$.

Moreover, a covariant cone similar to (25) is found as the locus of the polar of $z = 0$ with respect to a cone in the pencil above quoted, namely,

$$(36)\qquad 16p_{xx}q_{yy}(p_y + 2q_x)(q_x + 2p_y)z^2$$
$$= [2(p_y^2 - q_x^2)(2p_y + q_x)x + \{2(p_y + 2q_x)p_{yy} - (q_x + 5p_y)q_{xy}\}z]$$
$$\cdot\ [2(q_x^2 - p_y^2)(2q_x + p_y)y + \{2(q_x + 2p_y)q_{xx} - (p_y + 5q_x)p_{xy}\}z];$$

the polar of $z = 0$ is also the line $\mathcal{L}$.

For the Darboux tangents we obtain similarly four other covariant lines determined by the quadratic cones:

$$(37)\qquad (p_y - q_x)(2p_y + q_x)x^2 + 18q_{yy}yz$$
$$- \frac{xz}{p_y + q_x}\{(5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}\}$$
$$+ \frac{z^2}{(p_y + q_x)^2}[3(p_y + q_x)\{p_{yyy}$$
$$+ 3q_{xyy} + 3q_{yz}(p_y + q_x) - \frac{p_{xxx}}{p_{xx}}\, q_{yy}\}$$
$$- 4(2q_{xy} + p_{yy})^2] = 0,$$

$$(38)\qquad (q_x - p_y)(2q_x + p_y)y^2 + 18p_{xx}xz$$
$$- \frac{yz}{p_y + q_x}\{(5q_x + p_y)p_{xy} - 2(q_x + 2p_y)q_{xx}\}$$
$$+ \frac{z^2}{(p_y + q_x)^2}[3(p_y + q_x)\{q_{xxx} + 3p_{xxy} + 3p_{xz}(p_y + q_x) - \frac{q_{yyy}}{q_{yy}}\, p_{xx}\}$$
$$- 4(2p_{xy} + q_{xx})^2] = 0.$$

Another quadratic cone is derived from the pencil of these cones, namely,

$$
\begin{aligned}
(39) \qquad & 324 p_{xx} q_{yy} (p_y + q_x)^2 z^2 \\
&= [2(p_y^2 - q_x^2)(2p_y + q_x)x + \{2(p_y + 2q_x)p_{yy} - (5p_y + q_x)q_{xy}\}z] \\
&\quad \cdot [2(q_x^2 - p_y^2)(2q_x + p_y)y + \{2(q_x + 2p_y)q_{xx} - (5q_x + p_y)p_{xy}\}z].
\end{aligned}
$$

The line $\mathfrak{L}$ may be defined as the polar of the tangent plane $z = 0$ with respect to the cone (39) or the intersection of the polar planes of $y = z = 0$ and $x = z = 0$ with respect to the cones (37) and (38) respectively.

Further we remark one more relation between the line $\mathfrak{L}$ and the cones K_{12} associated with the Segre tangents: *Let* π_r $(r = 1, 2, 3)$ *be the polar plane of a Segre tangent* S_r *with respect to the cone* K_{12} *associated with the remaining Segre tangents; then the harmonic of the tangent plane* $z = 0$ *with respect to the trihedron* $\{\pi_1\pi_2\pi_3\}$ *is the line* $\mathfrak{L}$.

8. As has been pointed out by E. Bortolotti,[8] a line can be defined which is associated with a pencil of quadrics

$$
(40) \qquad z = \tfrac{1}{2}(p_y + q_x)xy + (\alpha x + \beta y + \gamma z)z.
$$

It is given by the equations

$$
(41) \qquad \begin{cases} y = 2\, \dfrac{2p_{xy} + q_{xx} - 3\alpha(p_y + q_x)}{p_y^2 - q_x^2}\, z, \\[2ex] x = 2\, \dfrac{2q_{xy} + p_{yy} - 3\beta(p_y + q_x)}{q_x^2 - p_y^2} \cdot z. \end{cases}
$$

Conversely, a pencil of quadrics having at O a contact of the second order with V_3^2 is determined provided that a certain covariant line through O be given.

A comparison of (26) and (41) gives

$$
(42) \qquad \begin{cases} \alpha = \dfrac{(25p_y + 53q_x)p_{xy} + 2(4p_y + 11q_x)q_{xx}}{36(p_y + q_x)(p_y + 2q_x)}, \\[2ex] \beta = \dfrac{(25q_x + 53p_y)q_{xy} + 2(4q_x + 11p_y)p_{yy}}{36(p_y + q_x)(q_x + 2p_y)}. \end{cases}
$$

Hence *the pencil of quadrics associated with the covariant line* $\mathfrak{L}$ *furnishes a new generalization of the Darboux pencil to a non-holonomic surface.*

NATIONAL UNIVERSITY OF CHEKIANG,
KWEICHOW, CHINA.

---

[8] Cf. Bortolotti, *loc. cit.*, p. 322.

THE PROJECTIVE THEORY OF SURFACES IN RULED SPACE, I.*

By Chenkuo Pa.

Introduction. The object of this paper is to study the projective properties of a surface in ruled space. In this treatment certain asymptotic ruled surfaces play an important role. The usual method for investigating these ruled surfaces requires, in general, some lengthy computation. We can however overcome the difficulty by means of a new process which seems simpler than that we have so far used and is particularly useful in the determination of equations of some geometrical elements associated with these ruled surfaces in the normal coordinates of Fubini.

The contents of this paper are divided into two parts. The first part is devoted to the derivation of some invariant quadric surfaces, invariant cones, and invariant curves projectively connected with a point of a general surface. In the last section of this part we further obtain a new definition for any ray in the second canonical pencil such that it may be considered as the principal line of Bompiani[1] for a pair of curves, suitably chosen on the surface at the point. The notions of the Darboux quadrics and of Bompiani-Lane's principal quadrics[2] have been generalized.

In the second part of this paper the equation of the osculating linear complex of every asymptotic osculating ruled surface is first determined, some new invariant quadrics are then introduced as analogues of the Lie quadric and certain special surfaces are finally defined. Among other things we have improved a theorem of Sullivan.[3] A general treatment of the asymptotic ruled surfaces and other ruled surfaces associated with a point such as the asymptotic chord surfaces[4] will be established in a sequel paper. Furthermore we remark that a certain quadric possessing many remarkable properties is defined in a simple way, and that a new property of the Darboux pencil of quadrics has been added. By means of them we can give a simple characterization to the

---

* Received June 1, 1942.

[1] E. Bompiani, "Invarianti d'intersezione di due curve schembe," *Rendiconti dei Lincei*, (VI), vol. 14 (1931), pp. 456-461.

[2] E. Bompiani, *loc. cit.*; E. P. Lane, "Invariants of intersection of two curves on a surface," *American Journal of Mathematics*, vol. 54 (1932), pp. 699-706.

[3] C. T. Sullivan, "Properties of surfaces whose asymptotic curves belong to linear complexes," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 167-196.

[4] E. Bompiani, "Sugli elementi di 2° ordine delle curve di una superficie," *Rendiconti dei Lincei*, vol. 9 (1929), pp. 288-294.

surfaces whose asymptotic curves belong to linear complexes, the isothermal-asymptotic surfaces, the surfaces whose Lie quadrics all touch a fixed plane[5] etc.

PART A.

The Asymptotic Ruled Surface Q_u^h or Q_v^h.

Let S be an analytic surface in ordinary space referred to its asymptotic net (u, v). It is well known that the non-homogeneous coordinates x, y, z of a generic point P on the surface are solutions of the following system of differential equations,

$$\begin{aligned} x_{uu} &= \theta_u x_u = \beta x_v, \\ x_{vv} &= \theta_v x_v + \gamma x_u, \end{aligned} \tag{1}$$

and similar equations for y and z.

The integrability conditions of this system are found to be

$$\begin{aligned} (\theta_{uv} + \beta\gamma)_u - \theta_u(\theta_{uv} + \beta\gamma) &= (\beta_v + \beta\theta_v)_v - \beta(\gamma_u + 2\gamma\theta_u), \\ (\theta_{uv} + \beta\gamma)_v - \theta_v(\theta_{uv} + \beta\gamma) &= (\gamma_u + \gamma\theta_u)_u - \gamma(\beta_v + \beta\theta_v). \end{aligned} \tag{2}$$

For the sake of convenience we normalize the coordinates so that at P $(u = 0,\ v = 0)$

$$x = y = z = x_v = z_v = y_u = z_u = y_{uv} = x_{uv} = 0, \quad x_u = y_v = z_{uv} = 1. \tag{3}$$

Then the equation of S in the neighborhood of P can be expanded as follows[6]

$$\begin{aligned} z = xy - \tfrac{1}{3}(\beta x^3 + \gamma y^3) + \tfrac{1}{12}[&(2\beta\theta_u - \beta_u)x^4 - 4(\beta_v + \beta\theta_v)x^3 y \\ &- 6\theta_{uv}x^2y^2 - 4(\gamma_u + \gamma\theta_u)xy^3 + (2\gamma\theta_v - \gamma_v)y^4] + (5). \end{aligned} \tag{4}$$

Suppose that a curve C_u^h of the surface S passes through P and touches at P the asymptotic tangent t_0 $(u = 0)$; it is evidently given by the expression

$$u = -(h/2)\gamma_v^2 + (3), \qquad (h = \text{const.}), \tag{5}$$

where we have put $h = 1 - j$, j being the contact invariant of C_u^h and the asymptotic curve $u = 0$ at P.[7]

Consider the asymptotic ruled surface Q_u^h composed of the tangents to the asymptotic curves $v = \text{const.}$ at the points of the curve $C_u{}^h$, and, similarly, the asymptotic ruled surface $Q_v{}^h$. A generic point $(\bar{x})$ of $Q_u{}^h$ has the coordinates

---

[5] These surfaces were first studied from a projective point of view by Buchin Su. Cf. his paper " On the surfaces whose Lie quadrics all touch a fixed plane, *Science Reports, Univ. of Chekiang*, vol. 2 (1936), pp. 53-61.

[6] Cf. Fubini et Čech, *Introduction à la Géométrie Projective Différentielle des Surfaces*, Paris (1931), p. 73.

[7] Fubini et Čech, *loc. cit.*, p. 56.

$$\bar{x} = \lambda x_u + x, \text{ etc.} \tag{6}$$

Since $\lambda = 0$ at P, we can easily show that the orders of the neighborhoods of $Q_u{}^h(Q_v{}^h)$ and the surface S at P are equal to each other. In what follows we shall give a process for investigating the geometry of these ruled surfaces associated with the point P.

1. **The fundamental form of $Q_u{}^h$.** From (6), we have

$$\bar{x}_\lambda = x_u, \quad \bar{x}_{\lambda\lambda} = 0, \quad \bar{x}_{\lambda v} = x_{uv} + (\theta_u x_u + \beta x_v)\frac{du}{dv},$$

$$\bar{x}_v = (*)x_u + \lambda x_{uv} + \left(\lambda\beta\frac{du}{dv} + 1\right)x_v, \quad \bar{x}_{vv} = \bar{A}x_v + \bar{B}x_{uv} + (*)x_u,$$

where the quantities $\bar{A}$, $\bar{B}$ are given by

$$\bar{A} = \lambda(\theta_{uv} + \beta\gamma) + 2\lambda(\beta_v + \beta\theta_v)\frac{du}{dv} + \lambda\left(\frac{du}{dv}\right)^2(\beta_u + \beta\theta_u) + \lambda\beta\frac{d^2u}{dv^2} + \theta_v + \beta$$

$$\bar{B} = \lambda\theta_v + 2\lambda\theta_u\frac{du}{dv} + \lambda\beta\left(\frac{du}{dv}\right)^2 + 2\frac{du}{dv}.$$

Therefore

$$D'' = (\bar{x}_v\bar{x}_\lambda\bar{x}_{\lambda\lambda}) = 0, \qquad D' = (\bar{x}_v\bar{x}_\lambda\bar{x}_{\lambda v}) = (x_v x_u x_{uv}),$$

$$D = (\bar{x}_v\bar{x}_\lambda\bar{x}_{vv}) = \left[-\lambda\bar{A} + \left(\lambda\beta\frac{du}{dv} + 1\right)\bar{B}\right](x_v x_u x_{uv}),$$

so that the differential equation of the curved asymptotic lines on $Q_u{}^h$ is

$$\begin{aligned} \frac{d\lambda}{dv} = \tfrac{1}{2}\lambda^2 \Big[\theta_{uv} + \beta\gamma + 2\beta_v\frac{du}{dv} + (\beta_u - \beta\theta_u)\left(\frac{du}{dv}\right)^2 \\ + \beta\frac{d^2u}{dv^2} + \beta\frac{du}{dv}\left(\theta_v - \beta\left(\frac{du}{dv}\right)^2\right)\Big] - \lambda\left[\theta_u\frac{du}{dv} + \beta\left(\frac{du}{dv}\right)^2\right] - \frac{du}{dv}. \end{aligned} \tag{7}$$

Let "d" denote differentiation along one of these curved asymptotic lines, then

$$\frac{d\bar{x}}{dv} = lx_u + mx_v + \lambda x_{uv},$$

where we have put

$$l = \lambda\theta_u\frac{du}{dv} + \frac{du}{dv} + \frac{d\lambda}{dv}, \qquad m = \lambda\beta\frac{du}{dv} + 1. \tag{8}$$

We denote, for brevity, the quantity $d\bar{x}/dv$ by $\bar{x}_v$, although this notation has been already used in a different sense.

Moreover, we put

$$e = Sx_u{}^2, \quad f = Sx_u x_v, \quad g = Sx_v{}^2, \quad r = Sx_u x_{uv}, \quad s = Sx_v x_{uv}, \quad t = Sv_{uv}{}^2,$$
$$E' = S\bar{x}_\lambda{}^2 = Sx_u{}^2 = e, \quad F' = S\bar{x}_\lambda \bar{x}_v = le + mf + \lambda r,$$
$$G' = S\bar{x}_v{}^2 = l^2 e + m^2 g + \lambda^2 t + 2lmf + 2l\lambda r + 2m\lambda s,$$

and introduce a new asymptotic parameter w of $Q_u{}^h$ instead of λ such that at P

$$\left(\frac{\partial\lambda}{\partial w}\right)_0 = 1.$$

On account of (7) it is readily seen that

$$(9) \qquad \left(\frac{\partial^2\lambda}{\partial w \partial v^2}\right)_0 = 0, \quad \left(\frac{\partial^3\lambda}{\partial w^2 \partial v}\right) = \theta_{uv} + (1-h)\beta\gamma, \quad \left(\frac{\partial^3\lambda}{\partial w \partial v^2}\right)_0 = \gamma h \theta_u.$$

Therefore the generic point $(\bar{x})$ of the surface $Q_u{}^h$ satisfies the following differential equations:

$$(10) \qquad \bar{x}_{ww} = \frac{\partial^2\lambda}{\partial w^2} \frac{1}{\dfrac{\partial\lambda}{\partial w}} \bar{x}_w, \qquad \bar{x}_{vv} = \bar{\theta}_v \bar{x}_v + \gamma \bar{x}_w,$$

where

$$(11) \quad \bar{\theta}_v = \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = \frac{EG_v + FG_w - 2FF_v}{2H^2}, \quad \bar{\gamma} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = \frac{-FG_v - GG_w + 2GF_v}{2H^2}$$

and

$$(12) \quad E = S\bar{x}_w{}^2 = \left(\frac{\partial\lambda}{\partial w}\right)^2 E', \quad F = S\bar{x}_w \bar{x}_v = \left(\frac{\partial\lambda}{\partial w}\right) F', \quad G = S\bar{x}_v{}^2 = G',$$
$$H^2 = EG - F^2.$$

We notice that [8]

$$\bar{x}_w{}^0 = 1, \quad \bar{y}_w{}^0 = 0, \quad \bar{z}_w{}^0 = 0, \quad \bar{x}_v{}^0 = 0, \quad \bar{y}_v{}^0 = 1, \quad \bar{z}_v{}^0 = 0, \quad \bar{x}_{wv}{}^0 = 0,$$
$$\bar{y}_{wv}{}^0 = 0, \quad \bar{z}_{wv}{}^0 = 1, \quad \bar{x}^0 = \bar{y}^0 = \bar{z}^0 = 0,$$

so that the coordinates of $(\bar{x})$ also satisfy the relations (3) at the point P. That is, when $Q_u{}^h$ is referred to the same coordinate system as S at the point P, its expansions in the neighborhood at P *can be obtained by replacing in* (4) *the quantities* β, γ, β_u, β_v, γ_u, γ_v, θ_u, θ_v *etc. by* $\bar{\beta}$, $\bar{\gamma}$, $\bar{\beta}_w$, $\bar{\beta}_v$, $\bar{\gamma}_w$, $\bar{\gamma}_v$, $\bar{\theta}_w$, $\bar{\theta}_v$ *etc., respectively.* But a simple calculation, gives

$$E^0 = 1, G^0 = 1, F^0 = 0, E_w{}^0 = 2\theta_u, E_v{}^0 = 0, G_w{}^0 = 0, G_v{}^0 = 2\theta_v, F_w{}^0 = 0, F_v{}^0 = \gamma,$$
$$(13) \quad F^0{}_{wv} = 1 + \gamma_u + \gamma\theta_u, F^0{}_{vv} = \gamma_v + \gamma\theta_v + \theta_{uv} + (1-h)\beta\gamma, \quad G^0{}_{ww} = 2,$$
$$G^0{}_{wv} = 2[\theta_{uv} + (1-h)\beta\gamma],$$

---

[8] We put $\bar{x}^0 = (\bar{x})_0$, etc. to represent the value of $\bar{x}$, etc. at P.

where we have taken λ such that

$$\left(\frac{\partial^2\lambda}{\partial w^2}\right)_0 = \theta_u.$$

Thus at P we obtain

$$(14)\quad \bar{\beta} = 0,\ \bar{\gamma} = \gamma,\ \theta_v = \theta_v,\ \bar{\theta}_w = \theta_u,\ \bar{\gamma}_v = \gamma_v,\ \bar{\gamma}_w = \gamma_u,\ \bar{\theta}_{wv} = \theta_{uv} + (1-h)\beta\gamma.$$

2. Certain invariant quadrics associated with Q_u^h and Q_v^h. In this section we shall give an application of the above result to the determination of certain invariant quadrics associated with Q_u^h and Q_v^h such as the asymptotic osculating quadrics, asymptotic chord quadrics[9] and the Moutard quadrics of these surfaces. The equation of the Moutard quadric belonging to a tangent whose equations are $z = 0$, $y - nx = 0$ takes the form,

$$\begin{gathered}[4(\beta + \gamma_u^3)^2 + 3n(\beta_u + 4\beta_v n + 4\gamma_u n^3 + \gamma_v n^4)]z^2 + 36n^3(z - xy + \tfrac{1}{2}\theta_{uv}z^2) \\ - 12n(\beta - 2\gamma_u^3)(y + \tfrac{1}{2}\theta_u z)z - 12n^2(\gamma_u^3 - 2\beta)(x + \tfrac{1}{2}\theta_v z)z = 0.\end{gathered}$$

Replacing β, γ, β_u, β_v, γ_u, γ_v, θ_u, θ_v, θ_{uv} in the above equation by $\bar{\beta}$, $\bar{\gamma}$, $\bar{\beta}_w$, $\bar{\beta}_v$, $\bar{\gamma}_w$, $\bar{\gamma}_v$, $\bar{\theta}_w$, $\bar{\theta}_v$, $\bar{\theta}_{wv}$, respectively, and then using the relations (14), we immediately obtain the equation of the Moutard quadric of Q_u^h belonging to the tangent in question, namely,

$$\begin{gathered}(15)\quad [4\gamma^2 n^3 + 3n(4\gamma_u + \gamma_v n)]z^2 + 36[z - xy + \tfrac{1}{2}\{\theta_{uv} + (1-h)\beta\gamma\}z^2] \\ + 24n\gamma(y + \tfrac{1}{2}\theta_u z)z - 12n^2\gamma(x + \tfrac{1}{2}\theta_v z)z = 0.\end{gathered}$$

Interchanging x and y; β and γ; u and v; n and $1/n$ in the above equation we have the equation of the Moutard quadric of Q_v^h belonging to the same tangent

$$\begin{gathered}(15')\quad [4\beta^2 + 3n(4\beta_v n + \beta_u)]z^2 + 36[z - xy + \tfrac{1}{2}\{\theta_{uv} + (1-h)\beta\gamma\}z^2]n^3 \\ + 24n^2\beta(x + \tfrac{1}{2}\theta_v z)z - 12n\beta(y + \tfrac{1}{2}\theta_u z)z = 0.\end{gathered}$$

For $h = 0$, the corresponding quadric has already been obtained by E. Čech.[10] George Wu[11] has recently determined these equations in a more general form.

If $n \to 0$ in the equation (15), we have the equation of a Darboux quadric D_h. Therefore the quadric D_h in the Darboux pencil is the limit of the Moutard quadric of Q_u^h or Q_v^h. More generally, we have the following theorem:

---

[9] Cf. E. Bompiani, *loc. cit.*

[10] Fubini-Čech, *Geometria proiettiva differenziale*, vol. 2, Bologna (1927), pp. 529-533.

[11] George Wu, "Systems of quadrics associated with a point of a surface."

The osculating quadric of a non-developable ruled surface along a generator is the limiting position of the Moutard quadric belonging to a non asymptotic tangent t at a point P on this generator when t tends to the tangent of the curved asymptotic line through P.

One of the asymptotic osculating quadrics associated with the curve $dv - \lambda du = 0$ is given by the equation

$$2\lambda^3(xy - z) + 2\beta\lambda yz - 2\beta\lambda^2 xz + [-\lambda^3(\theta_{uv} + \beta\gamma) + \beta\{\lambda' - \beta - (\frac{\beta_u}{\beta} - \theta_u)\lambda - (2\frac{\beta_v}{\beta} + \theta_v)\lambda^2\}]z^2 = 0,$$

where $\lambda' = d^2v/du^2$ while the other is given by

$$2\left(\frac{du}{dv}\right)^3(xy - z) + 2\gamma\frac{du}{dv}xz - 2\gamma\left(\frac{du}{dv}\right)^2 yz + \left[-\left(\frac{du}{dv}\right)^3 \{\theta_{uv} + (1 - h)\beta\gamma\} + \gamma\left\{\frac{d^2u}{dv^2} - \gamma - \left(\frac{\gamma_v}{\gamma} - \theta_v\right)\frac{du}{dv} - (2\frac{\gamma_u}{\gamma} + \theta_u)\left(\frac{du}{dv}\right)^2\right]z^2 = 0.$$

By the method we have so far used there is no difficulty in finding the equation of the asymptotic osculating quadrics of the ruled surface Q_u^h associated with the curve C_λ: $dv - \lambda du = 0$ on it.[12] Plainly, one of them is the quadric D_h of Darboux, while the other is given by the equation.[13]

$$(16)\quad 2\left(\frac{du}{dv}\right)^3(xy - z) + 2\gamma\frac{du}{dv}xz - 2\gamma\left(\frac{du}{dv}\right)^2 yz + \left[-\left(\frac{du}{dv}\right)^3 \{\theta_{uv} + (2 - h)\beta\gamma\} + \gamma\left\{\frac{d^2u}{dv^2} - \gamma - \left(\frac{\gamma_v}{\gamma} - \theta_v\right)\frac{du}{dv} - (2\frac{\gamma_u}{\gamma} + \theta_u)\left(\frac{du}{dv}\right)^2\right\}\right]z^2 = 0.$$

Similarly, we obtain, for the ruled surface Q_v^h, the quadric

$$(16')\quad 2(xy - z) + 2\beta\left(\frac{du}{dv}\right)^2 yz - 2\beta\frac{du}{dv}xz + \left[-\{\theta_{uv} + (2 - h)\beta\gamma\} + \beta\left\{-\frac{d^2u}{dv^2} - \beta\left(\frac{du}{dv}\right)^3 - \left(\frac{\beta_u}{\beta} - \theta_u\right)\left(\frac{du}{dv}\right)^2 - (2\frac{\beta_v}{\beta} + \theta_v)\frac{du}{dv}\right\}\right]z = 0.$$

---

[12] Of course, the curve C_λ must be on Q_n^h, but each quadric depends only on the curvilinear element of the second order of C_λ and Q_n^h at P has contact of the second order with S. Therefore this condition is not necessary.

[13] Noticing that $dv(d^2u + \theta_u du^2) - du(d^2v + \theta_v dv^2) = 0$ is intrinsic and $\bar{\theta}_v = \theta_v$,

It is remarkable that *the quadrics* (16) *and* (16′) of the surfaces Q'_u *and* Q'_v *for a curve through* P *are precisely the asymptotic osculating quadrics of* S *for a curve of* S *which has a contact of the second order with the given curve at* P.

Since the residual conic of intersection of these two quadrics (16) and (16′) for $Q_u^{h_1}$ and $Q_v^{h_2}$ lies on the plane π_λ whose equation is

$$\begin{aligned}
&\Big[-\frac{d^2u}{dv^2}\Big\{\gamma+\beta\Big(\frac{du}{dv}\Big)^3\Big\}-\beta^2\Big(\frac{du}{dv}\Big)^6+\gamma^2+(\gamma_v-\gamma\theta_v)\frac{du}{dv}\\
&+(2\gamma_u+\gamma\theta_u)\Big(\frac{du}{dv}\Big)^2-(\beta_u-\beta\theta_u)\Big(\frac{du}{dv}\Big)^3+\beta\gamma\Big(\frac{du}{dv}\Big)^3(h_2-h_1)\\
&-(2\beta_v+\beta\theta_v)\Big(\frac{du}{dv}\Big)^4\Big]z-2\frac{du}{dv}\Big(\gamma+\beta\Big(\frac{du}{dv}\Big)^3\Big)x+2\Big(\frac{du}{dv}\Big)^2\Big(\beta\Big(\frac{du}{dv}\Big)^3+\gamma\Big)y=0
\end{aligned}$$

and the osculating plane P_λ of C_λ at P is given by

$$2\Big(\frac{du}{dv}\Big)^2 y-2\frac{du}{dv}x-\Big[\beta\Big(\frac{du}{dv}\Big)^3-\gamma+\frac{du}{dv}\theta_v-\Big(\frac{du}{dv}\Big)^2\theta_u-\frac{d^2u}{dv^2}\Big]z=0,$$

we may conclude that they coincide when and only when the elements of C_λ at P satisfy the following relation:

$$\begin{aligned}
2\frac{d^2u}{dv^2}\Big(\gamma+\beta\Big(\frac{du}{dv}\Big)^3\Big)=\beta\gamma\Big(\frac{du}{dv}\Big)^3(h_2-h_1)+\Big[\gamma_v-\beta_u\Big(\frac{du}{dv}\Big)^4\Big]\frac{du}{dv}\\
+2\Big(\frac{du}{dv}\Big)^2\Big[\gamma_u-\beta_v\Big(\frac{du}{dv}\Big)^2\Big].
\end{aligned}$$

For the sake of convenience we shall call a curve having this property *the associate curve.* Especially when $h_1=h_2$, these curves become pangeodesics.[14] Denoting a plane through P by the equation $u_1x+u_2y+u_3z=0$, we have for the osculating plane of the associate curve at P

$$\begin{aligned}
&\rho u_1=-2n,\quad \rho u_2=2n^2,\\
&\rho u_3=-\beta n^3+\gamma-n\theta_v+n^2\theta_u\\
&\qquad+\frac{1}{2(\gamma+\beta n^3)}\left[\beta\gamma n^3(h_2-h_1)+(\gamma_v-\beta_u n^4)n+2(\gamma_u-\beta_v n^2)n^2\right]
\end{aligned}$$

where $n=du/dv$. In consequence, this plane envelopes a cone Γ_6 of class 6

$\bar\theta_w=\theta_u$ at P, we may obtain the required equation (16) by substituting (14) into the corresponding equation of the asymptotic osculating quadric of S at P and changing d^2u/dv^2 into $d^2u/dv^2-\beta(du/dv)^3$, the other quantities being unchanged.

[14] For a new definition of pangeodesics cf. E. P. Lane, "Quadriche aventi per generatrici le tangenti asintotiche in un punto di una superficie," *Rendiconti dei Lincei*, (VI), vol. 5 (1927), pp. 100-104.

when the tangent varies at P. The equation of this cone in Fubini's normal coordinates is

$$(17)\quad 2u_1u_2u_3(\beta u_2^3-\gamma u_1^3)-u_1u_2\left\{\frac{\phi}{2}(\beta u_2^3-2\gamma u_1^3)u_2+\frac{\psi}{2}(2\beta u_2^3-\gamma u_1^3)u\right.$$
$$\left.-(\beta^2u_2^6-\gamma^2u_1^6)-\beta\gamma(h_2-h_1)u_2^3u_1^3=0,\right.$$

where we have placed[15]

$$(18)\quad \phi=2\theta_u-\frac{\beta_u}{\beta}=\frac{\gamma_u}{\gamma}+\theta_u,\quad \psi=2\theta_v-\frac{\gamma_v}{\gamma}=\frac{\beta_v}{\beta}+\theta_v,\quad (\theta=\log\beta\gamma).$$

Evidently, the cone (17) is that of Segre when $h_1=h_2$.

One of the asymptotic chord quadrics associated with a curve of S through P has the equation

$$(19)\quad xy-z-\tfrac{1}{2}\beta\frac{du}{dv}xz+\tfrac{1}{6}\beta\left(\frac{du}{dv}\right)^2yz$$
$$-\tfrac{1}{2}\left\{\theta_{uv}+\beta\gamma+\tfrac{1}{3}(\beta_u-\tfrac{1}{2}\beta\theta_u)\left(\frac{du}{dv}\right)^2+(\beta_v+\tfrac{1}{2}\beta\theta_v)\frac{du}{dv}+\tfrac{1}{2}\beta\frac{d^2u}{dv^2}\right\}z^2=0,$$

and the other the equation

$$(20)\quad xy-z-\tfrac{1}{2}\gamma\frac{dv}{du}yz+\tfrac{1}{6}\gamma\left(\frac{dv}{du}\right)^2xz-\tfrac{1}{2}\left\{\theta_{uv}+\beta\gamma+\tfrac{1}{3}(\gamma_v-\tfrac{1}{2}\gamma\theta_v)\left(\frac{dv}{du}\right)^2\right.$$
$$\left.+(\gamma_u+\tfrac{1}{2}\gamma\theta_u)\frac{dv}{du}-\tfrac{1}{2}\gamma\left(\frac{dv}{du}\right)^3\frac{d^2u}{dv^2}\right\}z^2=0.$$

For the curve $C_u^h(C_v^h)$ the first (second) quadric becomes a Darboux quadric $D_{\frac{1}{2}h}$:

$$(21)\quad xy-z-\tfrac{1}{2}\{\theta_{uv}+(1-\tfrac{1}{2}h)\}z^2=0.$$

If $C_u^h(C_v^h)$ has a point of inflexion at P, we obtain *the quadric* $D_{\frac{1}{2}}$; if C_u^h has a stationary osculating plane at P, we have *the quadric* $D_{1/6}$.

By the method we have so far used, we find the corresponding quadrics for the surface Q_u^h. In fact, one of them is a quadric of Darboux and the other is given by the equation

$$(22)\quad xy-z-\tfrac{1}{2}\gamma\frac{dv}{du}yz+\tfrac{1}{6}\gamma\left(\frac{dv}{du}\right)^2xz-\tfrac{1}{2}\left\{\theta_{uv}+(3/2-h)\beta\gamma+\tfrac{1}{3}(\gamma_v-\tfrac{1}{2}\gamma\theta_v)\left(\frac{dv}{du}\right)\right.$$
$$\left.+(\gamma_u+\tfrac{1}{2}\gamma\theta_u)\frac{dv}{du}-\tfrac{1}{2}\gamma\left(\frac{dv}{du}\right)^3\frac{d^2u}{dv^2}\right\}z^2=0.$$

---

[15] In fact, if an equation in our coordinate system contains only the quantities β, γ, θ_u, θ_v, β_u, β_v, γ_u, γ_v, θ_{uv}, then it can be transformed to Fubini's normal coördinate system by putting $\theta=\log\beta\gamma$ in the equation without changing other quantities.

Similarly, we obtain, for the ruled surface, the quadric

$$(23)\quad xy-z-\tfrac{1}{2}\beta\frac{du}{dv}xz+\tfrac{1}{6}\beta\left(\frac{du}{dv}\right)^2yz-\tfrac{1}{2}\left\{\theta_{uv}+(3/2-h)\beta\gamma+\tfrac{1}{3}(\beta_u-\tfrac{1}{2}\beta\theta_u)\left(\frac{du}{dv}\right)^2\right.$$

$$\left.+(\beta_v+\tfrac{1}{2}\beta\theta_v)\frac{du}{dv}+\tfrac{1}{2}\beta\frac{d^2u}{dv^2}\right\}z^2=0.$$

For $h=\frac{1}{2}$, *these quadrics are asymptotic chord quadrics of* S *at* P.

3. The asymptotic curves of Q_u^h and Q_v^h. The osculating quadric of Q_u^h (or Q_v^h) at P is the Darboux quadric D_h of S. As any Darboux quadric must pass through the asymptotic tangents of S at P, and Q_u^h (or Q_v^h) has at P, the same tangent plane as S, the two asymptotic directions of Q_u^h (or Q_v^h) at P coincide with those of S at the same point. Consequently, the curved asymptotic line of Q_u^h (or Q_v^h) at P is tangent to the asymptotic curve $u=0$ (or $v=0$) of S. We can, however, show a more precise result, namely,

The curved asymptotic line of Q_u^h (Q_v^h) *and the asymptotic curve* $u=0$ $(v=0)$ *of* S *have a common osculating linear complex at* P.

To prove this, let the Plücker coordinates of a line joining two points (x) and $(\bar{x})$ be denoted by

$$p_{12}=x\bar{y}-\bar{x}y,\ p_{13}=x\bar{z}-\bar{x}z,\ p_{23}=y\bar{z}-\bar{y}z,\ p_{14}=x-\bar{x},\ p_{24}=y-\bar{y},\ p_{34}=z-\bar{z}$$

then the asymptotic osculating linear complex of asymptotic curves $u=0$ of S at P is given by the following equation:

$$(24)\qquad p_{12}-p_{34}+\left(\theta_u+\frac{\gamma_u}{\gamma}\right)p_{13}=0.$$

In order to obtain the asymptotic osculating linear complex of the curved asymptotic curve of Q_u^h at P, we have to replace θ_u, γ_u, γ in the above equation by $\bar{\theta}_w=\theta_u$, $\bar{\gamma}_w=\gamma_u$ and $\bar{\gamma}=\gamma$, respectively, which obviously leaves the equation unchanged.

In a similar way we can establish the result for Q_v^h.

We come now to consider the expansion for the curved asymptotic line of Q_u^h (or Q_v^h) at P up to higher orders. For this purpose the fundamental quantities of the surface Q_u^h (or Q_v^h) are required.

Since

$$x_{vvv}=\frac{d^3x}{dv^3}+3\frac{d^2\lambda}{dv^2}\,\frac{dx_u}{dv}+3\frac{d\lambda}{dv}\,\frac{d^2x_u}{dv^2}+\lambda\frac{d^3x_u}{dv^3}+\frac{d^3\lambda}{dv^3}x_u.$$

and $\bar{\theta}_v^2+\bar{\theta}_{vv}=\theta_v^2+\theta_{vv}$, $\bar{\theta}_v=\theta_v$ at P, we have

$$\bar{\theta}_{vv} = \theta_{vv}, \tag{25}$$

and

$$(\bar{x}_{vvvv})_0 = \left(\frac{d^3 x_v}{dv^3} - 3 \frac{d^2 u}{dv^2} x_{uvv} \right)_0 = (x_{vvv})_0,$$

$$\bar{\gamma}_{vv} = \gamma_{vv}. \tag{25}$$

It is well known that the power series expansions for the asymptotic curve $u = 0$ of S in the neighborhood of P are of the form

$$\begin{aligned} x &= \bar{a}y^2 + \bar{b}y^3 + \bar{c}y^4 + (5), \\ z &= \bar{r}y^3 + \bar{s}y^4 + \bar{t}y^5 + (6), \end{aligned} \tag{27}$$

where we have placed

$$\begin{aligned} &\bar{a} = \tfrac{1}{2}\gamma,\ \bar{b} = \tfrac{1}{6}(\gamma_v - 2\gamma\theta_v),\ \bar{c} = \tfrac{1}{24}\{\gamma(-2\theta_{vv} + 6\theta_v^2 + \gamma\theta_u + \gamma_u) + \gamma_{vv} - 5\theta_v\gamma_v\}, \\ &\bar{r} = \tfrac{1}{6}\gamma,\ \bar{s} = \tfrac{1}{2}\bar{b}, \qquad \bar{t} = \tfrac{3}{5}\bar{c} - \tfrac{1}{60}\gamma(\gamma\theta_u + \gamma_u). \end{aligned} \tag{28}$$

An application of the preceding method to this case gives the result that *the curved asymptotic line of* Q_u^h *has at* P *a contact of the fourth order at least with the asymptotic curve* $u = 0$ *of* S.

More generally, we can prove the following theorem:

If a curve C_u^h *has at* P *a contact of order* m *with the asymptotic curve* $u = 0$ *of* S, *then the curved asymptotic line of the asymptotic ruled surface associated with* C_u^h *has with the asymptotic curve* $u = 0$ *of* S *at* P *a contact of order* $m + 3$ *at least.*

In fact, denote the n-th derivative of a function x along the curved asymptotic line of Q_u^h by $X^{(n)}$ and that along the asymptotic curve $u = 0$ of S by $X^{[n]}$. As this theorem has been proved in the case $m = 1$, we can put $m = n - 1 \geqq 2$ and from the given condition we have at P

$$\frac{du}{dv} = \frac{d^2u}{dv^2} = \cdots = \frac{d^{n-1}u}{dv^{n-1}} = 0,$$

so that at P:

$$\frac{d\lambda}{dv} = \frac{d^2\lambda}{dv^2} = \cdots = \frac{d^{n-1}\lambda}{dv^{n-1}} = 0, \quad \frac{d^n\lambda}{dv^n} = -\frac{d^nu}{dv^n}, \quad \frac{d^{n+1}\lambda}{dv^{n+1}} = -\frac{d^{n+1}u}{dv^{n+1}}.$$

Since $\bar{x} = x + \lambda x_u$ we have at P

$$\bar{x}^{(n)} = x^{(n)} + (\lambda x_u)^{(n)} = x^{(n)} + \lambda^{(n)} x_u = \left(x_u \frac{du}{dv}\right)^{(n-1)}$$

$$+ x_v^{(n-1)} + \lambda^{(n)} x_u = x_u \left(\frac{d^n u}{dv^n} + \lambda^{(n)}\right) + x_v^{(n-1)} = x_0^{(n-1)} = x_v^{[n-1]} = x^{[n]},$$

$$\bar{x}^{(n+1)} = x^{(n+1)} + (\lambda x_u)^{(n+1)} = \left(x_u \frac{du}{dv}\right)^{(n)} + x_v^{(n)} + \lambda^{(n+1)} x_u + (n+1)\lambda^{(n)} x_u^{(1}$$

$$= x_u \frac{d^{n+1}u}{dv^{n+1}} + n \frac{d^n u}{dv^n} x_u^{(1)} + x_{uv} \frac{d^n u}{dv^n} + x_{vv}^{[n-1]} + \lambda^{(n+1)} x_u + (n+1)\lambda^{(n)} x_u^{(}$$

$$= x_u \left(\frac{d^{n+1}u}{dv^{n+1}} + \lambda^{(n+1)}\right) + x_{uv}(n+1)\left(\frac{d^n u}{dv^n} + \lambda^{(n)}\right) + x^{[n+1]} = x^{(n+1)},$$

$$\bar{x}^{(n+2)} = x^{(n+2)} + (\lambda x_u)^{(n+2)} = \left(x_u \frac{du}{dv}\right)^{(n+1)} + x_v^{(n+1)} + \lambda^{(n+2)} x_u$$

$$+ (n+2)\lambda^{(n+1)} x_u^{(1)} + \frac{(n+2)(n+1)}{2} \lambda^{(n)} x_u^{(2)}$$

$$= x_u \frac{d^{n+2}u}{dv^{n+2}} + (n+1) x_{uv} \frac{d^{n+1}u}{dv^{n+1}} + \frac{(n+1)n}{2} x_u^{(2)} \frac{d^n u}{dv^n}$$

$$+ x_{uv} \frac{d^{n+1}u}{dv^{n+1}} + n x_{uvv} \frac{d^n u}{dv^n} + x_{uvv} \frac{d^n u}{dv^n}$$

$$+ x^{[n-2]} + \lambda^{(n+2)} x_u + (n+2)\lambda^{n+1} x_{uv} + \frac{(n+2)(n+1)}{2} \left[x_{uu} \frac{d^2 u}{dv^2} + x_{uvv}\right]\lambda^{(n}$$

As we have at P

$$\frac{d^{n+2}u}{dv^{n+2}} + \lambda^{(n+2)} = -(n+1)\lambda^{(n)} \frac{d^2 u}{dv^2} \theta_u = (n+1) \frac{d^n u}{dv^n} \frac{d^2 u}{dv^2} \theta_u,$$

it follows that

$$\bar{x}^{(n+2)} = x^{[n+2]} - (n-1)\beta \frac{d^2 u}{dv^2} \frac{d^n u}{dv^n} x_v.$$

For $m \geqq 2$ i. e. $n \geqq 3$, we have obtained

$$\bar{x}^{(n+2)} = x^{[n+2]},$$

which completes the proof.

Further, we can prove the following result

If the asymptotic osculating linear complex of the asymptotic curve $u = 0$ at P is stationary, so also is that of the curved asymptotic line of Q_u^0 at P.

For at P we have

$$\begin{aligned} \bar{x}_{vvvw} &= x_{uvvv} - \beta\gamma^2 h(h+2), \\ \bar{\gamma}_{wv} &= \gamma_{uv} - 2\gamma^2 h^2 \beta - \beta\gamma^2, \end{aligned} \tag{29}$$

and therefore

$$(29') \qquad -\frac{\partial^2 \log \bar{\gamma}}{\partial u \partial v} = k_1 - 2\beta\gamma h^2,$$

where we have placed $k_1 = \beta\gamma - (\partial^2 \log \gamma/\partial u \partial v)$.

By means of these relations we have simultaneously found a geometrical meaning for the invariant $k_1/\beta\gamma$, and similarly for $h_1/\beta\gamma$, where $h_1 = \beta\gamma - (\partial^2 \log \beta/\partial u \partial v)$.

4. Generalizations of Darboux's pencil of quadrics and Bompiani-Lane's principal quadrics. Since

$$\bar{x}_{ww} = \frac{\partial^2 \lambda}{\partial w^2} x_u, \quad \bar{x}_{wwv} = \frac{\partial^3 \lambda}{\partial v \partial w^2} x_u + \frac{\partial^2 \lambda}{\partial w^2}\left(x_{uu} \frac{du}{dv} + x_{uv}\right),$$

we have at P

$$\begin{aligned}\bar{x}_{wwvv} &= \frac{\partial^4 \lambda}{\partial v^2 \partial w^2} + \theta_u(-\theta_u \gamma h + \gamma_u + \gamma\theta_u) \\ &= (\theta_{uv} + \beta\gamma)_v + 3\beta_v \frac{d^2u}{dv^2} + \beta \frac{d^3u}{dv^3} + \beta\theta_v \frac{d^2u}{dv^2} - \theta_u^2 \frac{d^2u}{dv^2} + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma \\ &= \theta_v(\theta_{uv} + \beta\gamma) + (\gamma_u + \gamma\theta_u)_u - \gamma(\beta_v + \beta\theta_v) - 3\beta_v\gamma h \\ &\quad + \beta \frac{d^3u}{dv^3} - \beta\gamma h\theta_v + \theta_u^2\gamma h + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma\theta_u).\end{aligned}$$

But

$$\begin{aligned}(\bar{x}_{wwvv})_0 = \bar{\gamma}_{ww} + \gamma_u\theta_u + \gamma\bar{\theta}_{ww} + \theta_u(\gamma_u + \gamma\theta_u) \\ + \theta_v(\theta_{uv} + \beta\gamma) - \beta\gamma h\theta_v + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma\theta_u).\end{aligned}$$

It follows that at P

$$(30) \qquad \bar{\gamma}_{ww} = \gamma_{uu} - \gamma(\beta_v + \beta\theta_v) - 3\beta_v\gamma h + \beta\frac{d^3u}{dv^3} + \gamma(\theta_{uu} - \bar{\theta}_{ww}),$$

where $\bar{\theta}_{ww} = \partial^3\lambda/\partial w^3$. For the following discussion it is not necessary to choose the value of $\partial^3\lambda/\partial w^3$ at P.

Let $Q = x + \xi x_u + \eta x_v + \zeta x_{uv}$ be any point in space; then the conditions of immovability for Q are as follows:

$$(31) \qquad \begin{cases}\dfrac{\partial\xi}{\partial u} = -\theta_u\xi - \beta\gamma\Pi\zeta - 1, & \dfrac{\partial\eta}{\partial u} = -\beta\xi - (\beta_v + \beta\theta_v)\zeta, \quad \dfrac{\partial\zeta}{\partial u} = -\eta - \theta_u\zeta, \\ \dfrac{\partial\xi}{\partial v} = -\gamma\eta - (\gamma_u + \gamma\theta_u)\zeta, & \dfrac{\partial\eta}{\partial v} = -\theta_v\eta - \beta\gamma H\zeta - 1, \quad \dfrac{\partial\zeta}{\partial v} = -\xi - \theta_v\zeta.\end{cases}$$

The equation of the osculating quadric of Q_u^h at P is

$$\xi\eta - \zeta - \tfrac{1}{2}\bar{\theta}_{wv}\zeta^2 = 0.$$

When this quadric moves along the curved asymptotic line of $Q_u{}^h$ at P, the characteristic curve is given by the equations

$$(32)\quad \begin{cases} \zeta\eta-\zeta-\frac{1}{2}\bar{\theta}_{wv}\zeta^2=0, \\ \bar{\gamma}\eta^2+\bar{\gamma}\bar{\phi}\eta\zeta+\frac{1}{2}(\bar{\theta}_{wwv}-\bar{\theta}_v\bar{\theta}_{wv})\zeta^2=0, \quad (\bar{\gamma}\bar{\phi}=\bar{\gamma}_w+\bar{\gamma}\bar{\theta}_w=\gamma_u+\gamma\theta_u). \end{cases}$$

By virtue of the conditions of integrability it is easily shown that at P

$$\bar{\theta}_{wwv}-\bar{\theta}_v\bar{\theta}_{wv}=(\bar{\gamma}_w+\bar{\gamma}\bar{\theta}_w)_w=\bar{\gamma}_{ww}+\gamma\theta_u+\bar{\gamma}\bar{\theta}_{ww}\equiv T.$$

Therefore the second equation of (32) can be rewritten in the form

$$(33)\qquad \bar{\gamma}\eta^2+\bar{\gamma}\bar{\phi}\eta\zeta+\tfrac{1}{2}T\zeta^2=0,$$

which shows that the characteristic curve in question decomposes into one of the asymptotic tangents at P and two other straight lines R_1 and R_2 with the following equations:

$$R_1:\quad 1+\tfrac{1}{2}\theta_{wv}\zeta=k_1\xi,\quad \eta=k_1\zeta;$$
$$R_2:\quad 1+\tfrac{1}{2}\bar{\theta}_{wv}\zeta=k_2\xi,\quad \eta=k_2\zeta,$$

where

$$(34)\quad k_1=(-\bar{\gamma}\bar{\phi}+(\bar{\gamma}\bar{\phi}-2\bar{\gamma}T)^{1/2})/2\gamma,\ k_2=(-\bar{\gamma}\bar{\phi}-(\bar{\gamma}\bar{\phi}-2\bar{\gamma}T)^{1/2})/2\gamma.$$

Making use of the relations (14) and (26) we may also represent R_1 and R_2 by the equations

$$(35)\quad \begin{array}{ll} R_1: & 1+\frac{1}{2}[\theta_{uv}+(1-h)\beta\gamma]\zeta=k_1\xi,\quad \eta=k_1\zeta, \\ R_2: & 1+\frac{1}{2}[\theta_{uv}+(1-h)\beta\gamma]\zeta=k_2\xi,\quad \eta=k_2\zeta, \end{array}$$

where

$$\frac{k_1}{k_2}=\left(-\phi\pm\left(\phi^2-\frac{2}{\gamma}(r\phi)_u+2\beta\psi+6\beta_v h-\frac{2}{\gamma}\beta\frac{d^3u}{dv^3}\right)^{1/2}\right)/2\quad (\gamma\phi=\gamma_u -$$

In particular for $h=0$ and $d^3u/dv^3=0$ (35) becomes the equations of the sides of the quadrilateral of Demoulin. We notice that the points of intersection of R_1 and R_2 with the tangent plane of S at P are the flecnode points of $Q_u{}^h$. Denoting them by P_1 and P_2, we obtain

$$P_1:\quad \zeta=\eta=0,\quad \xi=1/k_1,$$
$$P_2:\quad \zeta=\eta=0,\quad \xi=1/k_2.$$

The conjugate point of P with respect to P_1 and P_2 is independent of the quantity h and is therefore a fixed point for all $Q_u{}^h$. In fact, it is the point of intersection of the second directrix of Wilczynski with the asymptotic tangent $z=y=0$. Thus we have proved the following theorem:

The correspondence between the two flecnode points of each Q_u^h *at* P *is an involution on the asymptotic tangent* $z = y = 0$, *and the double points of it are the point* P *and the point of intersection of the second directrix with the asymptotic tangent.*

When $h = 0$, this coincides with a theorem of Demoulin.[16]

Obviously, the straight lines R_1 and R_2 are the flecnode tangents of Q_u^h at P_1 and P_2 respectively. Now, let S_1 and S_2 be the flecnode tangents of Q_v^h so that R_1, R_2 and S_1, S_2 form a quadrilateral Ω_h in space. Then the line joining the two points of intersection of the diagonal L_1, L_2, respectively, of Ω_h with the tangent plane of S at P is *the second directrix*, and the common intersecting line of L_1 and L_2 through P is the first directrix.[17]

It is readily seen that the associate asymptotic ruled surface Q_u^h of the curve C_u^h defined by the equation

$$\frac{d^3u}{dv^3} = 3\gamma h \frac{\beta_v}{\beta}, \tag{37}$$

has at P the same flecnode points as the asymptotic osculating ruled surface Q_u^0. We now discuss some properties of this kind concerning the curves C_u^h and C_v^h, the latter being defined by interchanging the asymptotic curves u and v.

First we shall determine the power series expansion of this curve C_u^h in the neighborhood of P. A curve C traced on the surface and passing through the point P can be expanded in the form

$$x(u(t), v(t)) = x + A_1 t + \tfrac{1}{2} A_2 t^2 + \tfrac{1}{6} A_3 t^3 + \tfrac{1}{24} A_4 t^4 + (5);$$

similar equations hold for $y(u(t), v(t))$, and $z(u(t), v(t))$. When the curve C is not tangent to the asymptotic tangent $v = 0$ we may take v for the parameter t so that

$$A_1 = x_u \frac{du}{dv} + x_v,$$

$$A_2 = x_{uu} \left(\frac{du}{dv}\right)^2 + 2x_{uv} \frac{du}{dv} + x_{vv} + x_u \frac{d^2u}{dv^2},$$

$$A_3 = x_{uuu} \left(\frac{du}{dv}\right)^3 + 3x_{uuv} \left(\frac{du}{dv}\right)^2 + 3x_{uvv} \frac{du}{dv} + x_{vvv} + 3 \left[x_{uu} \frac{du}{dv} \frac{d^2u}{dv^2} + x_{uv} \frac{d^2u}{dv^2}\right] + x_u$$

. . . .

---

[16] A. Demoulin, "Sur la quadrique de Lie," *Comptes Rendus*, vol. 147 (1908), pp. 493-496.

[17] For the quadrilateral of Demoulin the result has been pointed out by Bompiani and Godeaux.

On account of the relations (3), we obtain the required expansions for the curve $C_u{}^h$, namely,

$$(38)\qquad \begin{cases} x = \frac{\gamma}{2}(1-h)v^2 + \frac{1}{6}\Big[\gamma_v + \gamma\theta_v + \frac{d^3u}{dv^3}\Big]v^3 + (4), \\ y = v + \frac{1}{2}\theta_v v^2 + \frac{1}{6}(\theta_{vv} + \theta_v{}^2)v^3 + (4), \\ z = \frac{\gamma}{6}(1-3h)v^3 + \frac{1}{12}\Big[\gamma\theta_v(1-3h) + \gamma_v + 2\frac{d^3u}{dv^3}v^4 + (5), \end{cases}$$

or

$$38')\qquad \begin{cases} x = \frac{\gamma}{2}(1-h)y^2 + \Big[-\frac{\gamma}{2}(1-h)\theta_v + \frac{1}{6}\Big(\gamma_v + \gamma\theta_v + \frac{d^3u}{dv^3}\Big)\Big]y^3 + (4), \\ z = \frac{\gamma}{6}(1-3h)y^3 + [\frac{1}{12}\Big\{\gamma\theta_v(1-3h) + \gamma_v + 2\frac{d^3u}{dv^3}\Big\} - \frac{1}{4}\gamma\theta_v(1-3h)]y^4 + (5 \end{cases}$$

When $h=1$ this curve has a point of inflexion at P; when $h=\frac{1}{3}$ the osculating plane at P is stationary; when $h=0$ this curve has a contact of the second order with the asymptotic curve $v=0$ at P, and when $h=2$ this curve has a harmonic contact with $u=0$ at P. It is easy to show that for any pair of curves $C_u{}^h$ and $C_v{}^h$ *the three principal points of Bompiani*[18] *are collinear.* In order to determine the line on which lie the principal points, we rewrite the expansions (38') for the curve (37) in the normal coordinate system of Fubini:

$$(39)\qquad \begin{cases} x = \frac{\gamma}{2}(1-h)y^2 + \frac{\gamma(3h-1)}{6}\,\frac{\partial \log \beta^2\gamma}{\partial v}y^3 + (4), \\ z = \frac{\gamma}{6}(1-3h)y^3 + \frac{\gamma(6h-1)}{12}\,\frac{\partial \log \beta^2\gamma}{\partial v}y^4 + (5). \end{cases}$$

Interchanging the pairs x, y; β, γ; u, v we have the expansions for the curve $C_v{}^h$. The line under consideration is given by the equation

$$(40)\qquad \frac{1-6h}{4(1-3h)}\,\frac{\partial \log \beta\gamma^2}{\partial u}x + \frac{1-6h}{4(1-3h)}\,\frac{\partial \log \beta^2\gamma}{\partial v}y + 1 = 0.$$

Hence we obtain the following theorem:

The principal points for any pair of curves $C_u{}^h$ and $C_v{}^h$ lie on a canonical line l_h of the second species. In particular, for the four cases we have so far quoted, the lines are

L_0: The edge of Green;[19]
$L_{1/3}$: The canonical tangent;
L_1: The line $C(-\frac{5}{8})$;
L_2: The canonical line $C(-\frac{11}{20})$.[20]

---

[18] E. Bompiani, *loc. cit.*

[19] Cf. E. Bompiani, *loc. cit.*

[20] Cf. Buchin Su, "Osculating conics of the plane sections through a point of a surface," *American Journal of Mathematics*, vol. 65 (1943), pp. 439-449.

The quadric which has at P a contact of the fourth order with both $C_u{}^h$ and $C_v{}^h$ can be determined as follows:

The equation of this quadric must necessarily be of the form

$$(40) \qquad xy + k_1 z + k_2 xz + k_3 yz + k_4 z^2 = 0.$$

Substituting the expansions (39) into (41), and demanding that the coefficients of y^0, y, y^2, y^3, y^4 in the result must be equal to zero, we have

$$\frac{\gamma}{2}(1-h) - \frac{\gamma}{6}(1-3h)k_1 = 0, \quad \frac{\gamma}{6}(1-3h)k_3 + \frac{\gamma(3h-1)}{6}\psi + \frac{\gamma(6h-1)}{12}k_1\psi = 0.$$

Similarly we obtain the conditions for another asymptotic curve:

$$\frac{\beta}{2}(1-h) - \frac{\beta}{6}(1-3h)k_1 = 0, \quad \frac{\beta}{6}(1-3h)k_2 + \frac{\beta(3h-1)}{6}\phi + \frac{\beta(6h-1)}{12}k_1\phi = 0.$$

Therefore there exist ∞^1 quadrics in question, each of them being given by the equation

$$(42) \quad xy - \frac{3(1-h)}{1-3h}z + \frac{9h-1}{2(1-3h)^2}\phi xz + \frac{9h-1}{2(1-3h)^2}\psi yz + kz^2 = 0.$$

When $h = 0$ the corresponding pencil consists of the principal quadrics.[21] If the curve $C_u{}^h$ and $C_v{}^h$ have P for a point of inflexion, the corresponding quadrics belong to a pencil of cones:

$$(43) \qquad xy + \phi xz + \psi yz + kz^2 = 0.$$

The polar line of the tangent plane at P with respect each of these cones is the first canonical line $C(-1)$.[22]

The common polar reciprocals of the quadrics (42) and the quadrics of Darboux are the line

$$(44) \qquad \frac{1-9h}{4(1-3h)}\phi x + \frac{1-9h}{4(1-3h)}\psi y + 1 = 0,$$

and its conjugate with respect to any quadric of Darboux. In particular when $h = 0$ the corresponding lines are the edges of Green.[23]

Consider the linear complexes each of which has at P a contact of the third order with $C_u{}^h$; they are given by the following equation:

$$(45) \quad (1-3h)^2 p_{12} - h(1+3h)\psi p_{23} - (1-h)(1-3h)p_{34} + kp_{13} = 0,$$

where k is an arbitrary constant.

---

[21] Cf. E. P. Lane, *loc. cit.* [22] Cf. E. Bompiani, *loc. cit.* [23] E. P. Lane, *loc. cit.*

Interchanging the indices 1, 2 and ϕ, ψ we have the corresponding linear complexes of C_v^h at P, namely,

$$(46) \quad (1-3h)^2 p_{12} + h(1+3h)\phi p_{13} + (1-h)(1-3h)p_{34} + k' p_{23} = 0,$$

k' being an arbitrary constant.

We come now to a generalization of the notion of the Darboux pencil. As any quadric passing through the asymptotic tangents at P may be represented by an equation of the form (41), the Plücker coordinates of a generator L_1 in one regulus are[24]

$$p_{12} = k_1(k_1 + k_3\eta), \quad p_{13} = 0, \quad p_{23} = k_1\eta,$$

$$p_{14} = \frac{k_1}{\eta}(k_1 + k_3\eta), \quad p_{24} = k_2(k_1 + k_3\eta) - k_4\eta, \quad p_{34} = -k_1.$$

and those of L_2 in the other regulus are obtained by interchanging the indices 1, 2 and the pairs of values k_2, k_3; ξ, η.

It can be shown that there exist ∞^1 quadrics passing through the asymptotic tangents such that each regulus belongs to one of the linear complexes (45) and (46) respectively. Each of them is found to be

$$(47) \quad xy - \frac{1-h}{1-3h}z + h\,\frac{1+3h}{(1-3h)^2}\phi xz + h\,\frac{1+3h}{(1-3h)^2}\psi yz + kz^2 = 0,$$

where k denotes a parameter. In particular, when $h = 0$, we obtain the Darboux pencil of quadrics. This generalizes the definition of the Lie quadric, as was shown by Calapso.[25] Moreover, we obtain the pencil of cones (43) by putting $h = 1$.

The common polar reciprocals with respect to the pencil of quadrics (47) and the quadric of Lie are the line

$$(48) \quad -\frac{1+3h}{2(1-3h)}\phi x - \frac{1+3h}{2(1-3h)}\psi y + 1 = 0, \quad z = 0,$$

and its conjugate. Especially we obtain

The canonical line $C(\frac{1}{2})$, if $h = 0$,
The edge of Green, if $h = 1$,
The canonical line $C(-\frac{7}{10})$, if $h = 2$.

We can further prove the following theorem:

---

[24] Compare my paper: "An analogue of Darboux pencil of quadrics," *Science Record, Academia Sinica*, vol. 1 (1942), no. 1.

[25] R. Calapso, "Sui sistemi nulli osculatori alle asintotiche di una data superficie," *Rendiconti dei Lincei* (VI), vol. 13 (1931), pp. 183-185.

The linear complexes which have at P a contact of the fourth order with the curve C_u^h $(h \neq 0)$ *defined by* (5) *and* (37) *and the asymptotic curve* $u = 0$, *respectively, have common lines in two planes, namely, the tangent plane and a plane through the first canonical line* $c\left(\frac{1+3h}{2(1-3h)}\right)$.

In fact, the common lines of these linear complexes also belong to the following three linear complexes:

$$\begin{cases} (1-3h)^2(x\bar{y}-\bar{x}y)+h\phi(1+3h)(x\bar{z}-\bar{x}z)+(1-h)(1-3h)(z-\bar{z})=0, \\ y\bar{z}-z\bar{y}=0, \\ x\bar{y}-\bar{x}y-z+\bar{z}=0. \end{cases}$$

Let Δ be the determinant formed by the coefficients of $\bar{x}$, $\bar{y}$, $\bar{z}$ in these equations; $\Delta = 0$ gives the required plane $z = 0$ and

$$y - \frac{1+3h}{2(1-3h)}\phi z = 0. \tag{49}$$

A similar result is obtained on combining the curve C_v^h and the asymptotic curve $v = 0$. In this case we obtain the plane

$$x - \frac{1+3h}{2(1-3h)}\psi z = 0. \tag{50}$$

In consequence, the lines of the first canonical pencil are completely determined.

On the contrary, if we combine the linear complex of C_u^h in question with the asymptotic osculating linear complex of the asymptotic curve $v = 0$ at P, there is obtained a similar result. The canonical line in this case is $c\left(-\frac{6h^2-3h+1}{2(1-3h)(1-2h)}\right)$.

PART B.

The Osculating Linear Complexes Associated with the Asymptotic Ruled Surface.

We now proceed to consider the equation of the osculating linear complex of the asymptotic osculating, ruled surface Q_u^0 (or Q_v^0) along the generator through the point P, that is, the linear complex which has at P a contact of the fifth order with Q_u^0 along the generator.

1. The osculating linear complexes. It is of some interest to determine the equation of the required linear complex by the method stated in **4** of Part A. We shall however give a direct computation. For this purpose, we first notice the following results

$$(p_{12})_0 = 0, \quad (p_{13})_0 = 0, \quad (p_{23})_0 = 0, \quad (p_{14})_0 = 0, \quad (p_{24})_0 = -1, \quad (p_{34})_0 = 0;$$

$$\left(\frac{\partial p_{12}}{\partial u}\right)_0 = 1, \ \left(\frac{\partial p_{13}}{\partial u}\right)_0 = 0, \ \left(\frac{\partial p_{23}}{\partial u}\right)_0 = 0, \ \left(\frac{\partial p_{14}}{\partial u}\right)_0 = 0, \ \left(\frac{\partial p_{24}}{\partial u}\right)_0 = 0, \ \left(\frac{\partial p_{34}}{\partial u}\right)_0 =$$

$$\left(\frac{\partial^2 p_{12}}{\partial u^2}\right)_0 = \theta_u, \ \left(\frac{\partial^2 p_{13}}{\partial u^2}\right)_0 = 2, \ \left(\frac{\partial^2 p_{23}}{\partial u^2}\right)_0 = 0, \ \left(\frac{\partial^2 p_{14}}{\partial u^2}\right)_0 = -(\theta_{u0} + \beta\gamma), \ \left(\frac{\partial^2 p_{34}}{\partial u^2}\right)_0 =$$

$$\left(\frac{\partial^3 p_{12}}{\partial u^3}\right)_0 = \theta_{uu} + \theta_u^2 + 3(\beta_v + \beta\theta_v), \ \left(\frac{\partial^3 p_{13}}{\partial u^3}\right)_0 = 6\theta_u,$$

$$\left(\frac{\partial^3 p_{23}}{\partial u^3}\right)_0 = 2\beta, \ \left(\frac{\partial^3 p_{14}}{\partial u^3}\right)_0 = -(\theta_{uv} + \beta\gamma)_u - 2\theta_u(\theta_{uv}\beta\gamma),$$

$$\left(\frac{\partial^3 p_{34}}{\partial u^3}\right)_0 = -\theta_{uu} - \beta_v - \beta\theta_v - \theta_u^2; \quad \left(\frac{\partial^4 p_{13}}{\partial u^4}\right)_0 = 8\theta_{uu} + 14\theta_u^2 + 4(\beta_v + \beta\theta_v),$$

$$\left(\frac{\partial^4(p_{12} + p_{34})}{\partial u^4}\right)_0 = 2\beta_{uv} + 2\beta_u\theta_v - 2\beta^2\gamma + 8\theta_u\beta_v + 8\beta\theta_u\theta_v,$$

$$\left(\frac{\partial^4 p_{14}}{\partial u^4}\right)_0 = -(\theta_{uv} + \beta\gamma)(3\theta_{uu} + 3\theta_u^2 + \beta_v + \beta\theta_v) - (\theta_{uv} + \beta\gamma)_{uu} - 3\theta_u(\theta_{uv} + \beta\gamma)$$

Substituting these values into the determinant

$$\left| p_{12}(p_{12})_0 \ \left(\frac{\partial p_{12}}{\partial u}\right)_0 \ \left(\frac{\partial^2 p_{12}}{\partial u^2}\right)_0 \ \left(\frac{\partial^3 p_{12}}{\partial u^3}\right)_0 \ \left(\frac{\partial^4 p_{12}}{\partial u^4}\right)_0 \ \right| = 0,$$

and reducing, we obtain

$$\begin{vmatrix} p_{12} + p_{34} & p_{23} & -2p_{14} - (\theta_{uv} + \beta\gamma)p_{13} \\ \beta_v + \beta\theta_v & \beta & (\theta_{uv} + \beta\gamma)_u - \theta_u(\theta_{uv} + \beta\gamma) \\ \beta_{uv} + \beta_u\theta_v - \beta^2\gamma & \beta_u & (\theta_{uv} + \beta\gamma)_{uu} - \theta_u(\theta_{uv} + \beta\gamma)_u \\ & & - \theta_{uu}(\theta_{uv} + \beta\gamma) - (\theta_{uv} + \beta\gamma)(\beta_v + \beta\theta_v) \end{vmatrix} = 0.$$

Put

$$(51) \qquad \beta\gamma H = \theta_{uv} + \beta\gamma, \quad \beta_v + \beta\theta_v = \beta\psi, \quad \gamma_u + \gamma\theta_u = \gamma\phi,$$

$$(52) \qquad \beta\gamma - \frac{\partial^2 \log \beta}{\partial u \partial v} = h_1, \qquad \beta\gamma - \frac{\partial^2 \log \gamma}{\partial u \partial v} = k_1,$$

so that

$$(53) \qquad \beta\gamma H = h_1 + \psi_u = k_1 + \phi_0,$$

$$(\beta\psi)_{uv} - \beta(\gamma\phi)_u \beta\psi(K + \psi_u) = \beta_{uvv} + \beta_u\theta_{vv} + \beta_{uv}\theta_v - 3\beta\gamma\beta_v - \beta^2\gamma\theta_u - \beta^2\gamma;$$

the required equation of the linear complex L_1 is

$$(54) \quad \left(\psi + \frac{\partial \log \beta h_1}{\partial v}\right)(p_{12} + p_{34} - \psi p_{23}) + (\beta\psi)_v p_{23} + \beta(2p_{14} + \beta\gamma H p_{13} - \gamma\phi p_{2}$$

By interchanging the pairs of quantities β, γ; u, v; h_1, k_1 and the indices 1, 2, we obtain the corresponding linear complex L_2 of Q_v^h at P, namely;

$$(55) \quad \left(\phi + \frac{\partial \log \gamma k_1}{\partial u}\right)(p_{34} - p_{12}\,\phi p_{13}) + (\gamma\phi)_u p_{13} + \gamma(2p_{24} + \beta\gamma H p_{23} - \beta\psi p_{13}) = 0.$$

2. A theorem on surfaces whose asymptotic curves belong to linear complexes. If a non-developable ruled surface S be not a quadric and be defined by the equation $\beta = 0$ $(\gamma \neq 0)$, then the osculating linear complex L_1 is indeterminate, while the other linear complex L_2 is given by the equation

$$(55') \quad \left(\phi + \frac{\partial \log \gamma k_1}{\partial u}\right)(p_{34} - p_{12} - \phi p_{13}) + (\gamma\phi)_u p_{13} + \gamma(2p_{24} + \theta_{uv} p_{23}) = 0,$$

where we have put

$$\phi = \frac{\gamma_u}{\gamma} + \theta_u.$$

If one curved asymptotic line $(u = u_0)$ of this ruled surface belongs to a linear complex, then $k_1 = 0$ at $u = u_0$, and the quantity $\partial k_1/\partial u = 0$ or $\neq 0$ according as all the asymptotic curves belong to linear complexes or not. In the latter case, the osculating linear complex of this ruled surface is stationary, and coincides with the osculating linear complex of the asymptotic curve $u = u_0$. In the former case where all curved asymptotic lines belong to linear complexes $(k_1 = 0,\ \partial k_1/\partial u = 0)$, the osculating linear complexes are indeterminate. Thus we have the following theorem:

If a curved asymptotic line on a non-developable ruled surface belongs to a linear complex, then the ruled surface must belong to a linear complex.

We can now improve a theorem of Sullivan[26] in the following form:

If the asymptotic curves of a surface of one system $(u = const.)$ belong to linear complexes, then all the curved asymptotic lines on any asymptotic osculating ruled surface $Q_u{}^0$ must necessary belong to linear complexes, and conversely.

In fact, if the asymptotic curves $u = \text{const.}$ belong to linear complexes, we have $k_1 = 0$ identically for u and v and it follows from (55) that all the osculating linear complexes of the asymptotic osculating ruled surfaces $Q_u{}^0$ are indeterminate. On account of (55′), we conclude that all the curved asymptotic lines of $Q_u{}^0$ belong to linear complexes, which completes the proof.

3. Certain invariant quadrics. The purpose of this section is to prove the following theorem:

---

[26] C. T. Sullivan, *loc. cit.*

At a point P of a surface there exist ∞^1 *tangent quadrics such that both reguli of each quadric belong to the osculating linear complexes* (54) *and* (55) *respectively. Each quadric of the family is completely determined by the contact invariant of the quadric and the surface at P. The quadric of Lie is characterized by the contact invariant I.*

To prove this, let us consider any tangent quadric of the surface S at P:

$$(x+\lambda y)(\mu x+y)+l_1 z+l_2(x+\lambda y)z+l_3(\mu x+y)z+l_4 z^2=0. \tag{56}$$

where λ, μ, l_1, l_2, l_3, l_4 are arbitrary constants.

The Plücker coordinate of a generator l_1 or l_2 in each regulus of the quadric are

$$(57)_1 \qquad L_1:\begin{cases} p_{12}=l_1(l_1+l_3Y), \quad p_{13}=-l_1\lambda Y, \quad p_{23}=l_1Y, \\ p_{14}=\left(\dfrac{l_1}{Y}+l_3\right)(l_1-\lambda l_2Y)+\lambda l_4Y, \\ p_{24}=\left(\dfrac{l_1}{Y}+l_3\right)(l_2Y-l_1\mu)-l_4Y, \\ p_{34}=l_1(\lambda\mu-1); \end{cases}$$

$$(57)_2 \qquad L_2:\begin{cases} p_{12}=-l_1(l_1+l_2X), \quad p_{13}=l_1X, \quad p_{23}=-l_1\mu X, \\ p_{14}=(l_1+l_2X)\left(l_3-\dfrac{l_1\lambda}{X}\right)-l_4X, \\ p_{24}=(l_1+l_2X)\left(\dfrac{l_1}{X}-\mu l_3\right)+\mu Xl_4, \\ p_{34}=l_1(\lambda\mu-1), \end{cases}$$

where X and Y denote parameters.

By means of the equations (54), (55), $(57)_1$, and $(57)_2$, we can easily determine the required quadrics. In fact, in order that the quadric (56) have the property in question, it is necessary and sufficient that

$$(58) \qquad \begin{aligned} \lambda=\mu=0, \qquad l_3&=\frac{l_1+1}{2}\left(\psi+\frac{\partial\log\gamma k_1}{\partial u}\right)\equiv\frac{l_1+1}{2}\Psi, \\ l_2&=\frac{l_1+1}{2}\left(\phi+\frac{\partial\log\beta k_1}{\partial v}\right)\equiv\frac{l_1+1}{2}\Phi, \\ l_4&=\frac{(l_1+1)(l_2+2)}{2}\Phi\Psi+\frac{l_1}{2}\beta\gamma H, \end{aligned}$$

where

$$(58') \qquad \Phi=\phi+\frac{\partial\log\beta h_1}{\partial v}, \qquad \Psi=\psi+\frac{\partial\log\gamma k_1}{\partial u}.$$

Therefore the quadrics are given by the equation

$$(59)\quad xy + l_1 z + \frac{l_1+1}{2}\Phi xz + \frac{l_1+1}{2}\Psi yz + \left\{ \frac{(l_1+1)(l_1+2)}{2}\Phi\Psi + \frac{l_1}{2}\cdot\beta\gamma H \right\} z^2$$

Plainly, $-l_1$ is the contact invariant of this quadric and the surface at P, nd on that account we denote this quadric by $Q(l_1)$. Thus the quadric (-1) is the quadric of Lie.

There are two distinct quadrics $Q(l_1)$ passing through a generic point $'(x', y', z')$ in space. They coincide when and only when the point P lies the following quadric

$$60)\quad [1 + \tfrac{1}{2}\Phi x + \tfrac{1}{2}\Psi y + \tfrac{1}{2}(\beta\gamma H + 3\Phi\Psi)z]^2 - 2\Phi\Psi[xy + \tfrac{1}{2}\Phi xz + \tfrac{1}{2}\Psi yz + \Phi\Psi z^2] = 0$$

We note that this quadric Q is the envelope of the quadrics $Q(l_1)$.

4. Some special surfaces and a pencil of invariant lines. Noticing that the discriminant of (60) is

$$\begin{vmatrix} \Phi^2 & -3\Phi\Psi & \Phi(\beta\gamma H + \Phi\Psi) & 2\Phi \\ -3\Phi\Psi & \Psi^2 & \Psi(\beta\gamma H + \Phi\Psi) & 2\Psi \\ \Phi(\beta\gamma H + \Phi\Psi) & \Psi(\beta\gamma H + \Phi\Psi) & (\beta\gamma H + 3\Phi\Psi)^2 - \gamma(\Phi\Psi)^2 & 2(3\Phi\Psi + \beta\gamma H) \\ 2\Phi & 2\Psi & 2(3\Phi\Psi + \beta\gamma H) & 4 \end{vmatrix} = 64($$

We obtain the following result:

The quadric is never singular unless it decomposes into a double plane.

The equation of this plane is

$$61)\qquad 1 + \tfrac{1}{2}\Phi x + \tfrac{1}{2}\Psi y + \tfrac{1}{2}(\beta\gamma H + 3\Phi\Psi)z = 0, \qquad (\Phi\Psi = 0),$$

nd *the surface is characterized by* $\Phi\Psi = 0$.

Let us now determine a class of surfaces whose quadrics $Q(l_1)$ all touch fixed plane. In this case the plane (61) must be stationary. Hence the required surfaces are defined by $\Phi = \Psi = H = 0$. Consequently, the surfaces of this class are characterized by the fact that all quadrics of Lie touch a fixed plane[27] and the two projective normals coincide with the two directrices of the two osculating linear complexes (54) and (55). The second projective normals lie in the same fixed plane.

Differentiating (61) with respect to u and using the conditions of immovability (31), we obtain the relations

---

[27] Buchin Su, *loc. cit.*

$$\frac{\partial\Phi}{\partial u}=\theta_u\Phi+\beta\Psi-\tfrac{1}{2}\Phi^2,\qquad \frac{\partial\Psi}{\partial u}=\beta\gamma H+{}^5\!/_2\Phi\Psi,$$

$$\frac{\partial}{\partial u}\,(\beta\gamma H+3\Phi\Psi)=\theta_u(\beta\gamma H+3\Phi\Psi)+\Psi(\beta_v+\beta\theta_v)+\beta\gamma H\Phi-\tfrac{1}{2}\Phi(\beta\gamma H+3$$

In a similar way we have

$$\frac{\partial\Phi}{\partial v}=\beta\gamma H+{}^5\!/_2\Phi\Psi,\qquad \frac{\partial\Psi}{\partial v}=\theta_v\Psi+\gamma\Phi-\tfrac{1}{2}\Psi^2,$$

$$\frac{\partial}{\partial v}\,(\beta\gamma H+3\Phi\Psi)=\theta_v(\beta\gamma H+3\Phi\Psi)+\Phi(\gamma_u+\gamma\theta_u)+\beta\gamma H\Psi-\tfrac{1}{2}\Psi(\beta\gamma H+3$$

Since $\Phi\Psi=0$, the unique solution of this system of differential equation is $\Phi=\Psi=H=0$.

The directrices of the two osculating linear complexes (54) and (55) are given by the equations

$$(62)\qquad \begin{array}{ll} L_1: & x+\tfrac{1}{2}\Psi z=0,\quad y+\tfrac{1}{2}\Phi z=0; \\ L_2: & \tfrac{1}{2}\Phi x+\tfrac{1}{2}\Psi y+1=0,\quad z=0, \end{array}$$

respectively; and are polar reciprocal with respect to the quadric of Lie. Fo the sake of convenience they are called the first and the second directrices For the surface $\Phi=\Psi=0$, these lines coincide with the first and secon projective normals. As the plane (61) intersects the tangent plane in th second directrix l_2, we may conclude that if all the quadrics $Q(l)$ for a surfac touch a fixed plane, the first and second projective normals coincide with th first and the second directrices l_1 and l_2, respectively, and all the secon directrices lie on a fixed plane.

By virtue of the conditions of integrability we have

$$\Phi=\psi+\frac{1}{h_1}\,\frac{\partial A^2}{\partial u},\qquad \Psi=\phi+\frac{1}{k_1}\,\frac{\partial B^2}{\partial u},$$

where

$$A^2=-\tfrac{1}{4}\left(2M+2\,\frac{\beta_{vv}}{\beta}-\frac{\beta_v{}^2}{\beta^2}\right),\qquad B^2=-\tfrac{1}{4}\left(2L+2\,\frac{\gamma_{uu}}{\gamma}-\frac{\gamma_u{}^2}{\gamma^2}\right.,$$

and

$$L=\theta_{uu}-\tfrac{1}{2}\theta_u{}^2-\beta\theta_v-\beta_v,\quad M=\theta_{vv}-\tfrac{1}{2}\theta_v{}^2-\gamma\theta_u-\gamma_u.$$

If all the directrices of the two osculating linear complexes (54) and (55) coincide with those of Wilczynski, then the surface must satisfy the following conditions:

$$(63)\qquad \frac{\partial A^2}{\partial u}=0,\qquad \frac{\partial B^2}{\partial v}=0.$$

; is evident that the surfaces of Demoulin and Godeaux[28] defined by $= B = 0$ belong to this class.

Now consider the congruence formed by the first directrices l_1; it is conugate to the surface if and only if

4)
$$\frac{\partial^2 \log \beta^2 h_1}{\partial u \partial v} = \frac{\partial^2 \log \gamma^2 k_1}{\partial u \partial v}.$$

he surfaces of this class may be seen as generalizations of the isothermally symptotic surfaces.

Let us determine a pencil of invariant lines. By a method used by R. alapso[29] we make use of the quadric $Q(l_1)$ instead the quadric of Lie. The orresponding line here obtained is given by the equations

$$x + \frac{\psi + (l_1 + 1)\Psi}{6 + 2l_1} z = 0, \quad y + \frac{\phi + (l_1 + 1)\Phi}{6 + 2l_1} z = 0. \tag{65}$$

All these lines corresponding to various values of l_1 lie in the following plane:

$$(2\phi - 4\Phi)x - (2\psi - 4\Psi)y + (\phi\Psi - \psi\Phi)z = 0. \tag{66}$$

'or $l = -1$ we obtain the edge of Green. This pencil coincides with the irst canonical pencil for the surface (63).

5. Further results on the linear complexes. This section is devoted to ie demonstration the following two results:

(i) *A necessary and sufficient condition that a space curve should belong o a linear complex is that all the osculating linear complexes have a straight ine in common.*

(ii) *Each regulus of any quadric passing through the quadrilateral of emoulin of a surface at a point P belongs to one of the two asymptotic sculating linear complexes at P.*

There is no difficulty in proving (i). We proceed, therefore, to establish he result (ii).

To this end we remark that each regulus of the quadric of Lie belongs to ne of the two asymptotic osculating linear complexes at the point P.[30] Let us denote the four characteristic points, other than P, on the correspondin quadric of Lie by A, B, C, D so that $\{ABCD\}$ is the quadrilateral of Demouli $\overline{AD}$, $\overline{BC}$ being the diagonals. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be the two asymptotic osculatin linear complexes such that $\overline{AC}$, $\overline{BC}$ and $\overline{AB}$, $\overline{CD}$ belong to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. It is required to prove that the lines p_1 and p_2 passing through generic point P' in space and intersecting $\overline{AB}$, $\overline{CD}$ and $\overline{AC}$, $\overline{BD}$, respectivel must belong to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively.

---

[28] Cf. L. Godeaux, "Sur les surfaces ayant mêmes quadriques de Lie (Première ote), *Academie Royale de Belgique* (5), vol. 14 (1928), pp. 158-173.

[29] R. Calapso, "Un teorema sullo spigolo di Green," *Rendiconti dei Lincei* (VI), ol. 13 (1931), pp. 266-268.

[30] R. Calapso, *loc. cit.*

As shown by Bompiani[31] and Godeaux,[32] the diagonals $\overline{AD}$, $\overline{BC}$ intersec the directrices of Wilczynski and therefore $\overline{AD}$, $\overline{BC}$ are common lines o $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Denote the point of intersection of p_1 with $\overline{CD}$ by l and the nul systems defined by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ by $\{\mathfrak{L}_1\}$ and $\{\mathfrak{L}_2\}$, respectively; then the nul plane of A in $\{\mathfrak{L}_1\}$ is the plane ACD and consequently $\overline{AL}$ belongs to $\mathfrak{L}_1$. Th null plane of l in $\{\mathfrak{L}_1\}$ is the plane ABL. Hence p_1 belongs to $\mathfrak{L}_1$ and similarly p_2 belongs to $\mathfrak{L}_2$. This proves the theorem.

From the above results follows immediately a theorem of Su.[33] If all the quadrilaterals of Demoulin of a surface lie on a fixed quadric then all the asymptotic curves of the surface must necessarily belong to linear complexes

THE NATIONAL UNIVERSITY OF CHEKIANG,
KWEICHOW, CHINA.

CALCUTTA UNIVERSITY CENTRAL LIBRARY

---

[31] E. Bompiani, "Fascio di quadriche di Darboux e normale proiettiva in un punto di una superficie," *Rendiconti dei Lincei* (VI), vol. 6 (1927), pp. 187-190.

[32] L. Godeaux, "Sur les congruences formées par les directrices de Wilczynski d'une surface," *Bull. de l'Acad. Roy. Belgique* (5), vol. 14 (1928), pp. 335-345.

[33] B. Su, "On the surfaces whose asymptotic curves belong to linear complexes," *Tôhoku Mathematical Journal*, vol. 40 (1935), pp. 408-420.